MILNOR FILLABLE CONTACT STRUCTURES ARE UNIVERSALLY TIGHT

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Let Y be an oriented and connected 3-dimensional smooth manifold.

Contact forms

A differential 1-form α on Y is called a **contact form** if

$$\alpha \wedge d\alpha$$

is a volume form on Y.

Contact structures

A 2-dimensional distribution ξ in TY is called a **contact structure** if it can be given as the kernel of a contact 1-form α.

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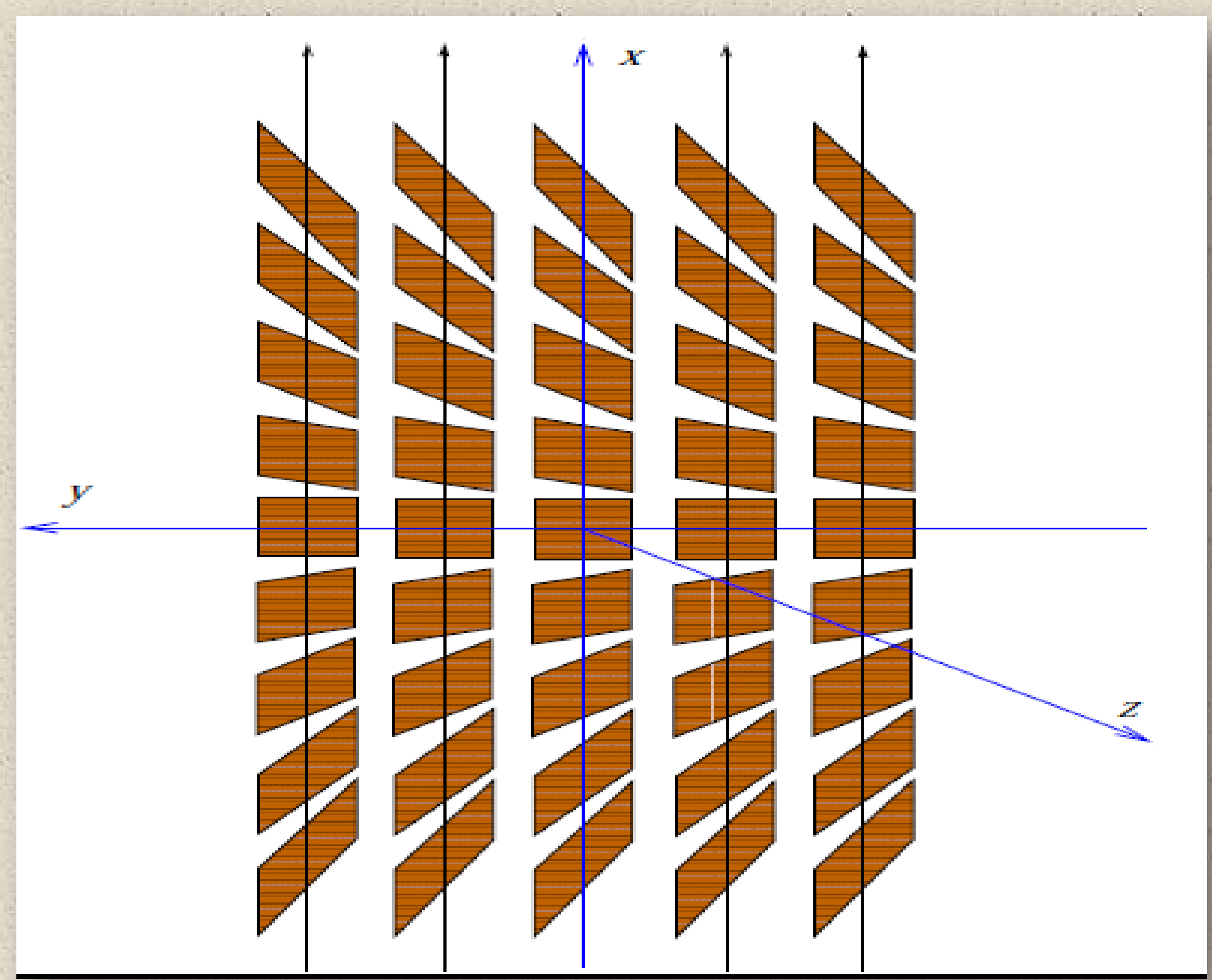
For any $p \in Y$, ξ_p is called the **contact plane**, which is oriented by $d\alpha_p$.

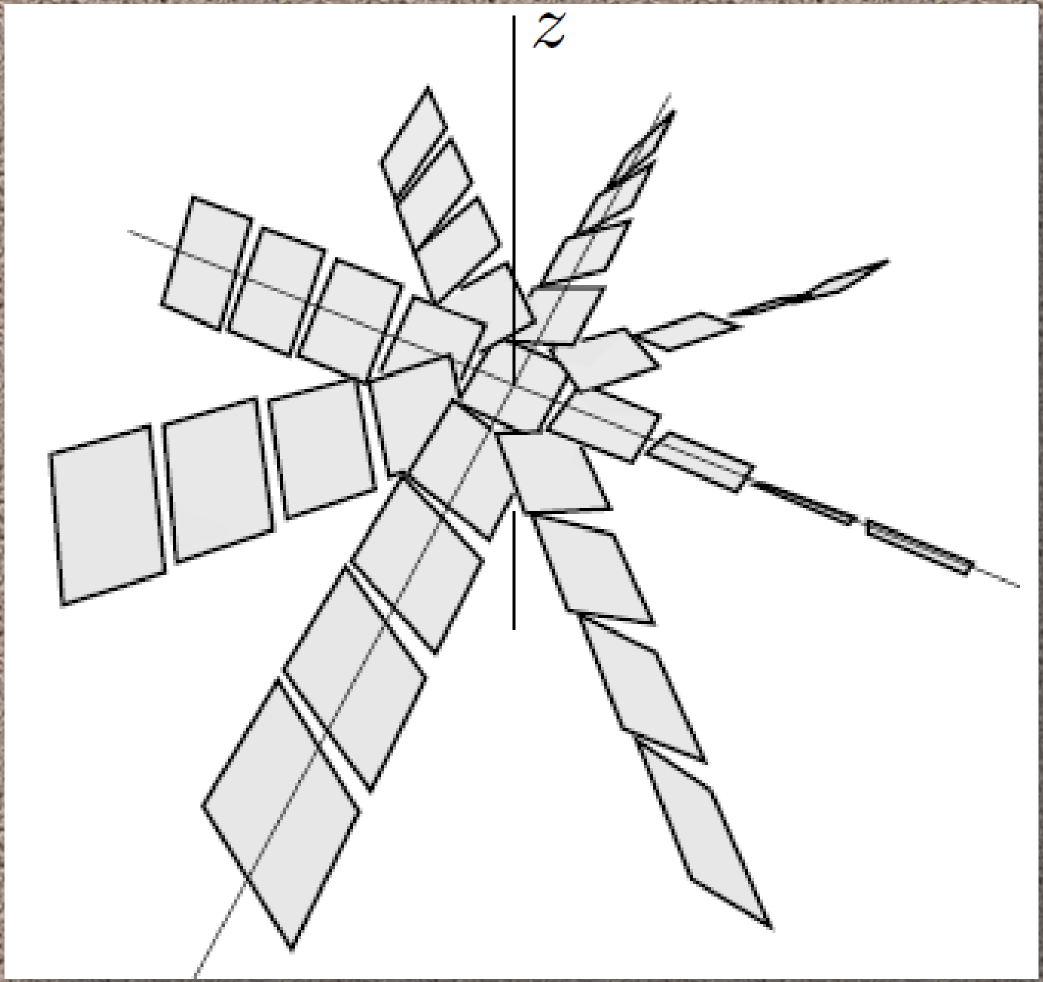
A contact structure ξ on Y is a “maximally non-integrable” oriented 2-plane field.

The standard contact structure ξ in $\mathbb{R}^3$

$$\xi = \ker(dz + x\,dy)$$



The contact structure $\ker(dz + r^2 d\theta)$ is isomorphic to the standard contact structure in $\mathbb{R}^3$.



Contact topology

Darboux: All contact structures look the same near a point, i.e., any point in a contact 3-manifold has a neighborhood isomorphic to a neighborhood of the origin in the standard contact $\mathbb{R}^3$.

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Any interesting phenomena in contact geometry should be related to the global topology of the manifold.

Fundamental dichotomy:

Tight versus overtwisted

Overtwisted disk

An overtwisted disk D is an embedded disk with Legendrian boundary so that $\xi_p = T_p D$ for every point $p \in \partial D$.

Fundamental dichotomy is a tautology

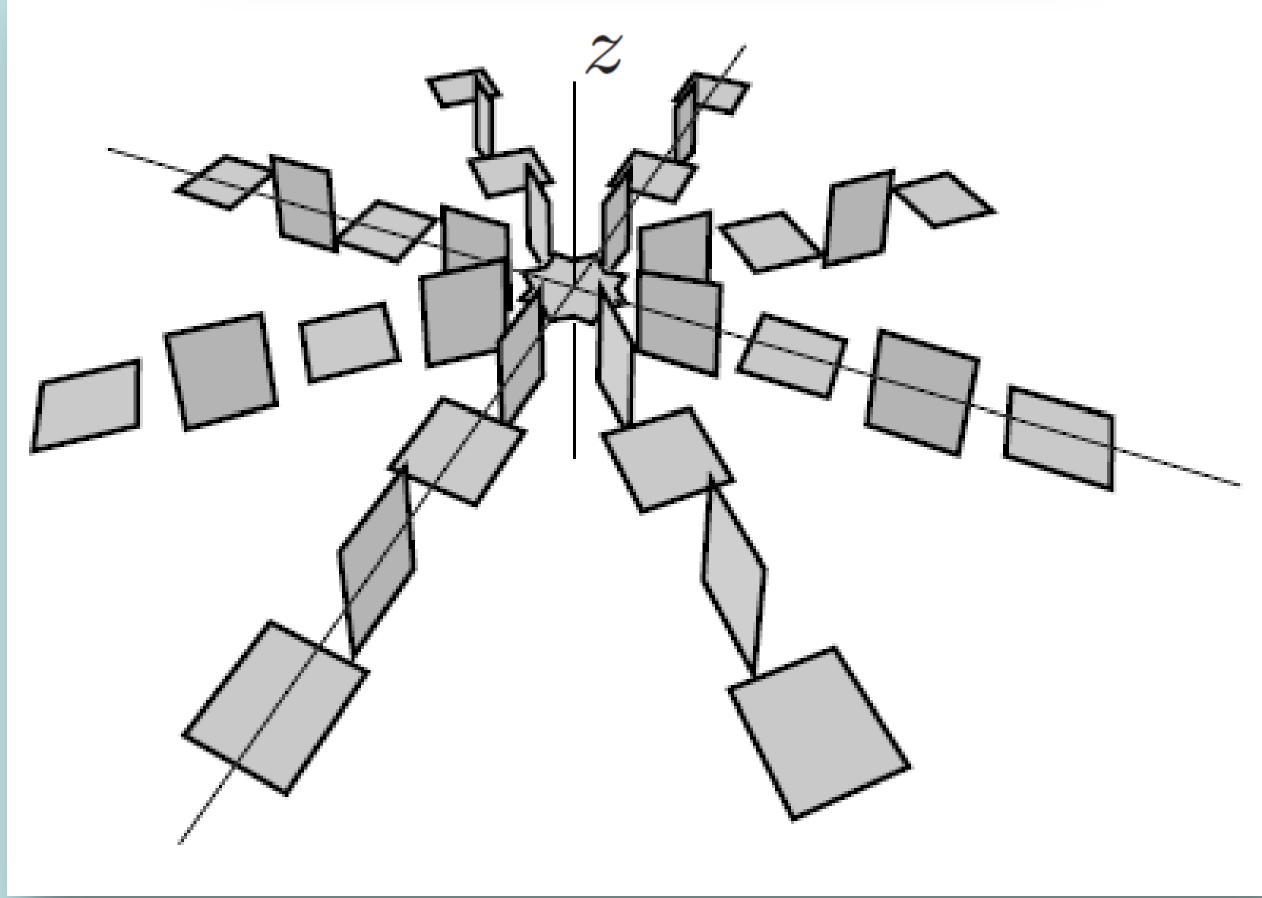
A contact structure in a 3-manifold Y is called overtwisted if Y contains an overtwisted disk.

Fundamental dichotomy is a tautology

A contact structure in a 3-manifold Y is called **overtwisted** if Y contains an overtwisted disk.

A contact structure is called **tight** if it is not overtwisted.

$$\ker(\cos r\,dz + r\sin r\,d\theta)$$



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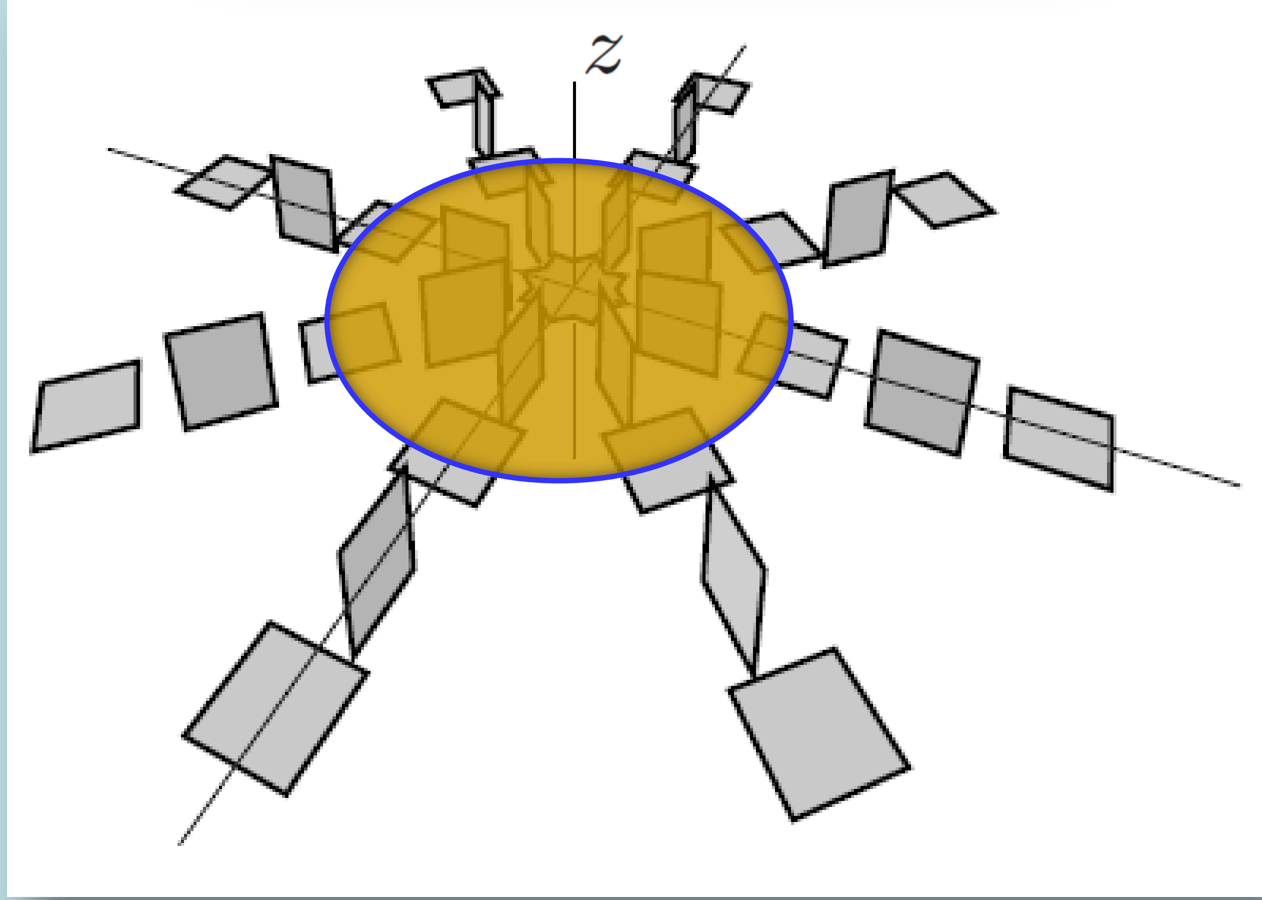
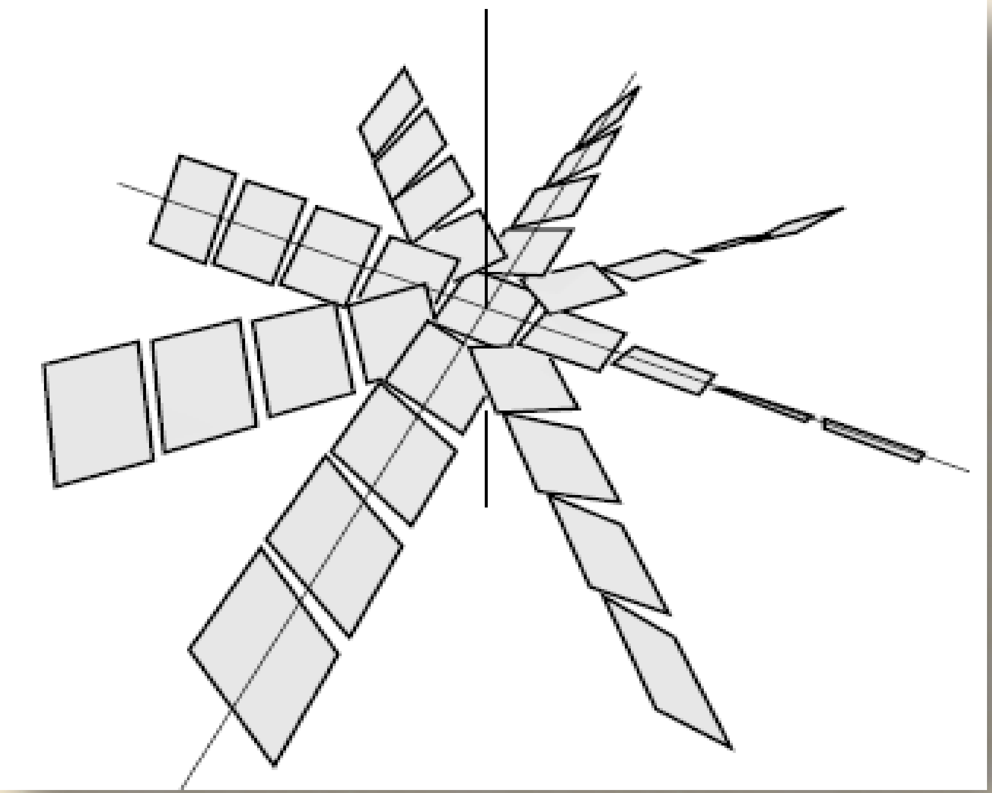
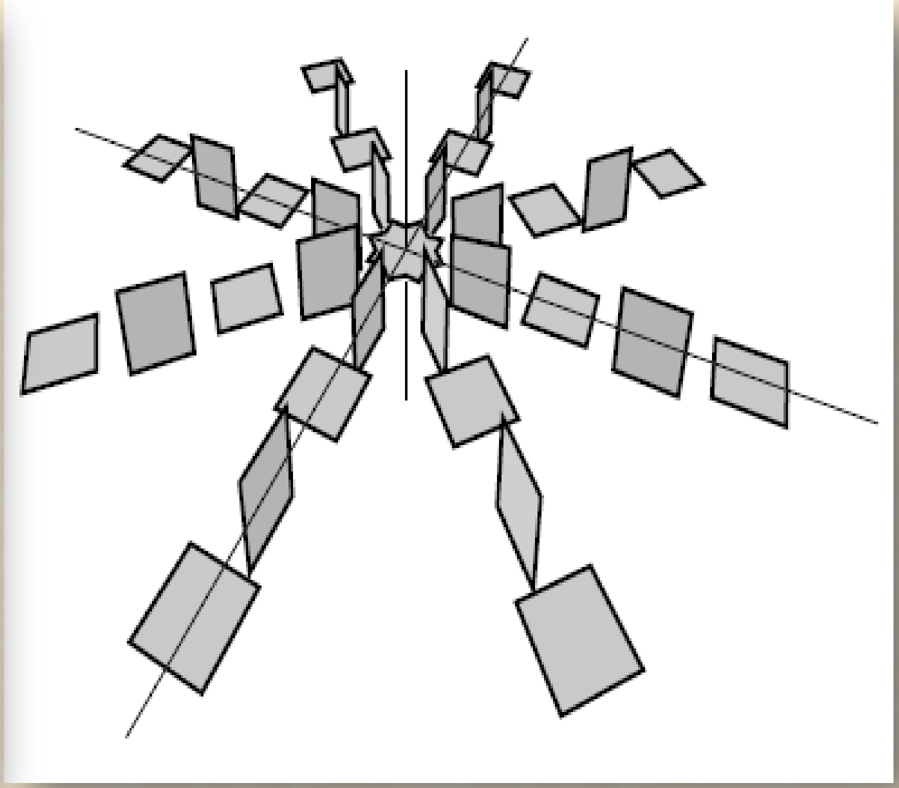


overtwisted

tight	overtwisted



$\ker(dz + r^2 d\theta)$



$\ker(\cos r\,dz + r \sin r\,d\theta)$

Classification of overtwisted contact structures

Martinet: (1971) Every closed oriented 3-manifold admits a contact structure.

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Eliashberg: (1989) Two overtwisted contact structures are isotopic if and only if they are homotopic as oriented plane fields.

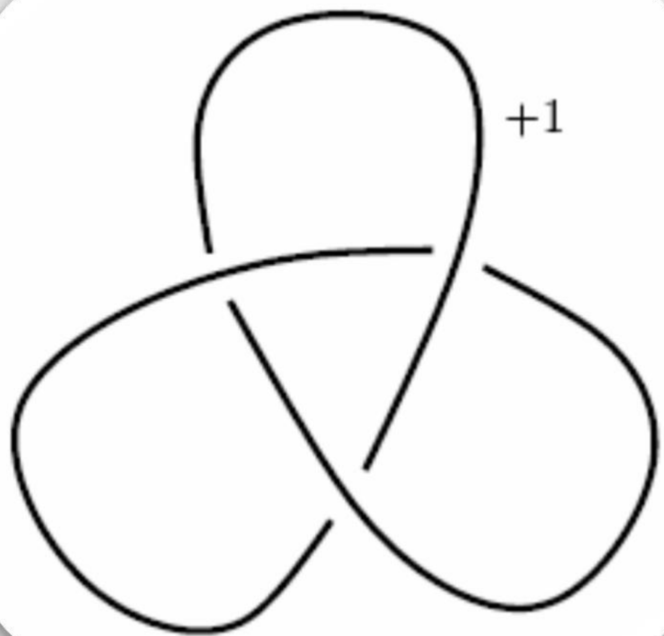
Classification of overtwisted contact structures

Martinet + Lutz + Eliashberg:

There is a unique overtwisted contact structure in every homotopy class of oriented plane fields.

Classification of tight contact structures?

Etnyre & Honda: (2001) The Poincaré homology sphere with its non-standard orientation does not admit a tight contact structure.



Colin & Giroux & Honda: (2008) Only finitely many homotopy classes of oriented plane fields carry tight contact structures on a closed oriented 3-manifold.

Dichotomy between universally tight

and

virtually overtwisted

(tight) contact structures

A tight contact structure is called universally tight if it remains tight when pulled back to the universal cover.

A tight contact structure is called **universally tight** if it remains tight when pulled back to the universal cover.

A tight contact structure is called **virtually overtwisted** if it becomes overtwisted when pulled back to some finite cover.

$L(p,1)$

Honda: There are $p-1$ tight contact structures (up to isotopy) on the lens space $L(p,1)$.

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Gompf: For $p \geq 4$, only two of these are universally tight, and the rest are virtually overtwisted.

The dichotomy between universally tight and virtually overtwisted contact structures

A group G is called residually finite if for every $g \neq 1$ in G, $\exists$ a normal subgroup of finite index not including g.

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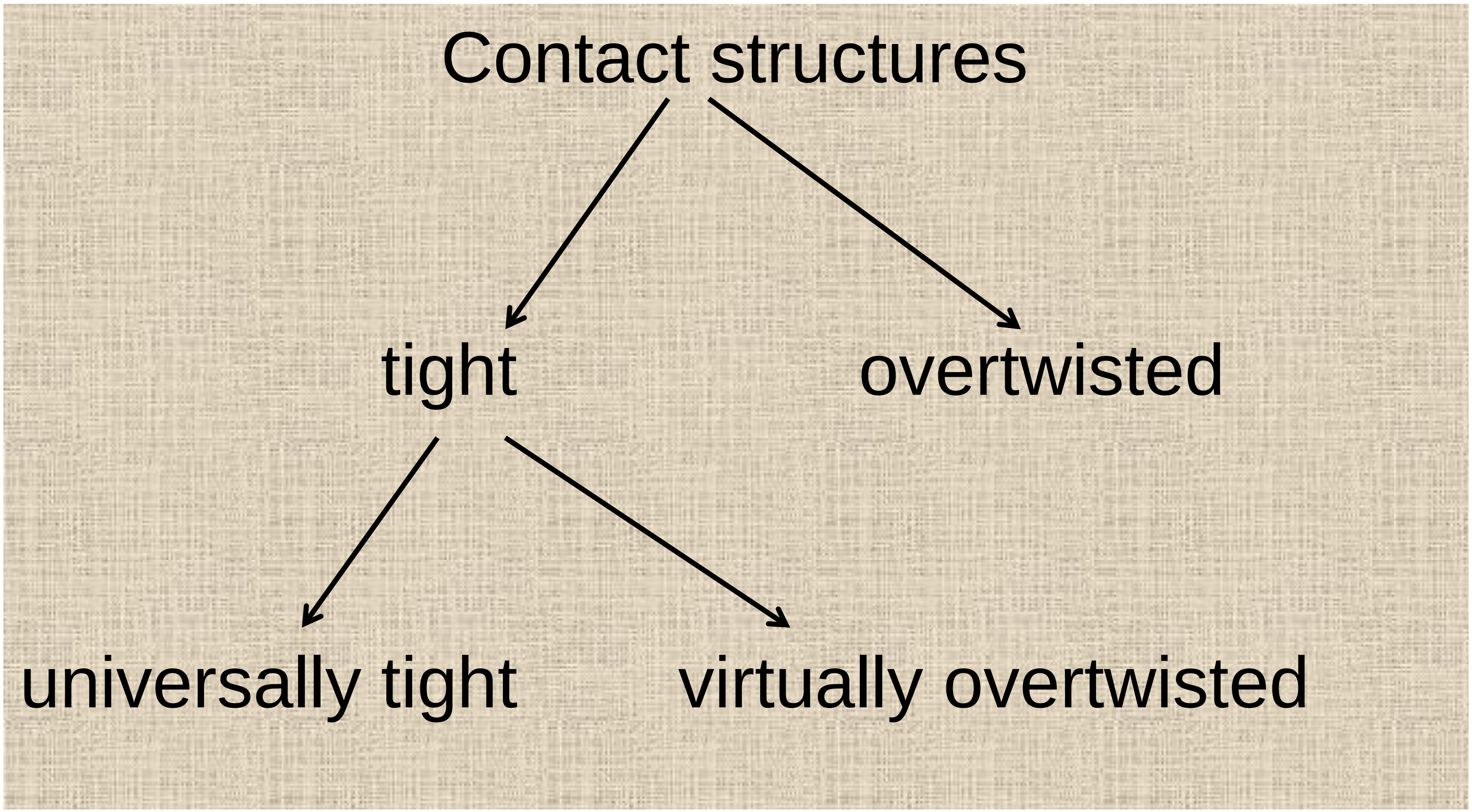
A group G is called **residually finite** if for every $g \neq 1$ in G, $\exists$ a normal subgroup of finite index not including g.

A consequence of geometrization is that the fundamental group of a closed 3-manifold Y is residually finite—which is equivalent to the following:

The dichotomy between universally tight and virtually overtwisted contact structures

For every compact subset K of the universal covering $\widetilde{Y}$, there is a connected finite index covering Y' of Y such that the natural projection $\widetilde{Y} \to Y'$ is injective on K.


Contact structures
tight
overtwisted
universally tight
virtually overtwisted

MILNOR FILLABLE CONTACT STRUCTURES

Complex surface singularities

A complex surface singularity $(X, 0)$ is defined as

$$(\{f_1 = f_2 = \cdots = f_m = 0\}, 0) \subset (\mathbb{C}^N, 0)$$

where each $f_i : (\mathbb{C}^N, 0) \to (\mathbb{C}, 0)$ is a germ of an analytic function with

$$r(p) = \textbf{rank}\Big[\frac{\partial f_i}{\partial z_j}(p)\Big] = N - 2$$

for all $p \in X - \{0\}$, and $r(0) < N - 2$.

The link of a singularity

Let $(X, 0)$ be a normal complex surface singularity.

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The intersection is a 3-manifold M^3 which is called the **link of the singularity**, whose diffeo type is independent of ϵ.

An example of a link of a singularity

The link of the singularity

$$(X,0)=(\{x^2+y^3+z^5=0\},0)\subset(\mathbb{C}^3,0)$$

is the Poincaré homology sphere.

Canonical contact structure

The complex hyperplane distribution ξ_{can} on $M^3 = X \cap S^{2N-1}$ induced by the complex structure on X is called the canonical contact structure.

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The contact 3-manifold (M, ξ_{can}) is called the contact boundary of $(X, 0)$.

Milnor fillable

A contact 3-manifold (Y, ξ) is said to be **Milnor fillable** if it is isomorphic to the contact boundary (M, ξ_{can}) of some isolated complex surface singularity $(X, 0)$.

Topological characterization

Mumford + Grauert: A 3-manifold carries a Milnor fillable contact structure if and only if it can be obtained by plumbing oriented circle bundles over Riemann surfaces according to a graph with negative definite intersection matrix.

Caubel & Nemethi & Popescu-Pampu: (2006) Any 3-manifold has at most one Milnor fillable contact structure up to isomorphism.

Our main result

Lekili & O. (2010): Every Milnor fillable contact structure is universally tight.

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Lekili & O. (2010): Every Milnor fillable contact structure is universally tight.

REMARK: Universal tightness of a contact structure is not implied by any other type of fillability:

A weakly/strongly symplectically fillable (or Stein fillable) contact structure is tight but not necessarily universally tight!

What was known?

Eliashberg & Gromov: (1989)

A weakly symplectically fillable contact structure is tight.

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Eliashberg & Gromov: (1989)

A weakly symplectically fillable contact structure is tight.

Bogomolov & de Oliveira: (1997)

A Milnor fillable contact structure is Stein fillable.

Milnor fillable → universally tight

↓

Stein fillable

↓

strongly symplectically fillable

↓

weakly symplectically fillable

↓

tight

There are no other implications!

Corollary (Lekili & O.): $\exists$ infinitely many closed, atoroidal, irreducible 3-manifolds with infinite π_1 which carry universally tight contact structures that are not perturbations of taut (or Reebless) foliations.

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Eliashberg & Thurston: (1998)

Any foliation on a closed 3-manifold (other than $S^1 \times S^2$) can be perturbed to a contact structure. If the foliation is taut, then the contact structure is universally tight and weakly symplectically fillable.

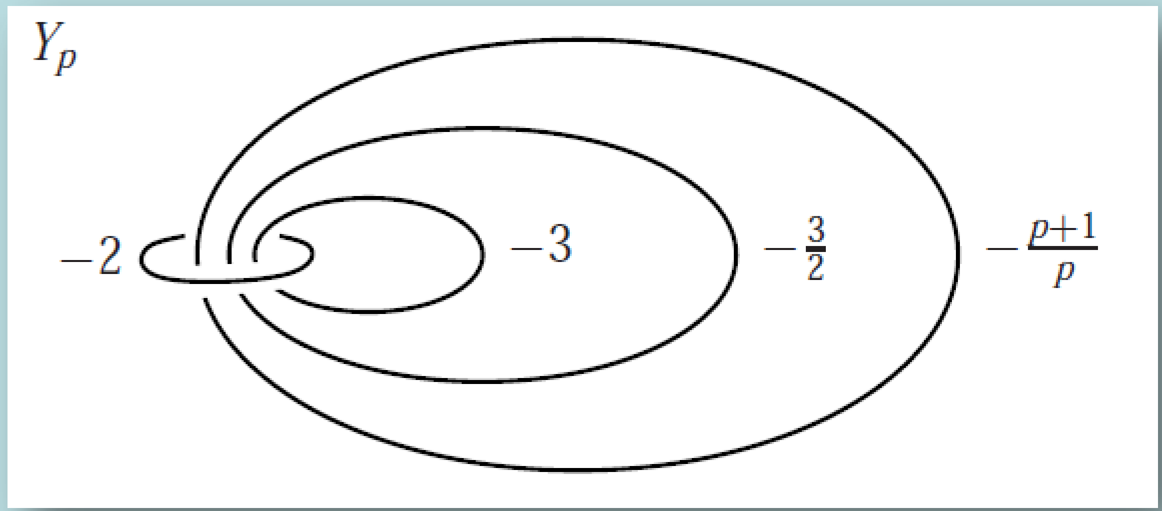
Corollary (Lekili & O.): ∃ infinitely many closed, atoroidal, irreducible 3-manifolds with infinite π_1 which carry universally tight contact structures that are not perturbations of taut (or Reebless) foliations.

The assumption on π_1: Every foliation on a closed 3-manifold with finite π_1 has a Reeb component (and hence is not taut) by a theorem of Novikov.

Corollary (Lekili & O.): $\exists$ infinitely many closed, atoroidal, irreducible 3-manifolds with infinite π_1 which carry universally tight contact structures that are not perturbations of taut (or Reebless) foliations.

Ghiggini (2006): $\exists$ toroidal 3-manifolds which carry universally tight contact structures that are not weakly fillable (and therefore can not be perturbations of taut foliations by a theorem of Eliashberg & Thurston).

Proof of the corollary



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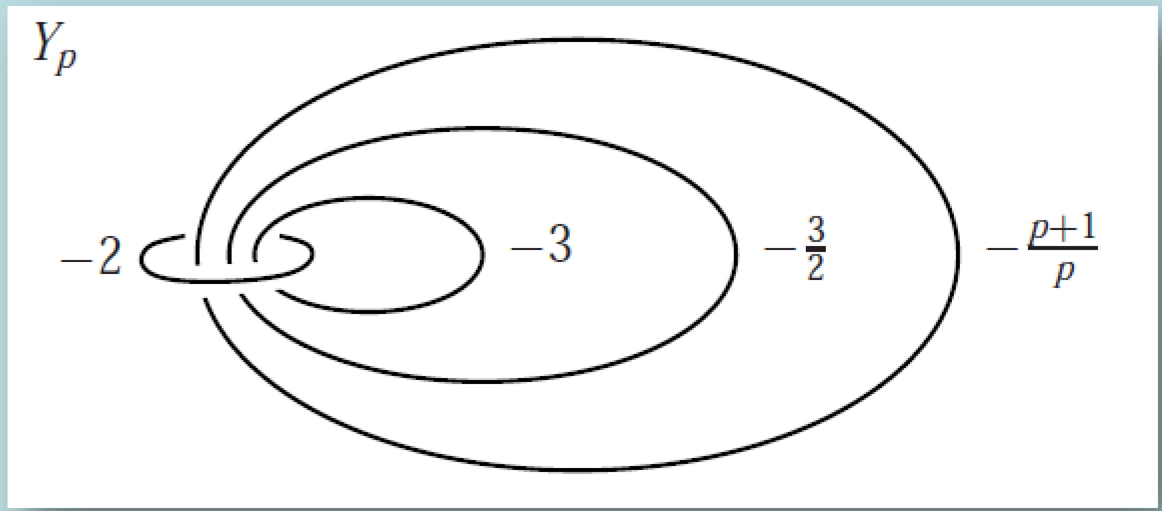

Small Seifert fibered 3-manifold Y_p is the link of a rational surface singularity.

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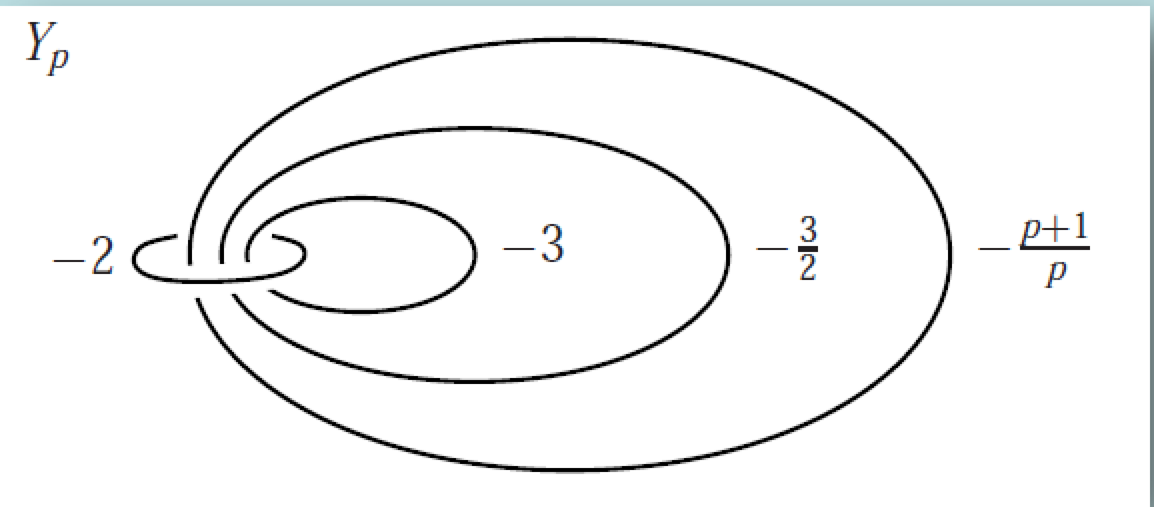

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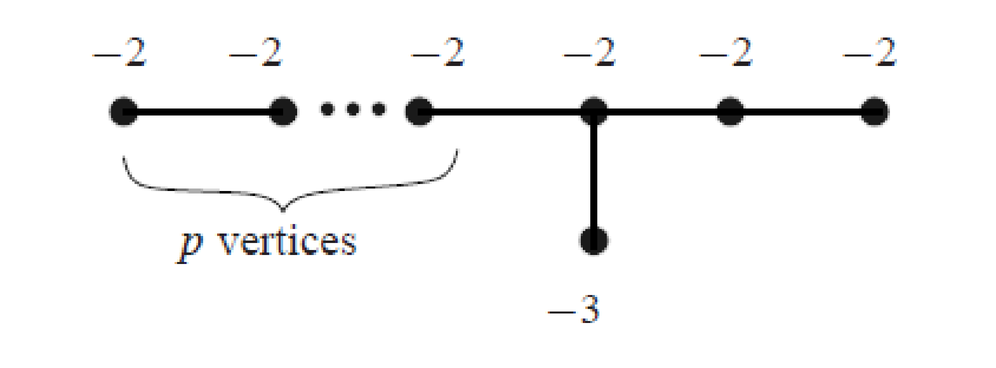
Dual resolution graph



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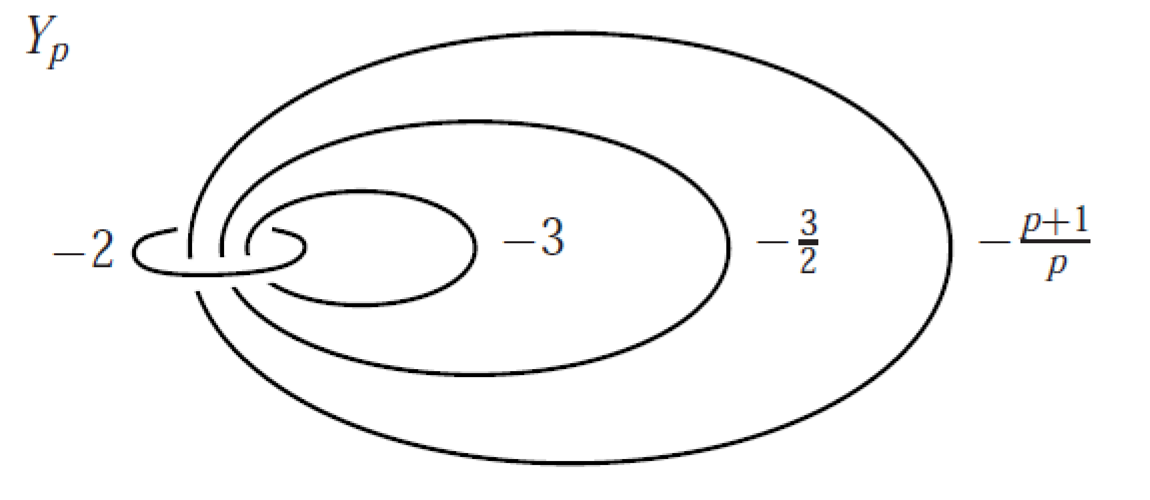

Y_p **is irreducible.**

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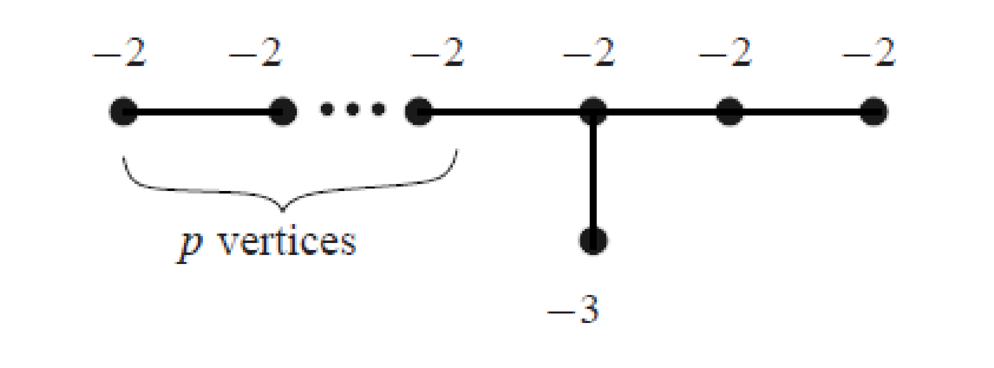
Dual resolution graph



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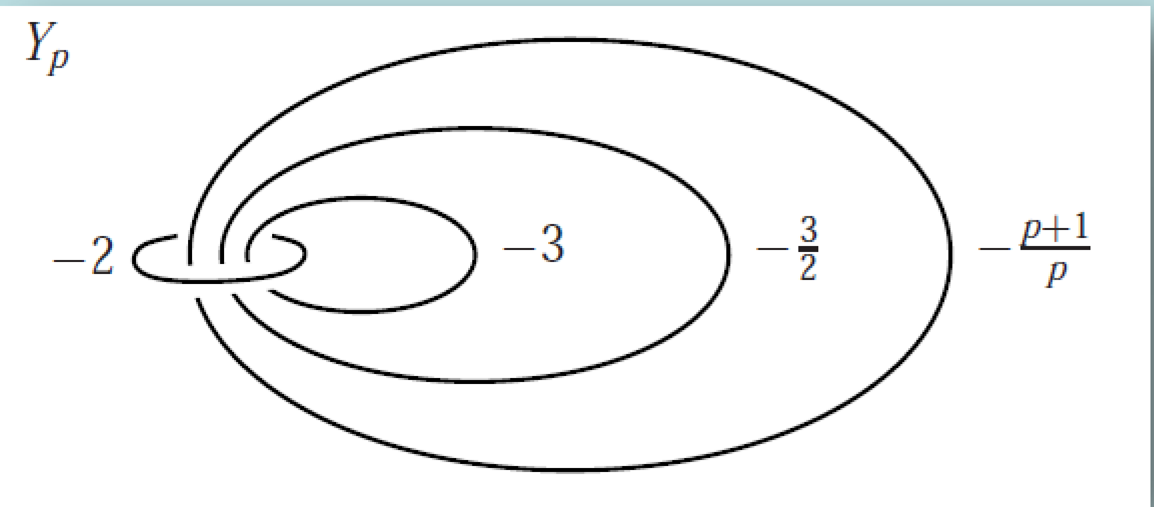

Y_p **is irreducible.**

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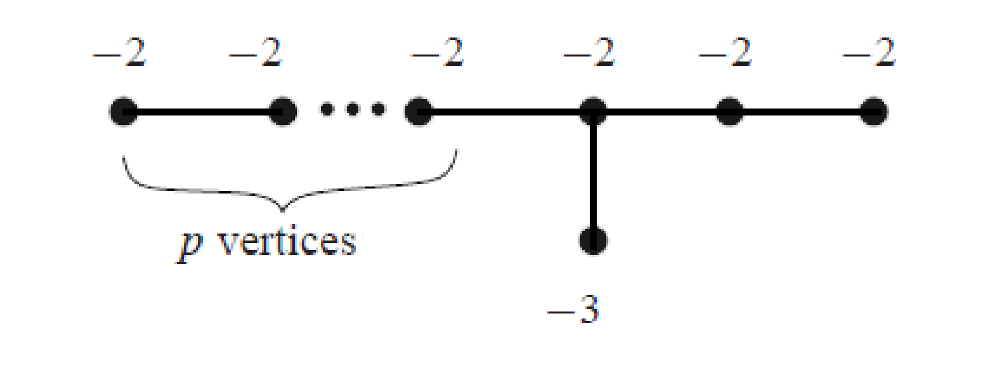

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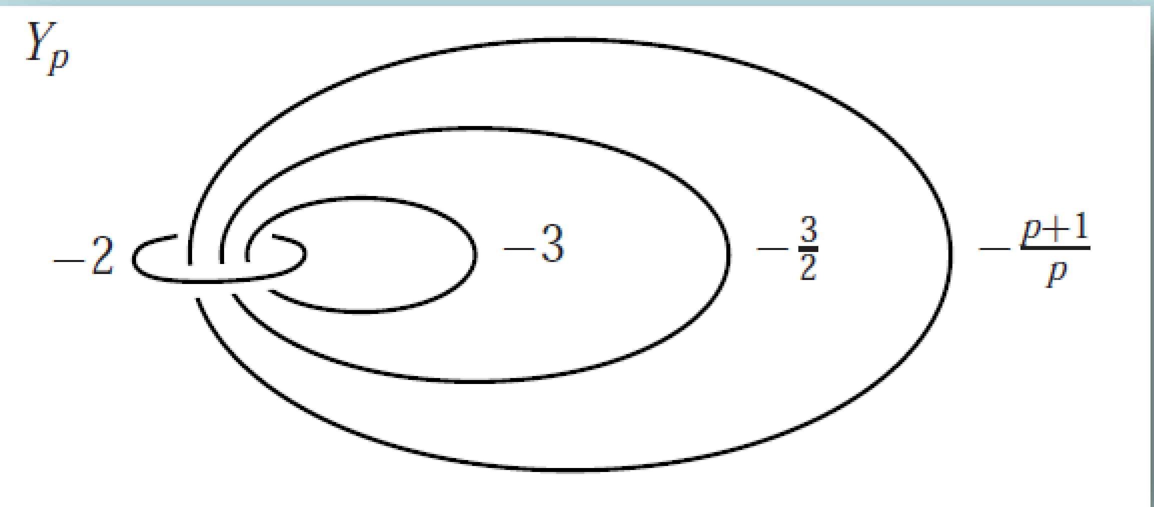


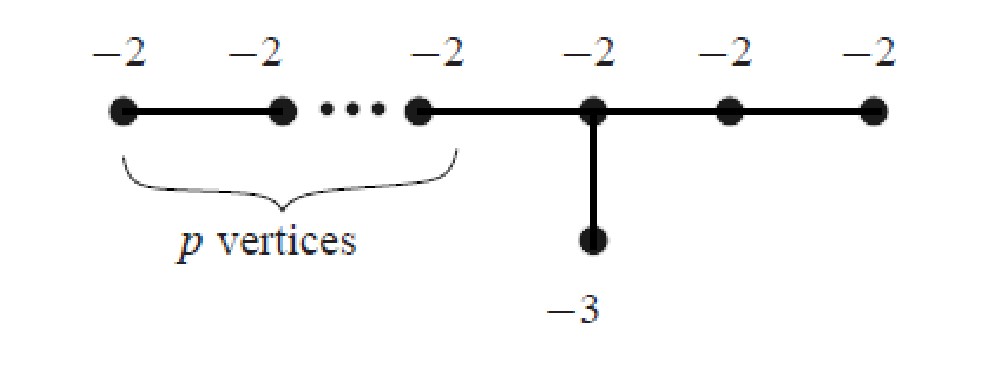
Y_p **is irreducible.**

Y_p **is atoroidal.**

$\pi_1(Y_p)$ **is infinite.**

Small Seifert fibered 3-manifold Y_p **is the link of a rational surface singularity.**

Dual resolution graph



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ξ_{can} on Y_p is universally tight.

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Outline of the proof:

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Let (M, ξ_{can}) be the contact boundary of a surface singularity $(X, 0)$.

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Let (M, ξ_{can}) be the contact boundary of a surface singularity $(X, 0)$.

For an analytic function $f\colon (X,0) \to (\mathbb{C}, 0)$, with an isolated singularity at 0, the open book decomposition $\mathcal{OB}_f$ of M with binding $L = M \cap f^{-1}(0)$ and projection

$$\pi = \frac{f}{|f|}\colon M \setminus L \to S^1 \subset \mathbb{C}$$

is called a **Milnor open book**.

We construct a universally tight contact structure ξ on M which is compatible with some Milnor open book $\mathcal{OB}$.

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It is known that ξ_{can} is compatible with any Milnor open book.

Since ξ_{can} and ξ are both compatible with $\mathcal{OB}$ we know that ξ_{can} is isotopic to ξ by a theorem of Giroux.

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We conclude that ξ_{can} on the singularity link M is universally tight.

Another approach ?

Is it true that a finite cover of a Milnor fillable contact structure is Milnor fillable?

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Neumann: Finite cover of a singularity link is a singularity link.

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Gompf: There are virtually overtwisted Stein fillable contact structures on $L(p,1)$!