Variation of extremal length on Teichmüller space

Weixu Su

Fudan University

August 11, 2012

Outline

1. Comparison between hyperbolic and extremal length.
2. Gromov's compactification of Teichmüller metric.
3. Variation of extremal length by harmonic maps

$S =$ closed surface of genus $g \geq 2$.
$T_g =$ Teichmüller space of marked conformal (hyperbolic) structures on S.

Let $X \in T_g$ and α be a simple closed curve on S.

Extremal length

$$\begin{aligned} \mathrm{Ext}_X(\alpha) &= \sup_{\rho} \frac{[\inf_{\alpha' \in [\alpha]}(\int_{\alpha'} \rho(z)|dz|)]^2}{Area_\rho(X)} \\ &= \inf_{A \sim \alpha} \frac{1}{\mathrm{mod}(A)}. \end{aligned}$$

Relation between extremal and hyperbolic length: Maskit, Masur, Minsky, Rafi $\cdots$

Teichmüller metric

Teichmüller metric on T_g:

$$d_T(X, Y) = \frac{1}{2} \inf_f \log K(f)$$

where $K(f)$ is the quasiconformal dilatation of $f : X \to Y$.
Teichmüller map

$$f_{k,\phi} : X \to X_{k,\phi}$$

where $\mu(f) = k\frac{|\phi|}{\phi}, \phi \in \mathrm{Q}(X), 0 \leq k < 1$.

Teichmüller geodesic ray with direction ϕ:

$$r_\phi(t) = X_{k,\phi}, t = \frac{1}{2} \log \frac{1+k}{1-k}.$$

Kerckhoff's Formula

$$d_T(X, Y) = \frac{1}{2} \log \sup_{\alpha} \frac{\mathrm{Ext}_X(\alpha)}{\mathrm{Ext}_Y(\alpha)}.$$

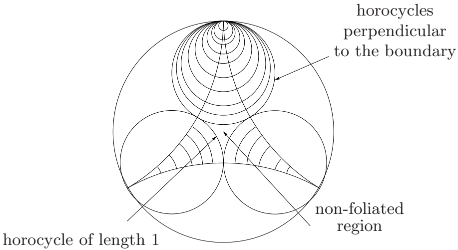
Theorem (Lenzhen-Rafi). Extremal length is quasi-convex (but not convex) along Teichmüller geodesics.
There is a constant $K \geq 1$ such that for any Teichmüller geodesic $r(t)$ and for any α,

$$\mathrm{Ext}_{r(s)}(\alpha) \leq K \max\{\mathrm{Ext}_{r(a)}(\alpha), \mathrm{Ext}_{r(b)}(\alpha)\}, a \leq s \leq b.$$

Thurston's Lipschitz metric

$$d_L(X,Y) = \log \sup_{\alpha} \frac{\ell_Y(\alpha)}{\ell_X(\alpha)}.$$

Stretch line



Question. Is hyperbolic or extremal length (quasi-)convex along stretch lines?

Comparison list between hyperbolic and extremal length

Hyperbolic length	Extremal length
Thurston's metric	Teichmüller metric
$\|\mu\|_L = \sup_{\lambda \in \mathcal{ML}} \frac{d\ell_\lambda(\mu)}{\ell_\lambda(X)}$	$\|\mu\|_T = \sup_{\lambda \in \mathcal{MF}} \frac{d\mathrm{Ext}_\lambda^{1/2}(\mu)}{\mathrm{Ext}_\lambda^{1/2}(X)}$
Thurston cataclysm coordinates $X \in T_g \mapsto F_\mu(X) \in \mathcal{MF}(\mu)$	The homeomorphism $X \in T_g \mapsto F_h(\Phi_F(X)) \in \mathcal{MF}(F)$
$\ell_\lambda(X) = i(\lambda, F_{\lambda^*}(X))$	$\mathrm{Ext}_\lambda(X) = i(\lambda, F_h(\Phi_\lambda(X)))$
$d\ell_\alpha(\mu) = \frac{2}{\pi}\mathrm{Re} < \Theta_\alpha, \mu >$	$d\mathrm{Ext}_\lambda(\mu) = -2\mathrm{Re} < \Phi_\lambda, \mu >$
...	...

Adding the list

Another possibility:
Weil-Petersson Funk metric (Yamada)

$$d(X, Y) = \sup_{\alpha} \frac{d_{WP}(Y, \overline{T}_\alpha)}{d_{WP}(X, \overline{T}_\alpha)}.$$

In this talk:

Hyperbolic length	Extremal length
Thurston's compactfication	**Gardiner-Masur compactification**
Kerckhoff, Wolpert $d^2\ell_\alpha(\mu, \mu)$	**Second variation** $d^2\mathrm{Ext}_\alpha(\mu, \mu)$

Thurston's compactification

$$T_g \hookrightarrow \mathbb{R}_+^{\mathcal{S}} \to P\mathbb{R}_+^{\mathcal{S}},$$

$$X \to \ell_X(\cdot) \to [\ell_X(\cdot)].$$

(Thurston) Thurston's compactification $\overline{T_g}^{Th}$ is a closed sphere of dimension $6g-6$. The action of Mod_g extends continuously to $\overline{T_g}^{Th}$.

(C. Walsh) The horofunction boundary of Lipschitz metric is Thurston's boundary.

Gardiner-Masur. The embedding

$$T_g \hookrightarrow \mathbb{R}_+^{\mathcal{S}} \to P\mathbb{R}_+^{\mathcal{S}},$$

$$X \to \mathrm{Ext}_X^{1/2}(\cdot) \to [\mathrm{Ext}_X^{1/2}(\cdot)]$$

has compact closure $\overline{T_g}^{GM}$. Moreover, Thurston's compactification $\overline{T_g}^{Th}$ is a proper subset of $\overline{T_g}^{GM}$.

Theorem (Miyachi). The action of Mod_g extends continuously to $\overline{T_g}^{GM}$.

Theorem (Liu-S). The horofunction boundary of Teichmüller metric is Gardiner-Masur's boundary.

Fixed a basis-point $X_0 \in (M, d)$ (locally compact). Assign each $X \in M$ with a function

$$\Psi_X(\cdot) = d(\cdot, X) - d(X_0, X).$$

(**Gromov**) $\Psi : M \to C(M), X \to \Psi_X$ is an embedding with compact closure $\overline{M}$.

Let $\overline{T_g}$ be Gromov's compactification of (T_g, d_T).

Theorem 1 (Liu-S). $\overline{T_g}^{GM} \cong \overline{T_g}$.

Normalized extremal length:

$$\mathcal{E}_X(\mu) = \frac{\mathrm{Ext}_X(\mu)^{1/2}}{K_X^{1/2}}, \mu \in \mathcal{MF}.$$

where $d_T(X_0, X) = \frac{1}{2}\log K_X$.

• $\mathcal{E}_{r(t)}(\mu)$ is a decreasing function along any Teichmüller geodesic $r(t)$.

Theorem (Miyachi). The normalized extremal length extends continuously to $\overline{T_g}^{GM}$: for any sequence (X_n) in T_g converging to $Z \in \overline{T_g}^{GM}$, there exists a subsequence (X_{n_j}) such that $\mathcal{E}_{X_{n_j}}(\cdot)$ converges to a positive multiple of $\mathcal{E}_Z(\cdot)$.

Proof of Theorem 1

We prove **Theorem 1** by showing that $Z \in \overline{T_g}^{GM} \to C(T_g)$

$$\Psi_Z(X) = \log \sup_{\mu \in \mathcal{MF}} \frac{\mathcal{E}_Z(\mu)}{\mathrm{Ext}_X(\mu)^{1/2}} - \log \sup_{\mu \in \mathcal{MF}} \frac{\mathcal{E}_Z(\mu)}{\mathrm{Ext}_{X_0}(\mu)^{1/2}}$$

is a homeomorphism onto $\overline{T_g}$.

- Note that when $Z \in T_g$,

$$\Psi_Z(X) = d_T(X, Z) - d_T(X_0, Z).$$

by Kerckhoff's formula.

Proof of Theorem 1

Let $\mathcal{L}_Z : \mathcal{MF} \to \mathbb{R}$,

$$\mathcal{L}_Z(\mu) = \frac{\mathcal{E}_Z(\mu)}{\sup_{\nu \in \mathcal{MF}} \frac{\mathcal{E}_Z(\nu)}{\mathrm{Ext}_{X_0}(\nu)^{1/2}}}.$$

Proposition.

$$\Psi_Z(X) = \log \sup_{\mu \in \mathcal{MF}} \frac{\mathcal{L}_Z(\mu)}{\mathrm{Ext}_X(\mu)^{1/2}}.$$

A sequence (Z_n) in $\overline{T_g}^{GM}$ converges to Z if and only if $\mathcal{L}_{Z_n}$ converges to $\mathcal{L}_Z$ uniformly on any compact set of $\mathcal{MF}$.

Proof of Theorem 1

Ψ is injective: $Z \neq Y$, $\exists\, X \in T_g$, $\Psi_Z(X) < \Psi_Y(X)$ (or $\Psi_Z(X) > \Psi_Y(X)$).

1. $\exists$ neighborhood $\mathcal{N} \subset \mathcal{PMF}$, $\mathcal{L}_Z(\nu) \leq A < B \leq \mathcal{L}_Y(\nu)$, $\forall\, \nu \in \mathcal{N}$.
2. $\exists\, \mu_0 \in \mathcal{N}$ uniquely ergodic, (Miyachi) $\mathcal{E}_{\mu_0}(\cdot) = i(\mu_0, \cdot)$.
3. Let $r(t)$ be a Teichmüller geodesic ray with vertical foliation μ_0, $\mathcal{L}_{r(t)} \to \mathcal{L}_{\mu_0} = i(\mu_0, \cdot)$.
4. For t sufficiently large, the supremum of $\frac{\mathcal{L}_Z(\cdot)}{\mathrm{Ext}_{r(t)}(\cdot)^{1/2}}$ is obtain in $\mathcal{N}$.

Corollary of Theorem 1

Corollary. The action of Mod_g extends continuously to $\overline{T_g}^{GM}$. Each Teichmüller (almost-)geodesic ray converges in the forward direction to a point in $\partial\overline{T_g}^{GM}$.

Compare with the following results:

(**Masur**) If a Teichmüller geodesic ray is uniquely ergodic or Strebel, then it converges to a limit in Thurston's boundary.

(**Lenzhen**) There exist Teichmüller geodesic rays which do not converge in Thurston's boudary.

Complex analytic theory of Teichmüller space

T_g is a $(3g-3)$ dim complex manifold (Bers embedding maps T_g to a bounded domain in $\mathbb{C}^{3g-3}$).

Let $X \in T_g$ with hyperbolic metric $\sigma^2(z)|dz|^2$.
$Q(X)$ = space of holomorphic quadratic differentials on X.

$$T_X T_g = \{\frac{\bar{\phi}}{\sigma^2} \mid \phi \in Q(X)\}$$

(space of harmonic Beltrami differentials).

Weil-Petersson metric

Weil-Petersson Riemannian metric:

$$< \frac{\bar{\phi}}{\sigma^2}, \frac{\bar{\psi}}{\sigma^2} >_{\mathrm{WP}} = \mathrm{Re} \int_X \frac{\phi\bar{\psi}}{\sigma^2}.$$

(Kähler, negative curvature, incomplete but geodesically convex).

Variation of hyperbolic length is important in W-P:

1. Wolpert's length-twist formula $\omega_{WP} = \sum d\ell_i \wedge d\tau_i$.
2. ℓ_α is convex along any W-P geodesic.
3. estimation of W-P curvature.

Pluri-subharmonic

A C^2 function F on a complex manifold is strictly pluri-subharmonic if its Levi 2-form $\partial\bar{\partial}F$ is positive definite.

Strictly pluri-subharmonic functions on T_g:

- (Tromba) Dirichlet energy of harmonic maps between hyperbolic surfaces, with varied domains and fixed targets.
- (Wolf, Yamada) Convexity of Dirichlet energy of harmonic maps, with fixed domain and varied targets.
- (S. K. Yeung) Bounded non-positive strictly plurisubharmonic exhaustion function on T_g (by using hyperbolic length functions).

Main result

Theorem 2 (Liu-S). Let $\Gamma_i(t), i = 1, 2$ be any two Weil-Petersson (or Teichmüller) geodesics on T_g with $\Gamma_1(0) = \Gamma_2(0) = X$ and $\frac{d}{dt}\Gamma_1(0) = \mu$, $\frac{d}{dt}\Gamma_2(0) = i\mu$, where i denotes the almost complex structure of T_g. Then

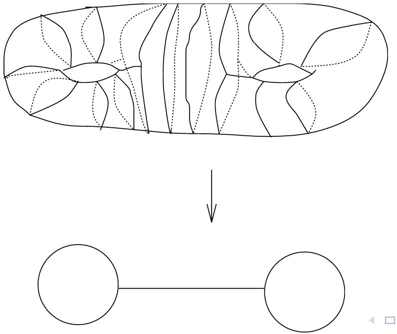
$$\frac{d^2}{dt^2}|_{t=0}\mathrm{Ext}_{\Gamma_1(t)}(\alpha) + \frac{d^2}{dt^2}|_{t=0}\mathrm{Ext}_{\Gamma_2(t)}(\alpha) > 0.$$

It follows that:
The extremal length of any simple closed curve is strictly pluri-subharmonic on T_g.

It seems that $Ext(\alpha)$ is not convex along Weil-Petersson geodesics.

Extremal length by harmonic maps

$\mathcal{F}$ = a measured foliation on S.
$\widetilde{\mathcal{F}}$ = the lift of $\mathcal{F}$ on the universal cover $\widetilde{S}$.
(T, d) = the leaf space of $\widetilde{\mathcal{F}}$
(T is a real tree with distance d induces by the measure of $\mathcal{F}$).
Let $\pi : \widetilde{S} \to T$ be the natural projection.



Extremal length by harmonic maps

Theorem (Wolf). Let $(S, \sigma) \in T_g$. There is a $\pi_1(S)$-equivariant harmonic map

$$\omega : (\widetilde{S}, \sigma) \to (T, d)$$

homotopic to π. The vertical measured foliation of the Hopf differential $\Phi =< \omega_z, \omega_z >_d dz^2$ is measured equivalent to $\mathcal{F}$.

(Kerckhoff)

$$\mathrm{Ext}_X(\alpha) = \int |\Phi_\alpha|,$$

where $\Phi_\alpha =$ the Hubbard-Masur differential with vertical foliation measured equivalent to α.

Corollary. The extremal length of $\mathcal{F}$ is equal to half of the energy $E(\omega, \sigma)$ of the harmonic map ω.

Extremal length by harmonic maps

(S, σ_t) : a smooth path in T_g.
Quasiconformal maps $z^t : (S, \sigma_0|dz|^2) \to (S, \sigma_t|dz_t|^2)$ with Beltrami differential $\mu(t) = t\mu + o(t)$.
More assumption: μ is a harmonic Beltrami differential $\frac{\overline{\Phi}}{\sigma_0^2}$ and $\ddot{\mu} = \frac{d^2}{dt^2}\mu(t)|_{t=0} \equiv 0$

(satisfied by Weil-Petersson or Teichmüller geodesic).

Second variation

$$\frac{d^2}{dt^2}E(\omega^t, \sigma_t) = -D_{11}^2 E(\omega^t, \sigma_t)[\dot{\omega}, \dot{\omega}] + D_{22}^2 E(\omega^t, \sigma_t)[\dot{\sigma}, \dot{\sigma}].$$

Denote $\omega^0 = \omega$.

$$E(\omega^t, \sigma_t) = \int_S |\frac{\partial \omega^t}{\partial z_t}|^2 dz_t d\overline{z_t}.$$

$$E(\omega, \sigma_t) = \int_S \frac{|\omega_z|^2 + |\mu(t)|^2 |\omega_{\overline{z}}|^2 - 2\mathrm{Re}(\mu(t)\omega_z \overline{\omega}_z)}{1 - |\mu(t)|^2} dz d\overline{z}.$$

Set $\omega^t = \omega + t\dot{\omega} + \frac{t^2}{2}\ddot{\omega} + o(t^2)$.

$$E(\omega^t, \sigma_0) = \int_S |\omega_z|^2 + t2\mathrm{Re}(\dot{\omega}_z \overline{\omega}_{\overline{z}}) + t^2(|\dot{\omega}_z|^2 + \mathrm{Re}(\omega \overline{\ddot{\omega}_{z\overline{z}}})) + o(t^2) dz d\overline{z}$$

Variational formulas

$$\frac{d}{dt}|_{t=0}E(\omega, g_t) = -2Re\int_S \mu\Phi, \Phi = \omega_z\bar{\omega}_z dz^2.$$

$$\frac{d^2}{dt^2}|_{t=0}E(\omega, \sigma_t) = 2\int_S |\mu|^2(|\omega_z|^2 + |\omega_{\bar{z}}|^2)dzd\bar{z}. \tag{1}$$

$$\frac{d^2}{dt^2}|_{t=0}E(\omega^t, \sigma_0) = 2\int_S |\dot{\omega}_z|^2 dzd\bar{z}. \tag{2}$$

$$\frac{d^2}{dt^2}|_{t=0}E(\omega^t, \sigma_t) = 2\int_S |\mu|^2(|\omega_z|^2 + |\omega_{\bar{z}}|^2)dzd\bar{z} - 2\int_S |\dot{\omega}_z|^2 dzd\bar{z}. \tag{3}$$

Understanding the vector field $\dot{\omega}$:

- $H(\omega^t, z_t) = \frac{\partial^2 \omega^t}{\partial z_t \partial \overline{z_t}} = 0$
- $-\frac{d}{dt}|_{t=0} H(\omega^t, z) = \dot{\omega}_{z\overline{z}}$
- $-\frac{d}{dt}|_{t=0} H(\omega, z_t) = \dot{\mu}\omega_{zz} + \overline{\dot{\mu}}\omega_{\overline{z}\overline{z}} + \overline{\dot{z}_{z\overline{z}}}\omega_{\overline{z}} + \dot{\mu}_z\omega_z$

(**Ahlfors** $\mu = \dot{z}_{\overline{z}}$)

$$= \frac{\partial}{\partial z}(\mu\omega_z) + \frac{\partial}{\partial \overline{z}}(\overline{\mu}\omega_{\overline{z}})$$

Equation of $\dot{\omega}$:

$$\dot{\omega}_{z\overline{z}} = \frac{\partial}{\partial z}(\mu\omega_z) + \frac{\partial}{\partial \overline{z}}(\overline{\mu}\omega_{\overline{z}}). \quad (4)$$

Using (4) and integration by part, we have

$$\begin{aligned} \int_S |\dot{\omega}_z|^2 dzd\overline{z} &= -\int_S \overline{\dot{\omega}}\dot{\omega}_{z\overline{z}} dzd\overline{z} \\ &= -\int_S \frac{\partial}{\partial z}(\mu\omega_z)\overline{\dot{\omega}} + \frac{\partial}{\partial \overline{z}}(\overline{\mu}\omega_{\overline{z}})\overline{\dot{\omega}} dzd\overline{z} \\ &= \int_S \mu\omega_z\overline{\dot{\omega}}_z + \overline{\mu}\omega_{\overline{z}}\overline{\dot{\omega}}_{\overline{z}} dzd\overline{z}. \end{aligned} \quad (5)$$

Proof of Theorem 2.

- By (3), $\frac{d^2}{dt^2}|_{t=0}E(\Gamma_1(t)) + \frac{d^2}{dt^2}|_{t=0}E(\Gamma_2(t))$

$$= 4\int_S |\mu|^2(|\omega_z|^2 + |\omega_{\bar{z}}|^2)dzd\bar{z}$$

$$-2\int_S |\dot{\omega}_z(\mu)|^2 dzd\bar{z} - 2\int_S |\dot{\omega}_z(i\mu)|^2 dzd\bar{z}.$$

- Apply (5) to establish to inequality (inspired by **Tromba**)

$$\int_S |\dot{\omega}_z(\mu)|^2 dzd\bar{z} + \int_S |\dot{\omega}_z(i\mu)|^2 dzd\bar{z}$$

$$< 2\int |\mu|^2(|\omega_z|^2 + |\omega_{\bar{z}}|^2)dzd\bar{z}.$$

Apply (5):

$$\int_S |\dot{\omega}_z|^2 dzd\bar{z} \leq 2\int_S |\mu|^2(|\omega_z|^2 + |\omega_{\bar{z}}|^2)dzd\bar{z}.$$

Theorem 3 (Liu-S). Let (S, σ_t) be a differential family of hyperbolic structures in T_g with Beltrami differential $\mu(t) = t\mu + o(t^2)$ at $t = 0$ and $\|\mu\|_\infty = 1$ (a.e. Teichmüller geodesic). Then we have

$$\begin{aligned} \frac{d^2}{dt^2}|_{t=0}\mathrm{Ext}_{\sigma_t}(\alpha) &\geq -\int_S(|\omega_z|^2 + |\omega_{\bar{z}}|^2)dzd\bar{z} \\ &= -2\int_S |\omega_z|^2 dzd\bar{z} = -4\mathrm{Ext}_{\sigma_0}(\alpha). \end{aligned}$$

Thanks for your attention!

Thanks Bob and Athanase!