The virtual Haken conjecture
Progress in Low-Dimensional Topology
QGM, Aarhus, August 2012

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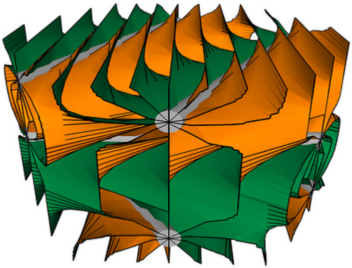


Thurston's questions

In Thurston's paper *Three Dimensional Manifolds, Kleinian groups, and hyperbolic geometry*, he asked 24 questions which have guided the last 30 years of research in the field. Four of the questions have to do with "virtual" properties of 3-manifolds:

- Question 15 (paraphrased): Are Kleinian groups LERF?
- Question 16: "Does every aspherical 3-manifold have a finite-sheeted cover which is Haken?" This question originated in a 1968 paper of Waldhausen.
- Question 17: "Does every aspherical 3-manifold have a finite-sheeted cover with positive first Betti number?"
- Question 18: "Does every hyperbolic 3-manifold have a finite-sheeted cover which fibers over the circle? This dubious-sounding question seems to have a definite chance for a positive answer."

Outline of the talk

The goal of this talk will be to explain the meaning of these questions, and to discuss their resolution.

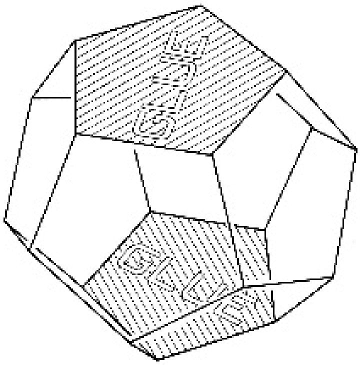
With the geometrization theorem proved by Perelman (Question 1 from Thurston's list), the most interesting case of questions 16-17 are for hyperbolic 3-manifolds, so we will focus for the most part on questions about virtual properties of hyperbolic 3-manifolds.

Hyperbolic 3-manifolds

A 3-manifold M is **hyperbolic** if it admits a complete Riemannian metric with constant curvature -1. Then $M = \mathbb{H}^3/\Gamma$, where $\mathbb{H}^3$ is hyperbolic 3-space, and Γ is a discrete torsion-free subgroup of $\mathrm{PSL}(2,\mathbb{C})$ (if M is orientable). If Γ is finitely generated, then it is called a **Kleinian group**.

Classic examples of hyperbolic 3-manifolds are the **Seifert-Weber dodecahedral space**, the **figure eight knot complement**, and the **Whitehead link complement**

Some hyperbolic spaces



Residual Finiteness

Definition

A group G is **residually finite (RF)** if for every $1 \neq g \in G$, there exists a finite group K and a homomorphism $\phi : G \to K$ such that $\phi(g) \neq 1 \in K$.

Alternatively,

$$\{1\} = \bigcap_{[G:H]<\infty} H. \quad (1)$$

Examples include

- finitely generated linear groups (Malcev)
- 3-manifold groups (Hempel + geometrization)
- mapping class groups of surfaces

Locally Extended Residual Finiteness

Definition

A subgroup $L < G$ is *separable* if for all $g \in G - L$, there exists $\phi : G \to K$ finite $\phi(g) \notin \phi(L)$.

Alternatively,

$$L = \bigcap_{L \leq H \leq G, [G:H] < \infty} H \tag{2}$$

Thus, residual finiteness means $1 < G$ is separable.

Definition

A subgroup $L < G$ is *weakly separable* if for all $g \in G - L$, there exists $\phi : G \to K$ such that $\phi(L)$ is finite and $\phi(g) \notin \phi(L)$ (K need not be finite).

Locally Extended Residual Finiteness

Definition

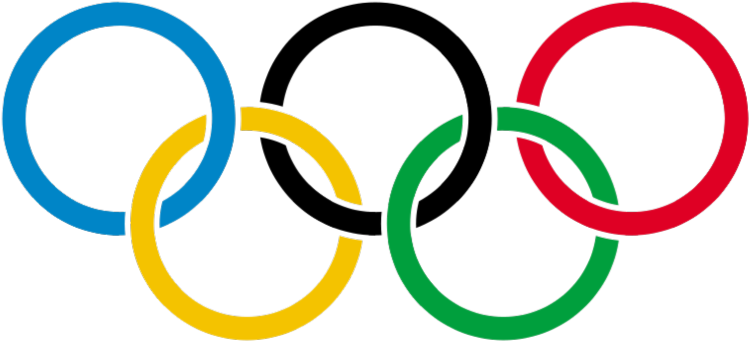
A group G is LERF if *finitely generated subgroups of G are separable.* (*local* means finitely generated)

Previously well-known examples include

- $\mathbb{Z}^n$
- free groups (Hall) and surface groups (Scott)
- Bianchi groups $\mathrm{PSL}(2, \mathbb{Z}[\sqrt{-d}])$ (Agol-Long-Reid) and certain other arithmetic subgroups of $\mathrm{PSL}(2, \mathbb{C})$.
- 3-dimensional hyperbolic reflection groups (Haglund-Wise)
- There are examples of 3-manifold groups which are not LERF which are *graph manifold* groups (Burns-Karrass-Solitar)

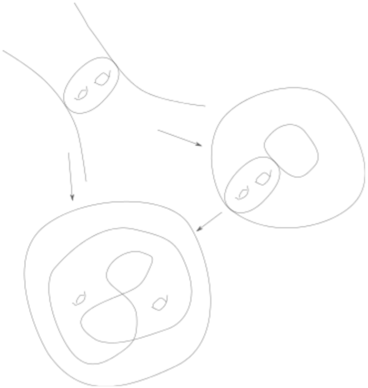
LERF

For example, the fundamental group of the complement of this link is not LERF (Niblo-Wise):



Locally Extended Residual Finiteness

LERF allows one to lift π_1-injective immersions to embeddings in finite-sheeted covers (Scott)



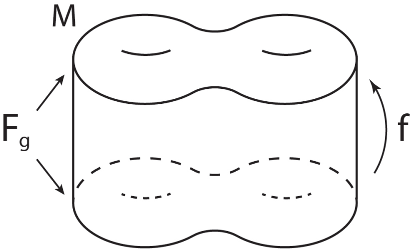
Virtual Haken

- A manifold M is **aspherical** if every $\pi_i(M) = 0$ for $i \geq 2$.
- A 3-manifold M is **Haken** if it is aspherical and M contains an embedded π_1-injective surface (e.g. a knot complement). The Sefert-Weber space is non-Haken (Burton- Rubinstein-Tillmann), as well as hyperbolic surgeries on the figure 8 knot complement (Thurston).
- A 3-manifold M is **virtually Haken** if there is a finite-sheeted manifold cover $\tilde{M} \to M$ such that $\tilde{M}$ is Haken, e.g. hyperbolic surgeries on the figure 8 knot complement are virtually Haken (Dunfield-Thurston)
- A 3-manifold M **contains an essential surface** if there is a map $f : F_g \to M$, such that f is π_1-injective (but f might not be an embedding), e.g. a virtually Haken 3-manifold
- Waldhausen conjectured that every aspherical 3-manifold M is virtually Haken (the *virtual Haken conjecture, Question 16).*

Fibered 3-manifolds

A manifold M **fibers over the circle** if there is a submersion $\eta : M \to S^1$. Each preimage $\eta^{-1}(x)$ is a codimension-one submanifold of M called a **fiber**. If M is 3-dimensional and fibers over S^1, then the fiber is a genus g surface F_g, and M is obtained as the mapping torus of a homeomorphism $f : F_g \to F_g$,

$$M \cong T_f = \frac{F_g \times [0,1]}{\{(x,0) \sim (\phi(x),1)\}}.$$



Virtually fibered 3-manifolds

- M is **virtually fibered** if there exists a finite-sheeted cover $\tilde{M} \to M$ such that $\tilde{M}$ fibers
- There are known examples of (aspherical) **Seifert-fibered spaces** and **graph 3-manifolds** which are not virtually fibered
- There have been several classes of 3-manifolds shown to virtually fiber, including 2-bridge links, some Montesinos links, and certain alternating links (Agol-Boyer-Zhang, Aitchison-Rubinstein, Bergeron, Chesebro-DeBlois-Wilton, Gabai, Leininger, Reid, Walsh, Wise).
- Thurston asked whether every hyperbolic 3-manifold is virtually fibered (Question 18)?

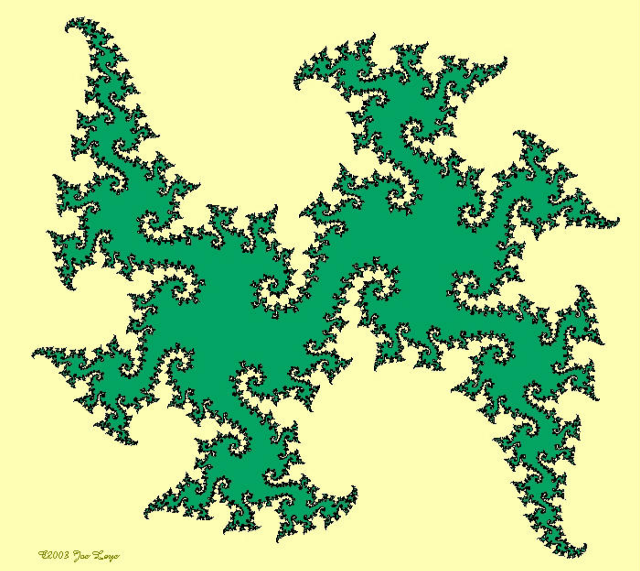
Surfaces in hyperbolic 3-manifolds

If M is a finite volume hyperbolic 3-manifold, and $f : \Sigma_g \to M$ is an essential immersion of a surface of genus $g > 0$, then there is a dichotomy for the geometric structure of the surface discovered by Thurston, and proven by Bonahon in general. Either f is

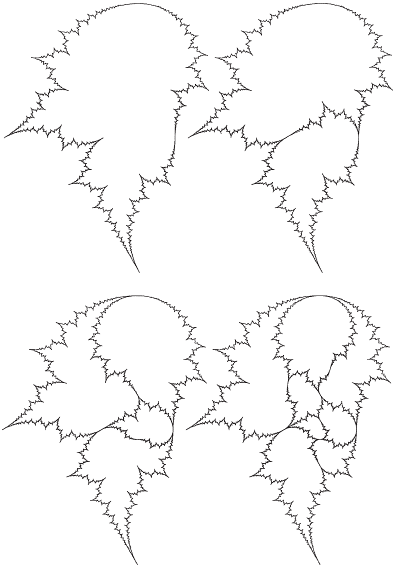
- **geometrically finite** or
- **geometrically infinite**.

The first case includes **quasifuchsian** surfaces. In the geometrically infinite case, the surface is **virtually the fiber** of a fibering of a finite-sheeted cover of M, and the subgroup $f_\#(\pi_1(\Sigma_g)) < \pi_1(M)$ is separable. The **Tameness theorem** (A., Calegari-Gabai) plus the **covering theorem** of Canary implies a similar dichotomy for finitely generated subgroups of $\pi_1(M)$: either a subgroup is geometrically finite, or it corresponds to a virtual fiber. This result is used in proving that certain Kleinian groups are LERF, since the subgroups corresponding to virtual fibers are separable.

Quasi-fuchsian surface group limit set

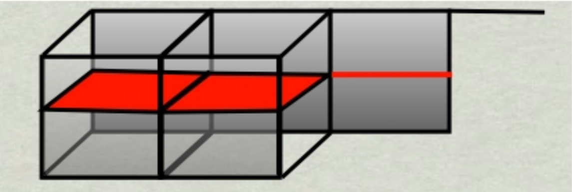


Part of the Peano curve "limit set" of the figure eight fiber



Cubulations

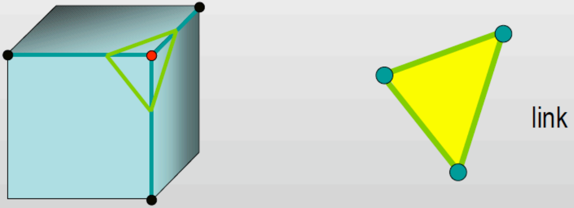
A topological space X is **locally CAT(0) cubed** if X is a cubical complex such that putting the standard Euclidean metric on each cube gives a locally CAT(0) metric (a form of non-positive curvature). Gromov showed that this condition is equivalent to a purely combinatorial condition on the links of vertices of X, they are **flag**. In a locally CAT(0) cube complex, there are canonical immersed codimension-one subcomplexes locally geodesic $W \looparrowright X$ called **walls** or **hyperplanes**, which are obtained by taking the union of midplanes in each cube.



Flag condition (a la Rob Ghrist)

gromov's link condition:

link: simplicial complex of incident cells...



theorem [gromov]:

cube complex is npc ⇔ link of each vertex is a flag complex

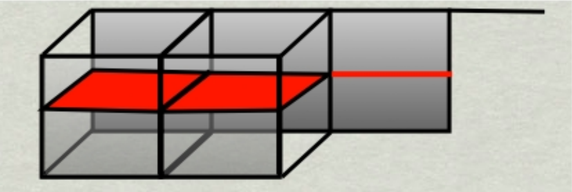
if the edges look like a k-simplex, there really is a k-simplex spanning them...

Cubulations

A topological space Y is **cubulated** if it is *homotopy equivalent* to a compact locally CAT(0) cube complex $X \simeq Y$. We will be interested in 3-manifolds which are cubulated.
Remark: If $Y = M^3$, and $X \simeq Y$ is a CAT(0) cubing, then $dim X$ may be > 3. Tao Li has shown that there are hyperbolic 3-manifolds Y such that there is no **homeomorphic** CAT(0) cubing $X \cong Y$.
Theorems of Sageev associate a cocompact action of $\pi_1(M)$ on a (globally) CAT(0) cube complex if M contains a quasi-fuchsian surface. Globally CAT(0) $\iff$ simply-connected and locally CAT(0)

Cubulations

Sageev's construction gives a cube complex in which each immersed essential surface in a 3-manifold corresponds to an immersed hyperplane.



In the case of a geometrically infinite surface, Sageev's construction gives rise to a crystallographic group action.

Essential surfaces in hyperbolic 3-manifolds

Theorem (Kahn-Markovic 2009)

Hyperbolic 3-manifolds contain immersed quasi-fuchsian surfaces which are arbitrarily close to being totally geodesic.

Previous work on this problem:

- Cooper-Long and Li proved that all but finitely many Dehn fillings on a cusped hyperbolic 3-manifold have essential surfaces.
- Masters-Zhang showed that cusped hyperbolic 3-manifolds contain essential quasifuchsian surfaces (no parabolics).
- Lackenby proved in 2008 that arithmetic hyperbolic 3-manifolds contain closed essential immersed surfaces (used work of Lewis Bowen).

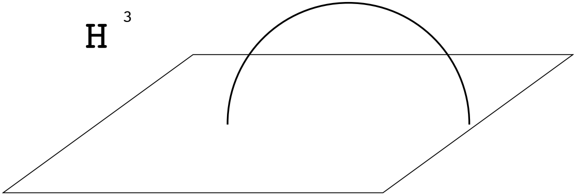
Essential surface meeting a geodesic

Theorem (Bergeron-Wise 2009)

Closed hyperbolic 3-manifolds are cubulated.

Bergeron and Wise show that the surfaces produced by Kahn-Markovic are sufficient to meet every geodesic in M, and therefore give rise to a cubulation.

If every geodesic is separated by an immersed quasi-fuchsian surface, then the action on the cube complex is proper and cocompact.



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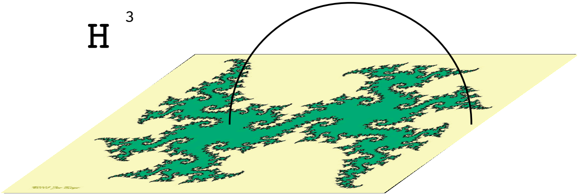


Right-Angled Artin Groups and special cubulations

Haglund and Wise introduced a notion of **special cubulations**. Take the standard cube structure on T^n by identifying opposite sides of a cube. A **Salvetti complex** is obtained by taking a subcomplex of T^n which is locally CAT(0). The fundamental group of a Salvetti complex will be a **Right-Angled Artin group (RAAG)**. A **special cube complex** has embedded walls and immerses locally isometrically into the Salvetti complex of a RAAG. A cubulation is **virtually special** if it has a finite-sheeted special cover.

Theorem (Haglund-Wise 2007)

If X is a virtually special cube complex with hyperbolic fundamental group $\pi_1 X$ (in the sense of Gromov), then $\pi_1 X$ virtually embeds in a RAAG and quasi-convex subgroups of $\pi_1 X$ are separable.

RAAGs (a la Vogtmann)

Right-angled Artin groups

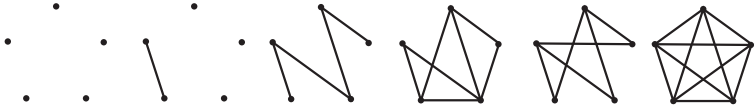
Γ = simplicial graph

The *right-angled Artin group* A_Γ is given by:

Generators: nodes of Γ

Relators: vw = wv if [v,w] is an edge of Γ

Examples:



F_5 $\mathbb{Z}^2 * F_3$ $\pi_1(M^3)$ G $F_2 \times F_3$ $\mathbb{Z}^5$

Theorem [Droms] A_Γ is a 3-manifold group if and only if Γ is a disjoint union of trees and triangles.

Salvetti complex for RAAG (a la Charney)

Properties of RAAGs.

(1) $K(A_\Gamma,1)$-spaces:

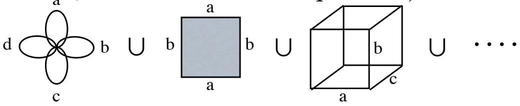
Last time: finite cell complex, Salvetti complex

$K(\pi,1)$-Conjecture: Salvetti complex is a $K(A,1)$-space.

Salvetti complex for A_Γ : $S_\Gamma = Sal(A_\Gamma)$

W_T is finite iff the generators in T all commute (so T spans a clique in Γ). In this case, the Coxeter cell C_T is a k-cube, k=|T|.

S_Γ = Rose $\cup_a$ (k-torus for each k-clique in Γ)



Salvetti complex for RAAG (a la Charney)

Examples:

A_Γ = free group (Γ discrete)

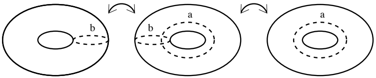
$\Rightarrow S_\Gamma$ = Rose, $\widetilde{S}_\Gamma$ = tree

A_Γ = free abelian group (Γ complete)

$\Rightarrow S_\Gamma$ = n-torus, $\widetilde{S}_\Gamma = R^n$

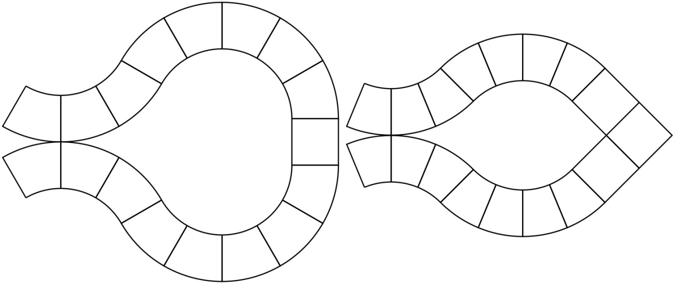
Γ =

S_Γ =



Special cube complexes

Special cube complexes are defined in terms of properties of their hyperplanes. Hyperplanes are embedded and 2-sided. Moreover, there are no self-osculating or inter-osculating hyperplanes:



These conditions are necessary for a locally isometrically immersed complex of a RAAG Salvetti complex. Haglund and Wise observed that these conditions also suffice for immersing isometrically.

Virtual retract property

This notion was defined independently by Long-Reid and Haglund:

Definition

A subgroup $L < G$ is a virtual retract if there exists $G' < G$ a finite-index subgroup such that $L < G'$ and a retract $r : G' \to L$, meaning $r_{|L} = Id$.

Claim: If G is residually finite, and L is a virtual retract of G, then L is separable in G.

Haglund proved that quasi convex subgroups of RAAGs are virtual retracts (a similar result was proved by Long-Reid for subgroups of right-angled reflection groups, and the method goes back to Scott). This is proved geometrically using "canonical completions" and "canonical retracts".

Virtual betti number and largeness of groups

Groups are assumed to be finitely generated here.

- A group G has **positive** $\mathbf{b_1}$ if $b_1(G) = rankH_1(G;\mathbb{Q}) > 0$. Equivalently there exists an epimorphism $\phi : G \twoheadrightarrow \mathbb{Z}$.
- A group G has **virtual positive** b_1 if there exists finite index $\tilde{G} < G$ with $b_1(\tilde{G}) > 0$.
- A group G is **large** if there is a finite index subgroup $\tilde{G} < G$ and a homomorphism $\phi : G \twoheadrightarrow \mathbb{Z} * \mathbb{Z}$.
- Question 17 of Thurston may be rephrased: does every aspherical 3-manifold group have virtual positive b_1?

There has been much work devoted to understanding Thurston's questions.

Previous work on these questions

- Freedman-Freedman and Cooper-Long proved that many Dehn fillings on cusped hyperbolic non-fibered 3-manifolds are virtually Haken
- Cooper-Long-Reid proved that non-cocompact non-elementary Kleinian groups are large.
- Masters showed that fibered manifolds with a fiber of genus 2 have virtual infinite b_1.
- For arithmetic 3-manifolds, the virtual b_1 conjecture is equivalent to the large conjecture by work of Cooper-Lackenby-Long-Reid. The virtual positive b_1 conjecture for arithmetic 3-manifolds would follow from conjectures in number theory, such as the Langlands functoriality conjecture. Previous work in various cases has proved this for certain classes of arithmetic 3-manifolds (Millson, Schwermer, et al).

Previous theorem on virtual fibering

Theorem (A.)

If M is virtually special cubulated, then M is virtually fibered.

Since M is cubulated, $M \simeq X$, where X is a CAT(0) compact cube complex.
There is a finite-sheeted cover $\tilde{X}$ which is **special**, and implies that $\pi_1(\tilde{X}) < RAAG$. A strong form of residual solvability for RAAG's called **RFRS** passes to $\pi_1(\tilde{X})$ and implies that M is virtually fibered (A.).
Together with the theorems of Kahn-Markovic and Bergeron-Wise, this implies that we need only show that $\pi_1(M)$ is LERF to show that M is virtually fibered.

Theorem on virtual fibering

Theorem (Wise 2011)

Virtually quasi-fuchsian Haken hyperbolic 3-manifolds are virtually special cubulated, and therefore Haken hyperbolic 3-manifolds are virtually fibered.

Wise's theorem gives a different approach to finding cubulations (based on work with Hsu) than the result of Bergeron-Wise, and holds in much greater generality.

RFRS and virtual fibering

The **rational derived series of a group** G is defined inductively as follows.
If $G^{(1)} = [G, G]$, then $G_r^{(1)} = \{x \in G | \exists k \neq 0, x^k \in G^{(1)}\}$.
If $G_r^{(n)}$ has been defined, define $G_r^{(n+1)} = (G_r^{(n)})_r^{(1)}$.
The rational derived series gets its name because
$G_r^{(1)} = \ker\{G \to \mathbb{Q} \otimes_{\mathbb{Z}} G/G^{(1)}\} = \ker\{G \to H_1(G; \mathbb{Q})\}$.
The quotients $G/G_r^{(n)}$ are solvable.

Definition

A group G is *residually finite* $\mathbb{Q}$*-solvable* or *RFRS* if there is a sequence of subgroups $G = G_0 > G_1 > G_2 > \cdots$ such that $\cap_i G_i = \{1\}$, $[G : G_i] < \infty$ and $G_{i+1} \geq (G_i)_r^{(1)}$.

RFRS and virtual fibering

By induction, $G_i \geq G_r^{(i)}$, and thus G/G_i is solvable with derived series of length at most i. We remark that if G is RFRS, then any subgroup $H < G$ is as well.
Examples of RFRS groups are free groups, surface groups, RAAGs.
For a 3-manifold M with RFRS fundamental group, the condition is equivalent to there existing a **cofinal tower** of finite-index covers

$$M \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots$$

such that M_{i+1} is obtained from M_i by taking a finite-sheeted cyclic cover dual to an embedded non-separating surface in M_i.
Equivalently, $\pi_1(M_{i+1}) = ker\{\pi_1(M_i) \to \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}\}$.
This condition implies that M has virtual infinite b_1, unless $\pi_1(M)$ is virtually abelian.

RFRS and virtual fibering

Theorem (A. 2007)

If M is aspherical and $\pi_1(M)$ is RFRS, then M virtually fibers.

The proof makes use of **sutured manifold theory**, an inductive technique for studying foliations of 3-manifolds introduced by Gabai. This criterion was one ingredient of the proof of

Theorem (Friedl-Vidussi 2008)

If M is a closed 3-manifold such that $M \times S^1$ is a **symplectic** *4-manifold, then M is fibered.*

The converse is an old result of Thurston, and this result was conjectured by Kronheimer and Taubes.

RFRS and virtual fibering

The RFRS condition for aspherical 3-manifolds has stronger implications for the structure of $H^1(M)$. For any $\alpha \in H^1(M)$, there is a finite-sheeted cover $\pi : \tilde{M} \to M$ (coming from the RFRS condition) such that $\tilde{\alpha} = \pi^*(\alpha) \in H^1(\tilde{M})$ is a limit of cohomology classes which correspond to fiberings of $\tilde{M}$. (this is actually the criterion that Friedl-Vidussi use)
This implies that $\tilde{\alpha}$ is dual to a **depth one taut foliation** $\mathcal{F}$ of $\tilde{M}$, with a closed leaf of $\mathcal{F}$ corresponding to the Poincare dual of $\tilde{\alpha}$.

Implications

Theorem (A., Groves, Manning, Martinez-Pedrosa 2008)

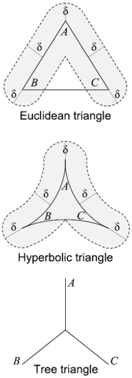
If **Gromov-hyperbolic groups** *are RF, then Kleinian groups are LERF*

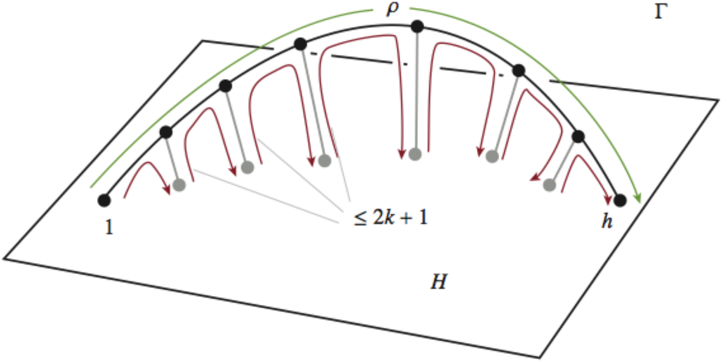
So it may be possible to show that hyperbolic 3-manifold groups are LERF by showing that Gromov-hyperbolic groups are RF

Caveat: This approach seems quite unlikely to work, since many experts believe that there are non-RF Gromov-hyperbolic groups.

Hyperbolic groups and quasi convex subgroups

A hyperbolic group may be defined by Rips' thin triangle condition $[B, C] \subset \mathcal{N}_\delta([A, B] \cup [A, C])$. A subgroup $H < G$ is λ-quasi convex if for every $h \in H$, $[1, h] \subset \mathcal{N}_\lambda(H)$.





Virtually special cubulations

Recently we have proved the conjecture of Wise:

Theorem (A. 2012)

Cubulations with hyperbolic fundamental group are virtually special.

Corollary

Let M be a closed hyperbolic 3-manifold. Then $\pi_1 M$ is LERF, large, and M virtually fibers.

The proof of this theorem makes use of Wise's results. Part of the argument is based on joint work with Groves and Manning:

Theorem (A.-Groves-Manning 2012)

Let G be a hyperbolic group, and $H < G$ a quasi-convex virtually special subgroup. Then H is weakly separable in G.

Virtually special cubulations: Outline of proof

Definition

A collection of subgroups $\{H_1, \ldots, H_m\}$, $H_i < G$ is *almost malnormal* if $|H_i^g \cap H_j| = \infty \implies i = j, g \in H_i$.

Theorem (Bowditch)

Let G be a hyperbolic group, and $\{H_1, \ldots, H_m\}$ a collection of subgroups. Then G is hyperbolic relative to $\{H_1, \ldots, H_m\}$ if and only if H_i is quasi convex and they form an almost malnormal collection.

The Malnormal Special Quotient Theorem

Theorem (Wise's Malnormal Special Quotient Theorem (MSQT))

Let G be hyperbolic, virtually special, and $\{H_1, \ldots, H_M\} < G$ a almost malnormal collection of quasi convex subgroups. Then there exists $\dot{H}_i \lhd H_i$ such that for any $H_i' < \dot{H}_i$, the quotient group $\overline{G} = G / << H_1', \ldots, H_m' >>$ is virtually special hyperbolic.

Remarks on the proof: Using hyperbolic Dehn filling results of Groves-Manning and Osin, one may conclude that $\overline{G}$ is hyperbolic whenever $H_i/\dot{H}_i$ has "large girth" (this is called *hyperbolic Dehn filling*, and is a descendent of small-cancellation theory). The difficult thing is showing that the quotient is cubulated. What Wise actually proves is that there is (virtually) a malnormal quasi convex hierarchy, and then applies his joint work with Haglund and Hsu to conclude that it is virtually special. This represents hundreds of pages of mathematics.

Outline of the proof

The proof of the theorem is by induction on dimension. Let X be a locally CAT(0) cube complex with $G = \pi_1 X$ hyperbolic. Let $W \looparrowright X$ be the immersed hyperplane complex. Then the maximal dimension of cubes in W is one less than those in X, so by induction we may assume that $\pi_1 W$ is virtually special (strictly, for each component of W). Then we may apply weak separability to conclude under these hypotheses that

Theorem

There exists $G'' \lhd G$, $G/G'' \cong \mathcal{G}$, $\tilde{X}/G'' \cong \mathcal{X}$ such that $\mathcal{X}$ has 2-sided embedded acylindrical compact hyperplanes.

If $\mathcal{X} \to X$ were a finite-sheeted cover, then we would be done.
Remark: The components of hyperplanes $\mathcal{W} \looparrowright \mathcal{X}$ will generally not be separating if X is not Haken.

Outline of proof

Definition (Crossing Graph)

Let $\Gamma(\mathcal{X})$ be a graph with vertex set $V(\Gamma(\mathcal{X})) = \mathcal{W}$ the hyperplanes of $\mathcal{X}$, and edges $(W_1, W_2) \in E(\Gamma(\mathcal{X}))$ if $W_1 \cap W_2 \neq \emptyset$ or if there is an essential cylinder going between W_1 and W_2.

Definition (Coloring space)

Let $[n] = \{1, \ldots, n\}$. Let

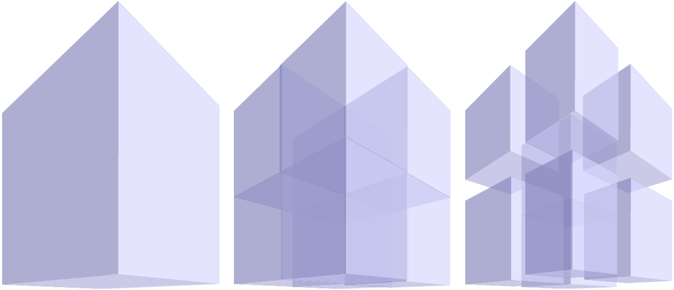
$$C_n(\Gamma) = \{c : V(\Gamma) \to [n] | c(W_1) \neq c(W_2), \forall (W_1, W_2) \in E(\Gamma)\}$$

denote the space of n-colorings of the graph Γ.

We regard $C_n(\Gamma)$ as a closed subspace of the Cantor set $[n]^{V(\Gamma)}$. If $deg(\Gamma) \leq k$, then $C_{k+1}(\Gamma) \neq \emptyset$.

Colorings and hierarchies

A coloring $c \in C_n(\Gamma(\mathcal{X}))$ gives rise to a hierarchy of $\mathcal{X}$: cut along the walls colored 1, then the walls colored 2, ..., and finally the walls colored n.



What is left at the ends are stars of the vertices of $\mathcal{X}$, with residues of the colorings remaining on the boundaries.

Colorings and hierarchies

Lemma

There exists a probability measure μ on $C_{k+1}(\Gamma(\mathcal{X}))$ which is $\mathcal{G}$-invariant.

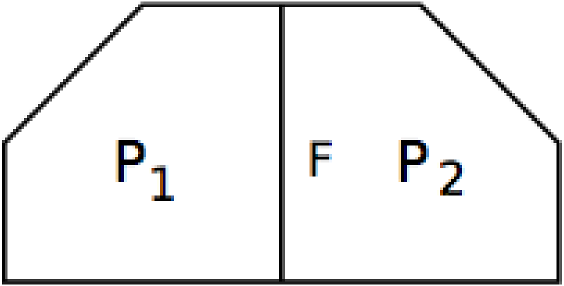
The proof of this lemma proceeds by coloring the vertices $V(\Gamma)$ randomly with n-colors, $n \geq k+1$. The probability that two endpoints of and edge $e \in E(\Gamma)$ have the same color is $1/n$. One can produce an $n-1$-coloring of the vertices, by sending each vertex colored n each color to the smallest color unused by its neighbors. By induction then, one produces a measure on $k+1$-colorings of $V(\Gamma)$ which have probability of coloring the endpoints of e the same color as $1/n$. Taking a weak-* limit of these measures, one obtains a $\mathcal{G} - invariant$ measure μ on $V(\Gamma)^{[k+1]}$ which is supported on the colorings of Γ.

Colorings and hierarchies

The probability measure is just an artifice to construct a solution to the *gluing equations*. We want to reverse engineer a hierarchy of a finite-sheeted cover. We have an infinite hierarchy associated to the cover $\mathcal{X}$. The probability measure allows us to extract some finiteness associated to this hierarchy.

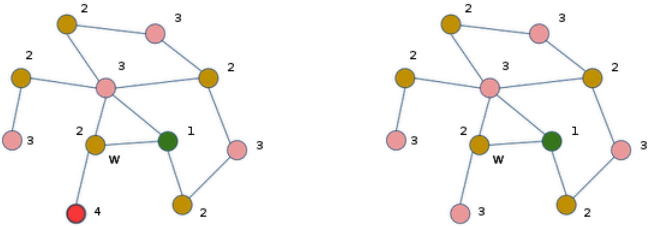
Polyhedra and facets

Let $\mathcal{P}$ denote the stars of vertices of $\mathcal{X}$, which we will call *polyhedra*. Let $\mathcal{F}$ denote the facets of $\mathcal{X}$, which are dual to each edge of $\mathcal{X}$, and are the facets of the polyhedra $\mathcal{P}$. Each facet $F \in \mathcal{F}$ will be contained uniquely in two polyhedra $P_1, P_2 \in \mathcal{P}$, $P_1 \cap P_2 = F$.



Supercoloring

Each polyhedron and facet of $\mathcal{X}$ will correspond uniquely to one of X via the covering $\mathcal{X} \to X$. We refine the colorings of the walls $\mathcal{W}$ by decreasing chains of colors emanating from a vertex, and call these "super colorings". The facets $F \in \mathcal{F}$ get super colored by their corresponding walls, and polyhedra will be super colored by their facets.



In these colorings, the vertex w will receive the same supercoloring.

Polyhedral gluing equations

The variables for the gluing equations will be super colored polyhedra, and the gluing equations will say that for a given super colored facet F, the super colorings of P_1 which induce the same super coloring of F must equal the super colorings of P_2 which induce the super coloring of F. We require that the variables are $\mathcal{G}$-invariant, in which case they are determined by finitely many variables corresponding to the polyhedra of X (or $\mathcal{G}$-orbits of super colored polyhedra). The $\mathcal{G}$-invariant measure μ gives us a solution to the gluing equations with non-negative weights. Then we can get an integral solution to the gluing equations with non-negative weights, since the equations are linear equations with integral coefficients.

We take the integral solution to the polyhedral gluing equations, and use them to glue up a finite-sheeted cover of X, which is "modeled" on the hierarchies associated to colorings of $\mathcal{X}$.

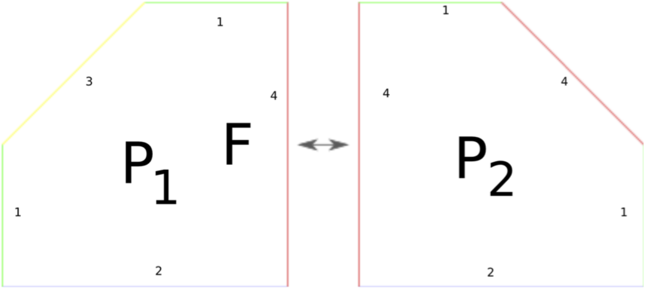
Gluing up the hierarchy

We construct a sequence of (usually disconnected) finite cube complexes $\mathcal{V}_j$, $k+1 \geq j \geq 0$, with boundary pattern $\{\partial_1(\mathcal{V}_j), \ldots, \partial_j(\mathcal{V}_j)\}$ which have the following properties:

1. there is a locally convex combinatorial immersion $\nu_j : \mathcal{V}_j \to X$
2. $\mathcal{V}_j$ is glued together from copies of $\mathcal{G}$-orbits of equivalence classes of super colored polyhedra from $\mathcal{P}$ in such a way that if polyhedra $P_1, P_2 \subset \mathcal{V}_j$ share a facet F, then the induced equivalence class of F is the same.
3. The multiplicities of $\mathcal{G}$-orbits of equivalence classes of colored polyhedra making up $\mathcal{V}_j$ satisfy the polyhedral gluing equations.
4. The boundary of $\mathcal{V}_j$ is the union of all facets F contained in precisely one polyhedron $\partial P_g \subset \mathcal{V}_j$. Moreover, the boundary pattern $\partial_i \mathcal{V}_j$ consists of the facets of $\mathcal{V}_j$ which are colored i, $i \leq j$.

Gluing polyhedra satisfying the gluing equations

The super coloring guarantees that when we glue together super colored polyhedra P_1 and P_2 along the face F so that they satisfy the gluing equations, then the resulting boundary pattern will be super colored in a consistent fashion, allowing a hierarchy to be constructed inductively.



Gluing up the hierarchy

The base case $\mathcal{V}_{k+1}$ is the collection of equivalence classes of polyhedra given by the solution to the polyhedral gluing equations.

We obtain $\mathcal{V}_i$ from $\mathcal{V}_{i+1}$ by finding a covering space $\tilde{\mathcal{V}}_{i+1} \to \mathcal{V}_{i+1}$ in which the boundary pattern $\partial_{i+1}\tilde{\mathcal{V}}_{i+1}$ may be matched up in pairs which reverse the coorientations and preserve super colorings. Constructing this cover requires another application of Wise's theorem.

The cube complex $\mathcal{V}_0$ will have no boundary pattern, and thus will give a finite-sheeted covering space $\mathcal{V}_0 \to X$ and which has by construction a malnormal quasi convex hierarchy.

One more application of Wise's theorem gives a cover $\tilde{V}_0 \to X$ which is special.

Non-positively curved manifolds

Questions 15-18 have extensions to compact 3-manifolds whose interior admits a metric of non-positive curvature (**NPC**), when interpreted properly.

Theorem (Przytycki-Wise 2012)

A compact 3-manifold admits an NPC metric if and only if it is virtually special.

Remark: The locally CAT(0) cube complex here might not be compact, but has finitely many walls. Leeb has determined which

aspherical 3-manifolds are NPC, and the only exceptions are certain graph manifolds. In particular, this implies virtual RFRS, virtual fibering, linearity of fundamental group, etc. For non-positively curved graph manifolds, this result is due to Yi Liu (2011). The manifolds are not quite LERF, but are QCERF.

Open questions

- (Long-Reid) Can two compact 3-manifolds fundamental groups which are non-isomorphic have the same profinite completion?
  Remark: This is equivalent to the question, given two 3-manifold groups, do they have the same collection of finite quotients?
- Are compact 3-manifold fundamental groups linear?
  Remark: The only case left is graph manifolds which don't admit an NPC metric.
- (Cooper) Is there an R so that hyperbolic 3-manifolds with injectivity radius $> R$ are Haken?
- Find a bound on the index of a cover of an aspherical 3-manifold which is Haken. The bound should be some computable function of some complexity of the 3-manifold, such as the minimal number of tetrahedra of a triangulation.

Open questions

- (Cannon) Is a hyperbolic group with boundary S^2 virtually a Kleinian group?
  Remark: Markovic '12 has shown that it suffices for any pair of points in S^2 to find a quasi-convex surface subgroup whose limit sets separates the pair.
- For any two hyperbolic 3-manifolds M_1, M_2, is there a finite-sheeted cover $M_1' \to M_1$, such that there is a non-zero degree map $M' \to M_2$? Are there fibered covers $M_i' \to M_i$ such that there is a non-zero degree map $M_1' \to M_2'$ which preserves the fibering?
- Do closed hyperbolic 3-manifolds contain immersed quasi-fuchsian surfaces of odd Euler characteristic?
- (Niblo-Wise) Which 3-manifold groups are LERF? No Seifert-Seifert gluings in JSJ?