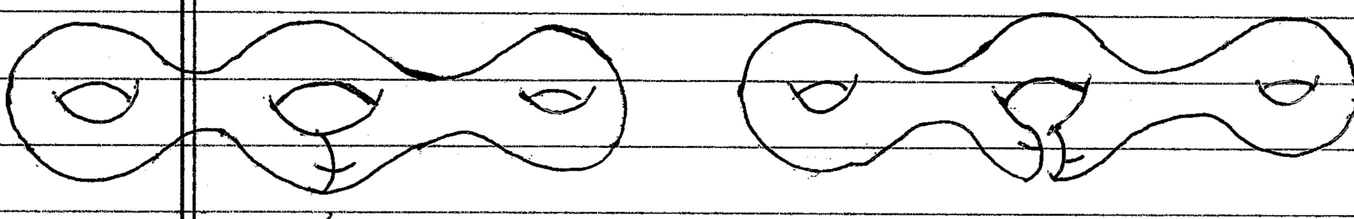
Products of twists, geodesic-lengths & Thurston shears

Goal: generalize formulas for Fenchel-Nielsen twists, geodesic-lengths & Weil-Petersson metric to setting of punctured surfaces triangulated by ideal geodesics

Fenchel-Nielsen twists & geodesic-lengths
hyperbolic surface and a geodesic α



geometry in neighborhood of α is determined by length l_α
measure displacement in hyperbolic distance
infinitesimal displacement FN twist t_α

WP Kähler metric for Teichmüller space

duality formulas $2t_\alpha = i\,\mathrm{grad}\, l_\alpha$ & $2\omega_{WP}(\ ,t_\alpha) = dl_\alpha$

Riera pairing formula

$$\langle \mathrm{grad}\, l_\alpha, \mathrm{grad}\, l_\beta \rangle = \frac{2}{\pi}\delta_{\alpha\beta}\, l_\alpha + \frac{2}{\pi}\sum_{\substack{\text{homotopy classes } \gamma \\ \text{connecting } \alpha \text{ to } \beta \text{ rel } \alpha \text{ rel } \beta}} R(u)$$

$$R(u) = u \log\left|\frac{u+1}{u-1}\right| - 2$$

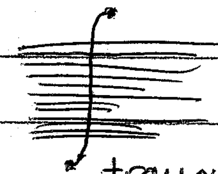
$u = \cosh l(\gamma)$ for α, β disjoint
$u = \cos\theta$ for α intersect β

$$t_\alpha l_\beta = 2\omega_{WP}(t_\alpha, t_\beta) = \sum_{p\in\alpha\cap\beta} \cos\theta_p \quad \text{angles from } \alpha \text{ to } \beta$$

Thurston cataclysm / Bonahon shear for compact surfaces

A geodesic lamination λ for a hyperbolic surface is a closed union of disjoint simple complete geodesics

local picture – Cantor set cross section



transverse

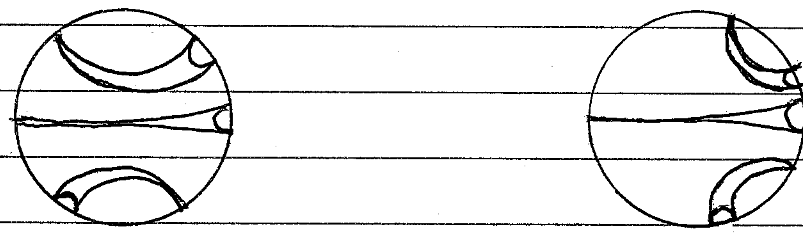
A measure for λ is an assignment for each smooth transverse arc τ with endpoints in λ^c – a non negative measure with support $\tau \cap \lambda$ where the measure only depends on the equivalence class of transverse
Given suitable τ – evaluation of mass $(\lambda \cap \tau)$ is a functional of λ – masses define a topology for
$\mathcal{MGL}$ space of measured geodesic laminations

(Thurston, Bonahon) $\mathcal{MGL}$ has a PL structure with suitable tangent spaces

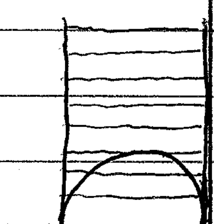
Tangent vectors – A transverse cocycle is a 'generalized measure' as above but only require finite additivity of mass with respect to disjoint unions of intervals
Essential construction – an embedding of Teichmüller space $\mathcal{T} \hookrightarrow$ Cone in space of transverse cocycles on a maximal geodesic lamination
complement is a union of ideal triangles

For a representation $\pi_1(\text{surface}) \hookrightarrow PSL(2;\mathbb{R})$ can restrict transformation to S^1_∞ and deformation is given by conjugation by a homeomorphism of S^1_∞

Lift maximal geodesic lamination to universal cover



geodesic lamination complement a union of ideal triangles
for deformation endpoints of leaves correspond by homeomorphism of S^1_∞ – ideal triangles have midpoints sides
essential – measure relative shifts between complementary triangles



method take tangent field to the horocycle foliation between pairs of interceding sides
geodesic laminations have measure zero
important tangent fields to horocycles of interceding sides extends to a Lipschitz vector field
flow lines define a projection between pairs of sides – measure displacement of midpoints of sides – relative displacements are positive or negative and finitely additive

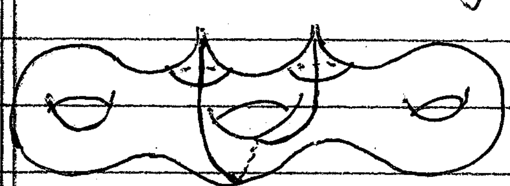
obtain a transverse cocycle encoding the hyperbolic structure (Bonahon-Sozen) for the shift transverse cocycles then the Thurston & WP symplectic forms are multiples

Penner lambda-length coordinates for punctured surfaces

Can decompose a surface with cusps into a union of ideal triangles with 'vertices' at the cusps and edges ideal geodesics

A decoration is a specification of a horocycle at each cusp of the surface

The decorated Teichmüller space is the space of homotopy marked pairs – a hyperbolic surface with decoration



a decorated surface with an ideal geodesic γ

Lambda-length of γ is $\lambda(\gamma) = e^{\frac{1}{2}\text{ signed distance between horocycles}}$

Given homotopy classes $\{[\gamma_1], \ldots, [\gamma_{6g-6+3n}]\}$ for ideal geodesics triangulating a surface then

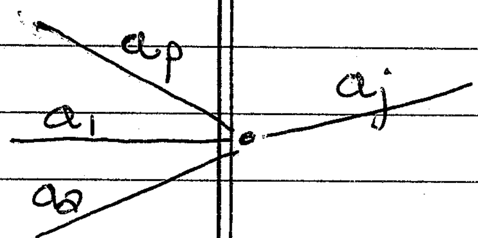
(Penner) The lambda-length mapping of decorated Teichmüller space is a real-analytic equivalence to $\mathbb{R}^{6g-6+3n}_{>0}$ and

$$\omega_{WP} = \sum_{\text{ideal triangles edges}} \frac{d\lambda(\alpha)}{\lambda(\alpha)} \wedge \frac{d\lambda(\beta)}{\lambda(\beta)} + \frac{d\lambda(\beta)}{\lambda(\beta)} \wedge \frac{d\lambda(\gamma)}{\lambda(\gamma)} + \frac{d\lambda(\gamma)}{\lambda(\gamma)} \wedge \frac{d\lambda(\alpha)}{\lambda(\alpha)}$$

(Papadopoulos - Penner) For the dual punctured null gon track the Thurston & WP symplectic forms are multiples

Situation to consider – surfaces with punctures triangulated by a configuration of ideal geodesics

Define balanced sums $\mathcal{a} = \sum_j a_j \alpha_j$ of ideal geodesics



a_j weights of approaching segments

partial sums $A_j = \sum_{k=1}^{j} a_k$

pairing $\omega(\{a_j\}, \{b_j\}) = \frac{1}{2} \sum_{j=1}^{p} (A_j + A_{j-1}) b_j$

pairing is alternating by summation by parts

Define

$\sigma_{\{a_j\}}$ shear shift deformation for data

cusps remain cusps

$L(\{a_j\})$ total length of weighted configuration

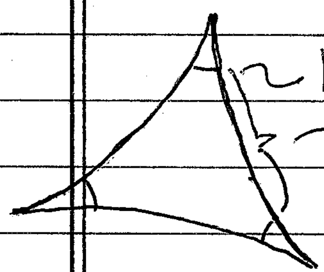
compute by decoration – independent of choice

Calculate infinitesimal deformations

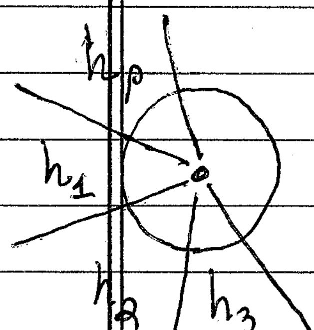
difficulty geometric terms – shear & total length are not defined for individual ideal geodesics

analytic terms – for individual ideal geodesics the associated quadratic differentials have poles of too large an order

Decorated Teichmüller space $\mathcal{D}\mathcal{T} \xrightarrow[\text{fibration}]{} \mathcal{T}$



decorated triangle



cusp with ideal geodesics

shear displacement of j^{th} geodesic is $\log \frac{h_j}{h_{j-1}}$

differential of shear $d\log h_j - d\log h_{j-1}$

Use shear displacement formula to lift

$$\sum_{\text{cusps}} \frac{1}{2} \sum_{j=1}^{p} (A_j + A_{j-1})\, b_j \quad \text{to } \mathcal{D}\mathcal{T} \qquad \underbrace{\sum_{\text{cusps}} \sum_{j=1}^{p} d\log h_j \wedge d\log h_{j+1}}_{\text{MCG invariant on } \mathcal{D}\mathcal{T}}$$

Approach – double surfaces across cusps, open cusps to become short geodesics and take limits of corresponding quantities for compact surfaces limits involve canceling infinities

Formulas

duality $\quad 2\omega_{WP}(\ \ ,\sigma_{\{a_j\}}) = dL(\{a_j\})$

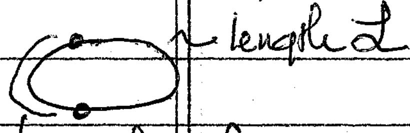
symplectic $\quad \omega_{WP}(\sigma_{\{a_j\}},\sigma_{\{b_j\}}) = \frac{1}{2}\sum_{\text{cusps}} \omega(\{a_j\},\{b_j\})$

and gradient

$\langle \operatorname{grad} L(\{a_j\}), \operatorname{grad} L(\{b_j\})\rangle$

$$= \sum_{j,k} a_j b_k \Big(\delta_{\alpha_j\beta_k}\frac{2}{\pi}\big(\text{reduced length}(\alpha_j)+2\big) + \frac{2}{\pi}\sum_{\text{cusps}}\ \sum_{\substack{\text{segments } \alpha_j,\beta_k \\ \cancel{\text{limiting to the cusp}}}} \log\lambda(\alpha_j,\beta_k) + \sum^{\text{reduced}}_{\alpha_j \text{ to } \beta_k} R \Big)$$

reduced length α is length of ideal geodesic segment between length one horocycles



invariant of point pair is
$$\lambda(a) = \frac{a(1-a)}{2\sin\pi a}$$

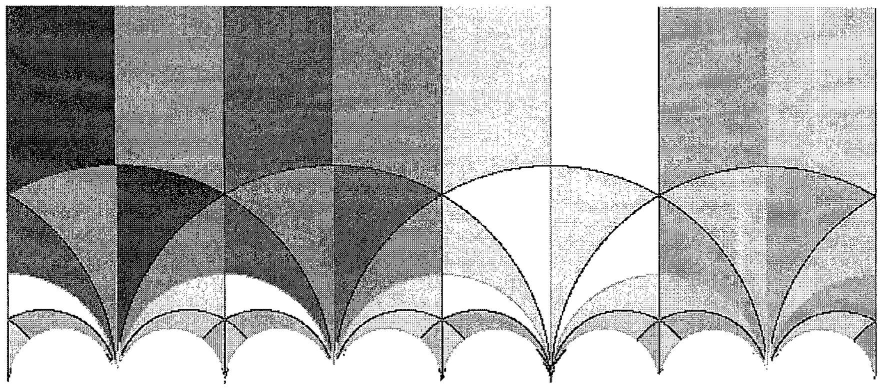


Figure 6: The Dedekind tessellation. Graphic created by and used with permission from Gerard Westendorp.

Example 21. A distances relation for the elliptic modular tessellation.

The Dedekind tessellation is the tiling of the upper half plane for the action of $PSL(2;\mathbb{Z})$. The light, respectively dark, triangle tiles form a single $PSL(2;\mathbb{Z})$ orbit. The tessellation vertices are fixed points for the group action. There are two orbits for vertices. There are also two orbits for ideal lines. The first consists of the lines containing a single order-2 fixed point. The second consists of the lines sequentially containing an order-2, an order-3 and an order-2 fixed point. We refer to the types as 2-lines and 323-lines. We consider the lines with weights: $w = +1$ for 323-lines and $w = -1$ for 2-lines. The system of weighted lines is $PSL(2;\mathbb{Z})$ invariant.

The formula of Theorem 19 provides a relation for the distances between lines for the Dedekind tessellation. For any choice $\tilde{a}$ of a 323-line and $\tilde{\alpha}$ of a 2-line we have

$$\sum_{\text{ultraparallels to } \tilde{a}} w(\eta)R(d(\tilde{a},\eta)) - \sum_{\text{ultraparallels to } \tilde{\alpha}} w(\eta)R(d(\tilde{\alpha},\eta)) = \log \frac{3^6\pi^3}{2^{36}}$$

for $R(d) = u\log((u+1)/(u-1)) - 2$ and $u = \cosh d$. Ultraparallels are the tessellation lines at positive distance.

We find the relation as an exercise in evaluating the formula of Theorem 19. We begin with the geometry of the tiling quotient. We work with the thrice-punctured sphere uniformized by the projectivized index 6 subgroup $P\Gamma_0(4) \subset PSL(2;\mathbb{Z})$ of matrices with lower left entry congruent to $0 \mod 4$. A fundamental domain for $P\Gamma_0(4)$ is given by the twelve light and dark triangles intersecting a given 323-line. The $P\Gamma_0(4)$ quotient has three 323-lines,