Dilogarithm identities in conformal field theory and cluster algebras

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[N09] T. Nakanishi, Dilogarithm identities for conformal field theories and cluster algebras: simply laced case, arXiv:0909.5480, to appear in Nagoya Math. J.

Summary

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In the late 80's Bazhanov, Kirillov, and Reshetikhin conjectured the dilogarithm identities for the central charges of the Wess-Zumino-Witten conformal field theories. We prove the identities and its functional generalizations using cluster algebras.

Outline

Euler dilogarithm (1)

$k = 1, 2, 3, \dots$ (polylogarithm)

$$\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} \qquad \text{(converges in } |x| < 1\text{)}$$

Li = Logarithmic Integral
$k = 1$ (logarithm)

$$\mathrm{Li}_1(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x)$$

$k = 2$ (Euler dilogarithm)

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Integral expression:

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-y)}{y} dy$$

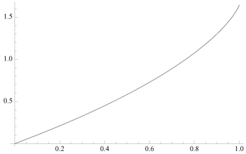
analytically continued to the universal covering of $\mathbb{P}^1 - \{0, 1, \infty\}$.
Special values:

$$\mathrm{Li}_2(0) = 0, \quad \mathrm{Li}_2(1) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} \quad \text{(Euler)}$$

Euler dilogarithm (2)

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = -\int_0^x \frac{\log(1-y)}{y} dy.$$

The function $\mathrm{Li}_2(x)$ looks boring.



However,

> *"Almost all of its appearances in mathematics, and almost all the formulas relating to it, have something of the fantastical in them, as if this function alone among all others possessed a sense of humor." — Don Zagier (1988)*

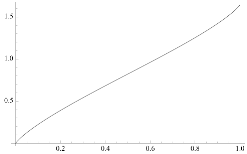
Rogers dilogarithm

Rogers dilogarithm function $L(x)$ $(0 \leq x \leq 1)$

$$L(x) = -\frac{1}{2}\int_0^x \left\{ \frac{\log(1-y)}{y} + \frac{\log y}{1-y} \right\} dy = \mathrm{Li}_2(x) + \frac{1}{2}\log x \log(1-x)$$

Again, $L(0) = 0, \quad L(1) = \dfrac{\pi^2}{6}$ (very important!)

Once again, the function looks boring (almost linear!).



Only few special values are known, e.g.,

$$\frac{6}{\pi^2}L\left(\frac{1}{2}\right) = \frac{1}{2}, \quad \frac{6}{\pi^2}L\left(\frac{-\sqrt{5}+3}{2}\right) = \frac{2}{5}, \quad \frac{6}{\pi^2}L\left(\frac{\sqrt{5}-1}{2}\right) = \frac{3}{5}.$$

Functional relation (1) (Euler) $L(x) + L(1-x) = \dfrac{\pi^2}{6}$.

Functional relation (2) (Abel, 5-term/pentagon relation)

$$L(x) + L(y) + L(1-xy) + L\left(\frac{1-x}{1-xy}\right) + L\left(\frac{1-y}{1-xy}\right) = \frac{\pi^2}{2} = 3\frac{\pi^2}{6}.$$

Outline

Dilogarithm identities in conformal field theories

X: any Dynkin diagram of A_r, D_r, E_6, E_7, or E_8 with index set I
$\ell \geq 2$: any integer
constant Y-system: $\{Y_m^{(a)} \mid a \in I; 1 \leq m \leq \ell - 1\}$: a family of positive real numbers

$$(Y_m^{(a)})^2 = \frac{\prod_{b:b\sim a} (1 + Y_m^{(b)})}{(1 + Y_{m-1}^{(a)}{}^{-1})(1 + Y_{m+1}^{(a)}{}^{-1})}, \qquad \text{(cY)}$$

$b \sim a$: b is adjacent to a in X, $Y_0^{(a)-1} = Y_\ell^{(a)-1} = 0$.
There exists a unique positive real solution of (cY). [Nahm-Keegan 09]

Conjecture 1 (Dilogarithm identities) [Bazhanov, Kirillov, Reshetikhin, 86–90]

For the unique positive real solution $\{Y_m^{(a)} \mid a \in I; 1 \leq m \leq \ell - 1\}$ of (cY),

$$\frac{6}{\pi^2} \sum_{(a,m)} L\left(\frac{Y_m^{(a)}}{1 + Y_m^{(a)}}\right) = \frac{\ell \dim \mathfrak{g}}{h + \ell} - r,$$

h: Coxeter number of X, $\mathfrak{g}$: simple Lie algebra of type X, r: rank of X.

(asymptotic of entropy of spin chains/S-matrix models) = (central charge of CFT)
Proved for $X = A_r$ [Kirillov 90].
Related to Rogers-Ramanujan-type identities, KR modules, hyperbolic 3-folds, etc.
Only partially proved in B.C. (=Before Cluster algebra [2000])

Functional dilogarithm identities

Y-system: [Zamolodchikov 91, Kuniba-Nakanishi 92, Ravanini-Tateo-Valleriani 93]
(X, X'): any pair of Dynkin diagrams A_r, D_r, E_6, E_7, or E_8
$\{Y_{ii'}(u) \mid i \in I, i' \in I', u \in \mathbb{Z}\}$: a family of variables

$$Y_{ii'}(u-1)Y_{ii'}(u+1) = \frac{\prod_{j:j\sim i}(1+Y_{ji'}(u))}{\prod_{j':j'\sim i'}(1+Y_{ij'}(u)^{-1})}, \qquad \text{(Y)}$$

where $j \sim i$: j is adjacent to i in X, $j' \sim i'$: j' is adjacent to i' in X'.

Conjecture 2 (Periodicity) [Ravanini-Tateo-Valleriani 93]

For $\{Y_{ii'}(u) \mid i \in I, i' \in I', u \in \mathbb{Z}\}$ satisfying (Y),

$$Y_{ii'}(u + 2(h + h')) = Y_{ii'}(u), \quad h : \text{Coxeter number of } X.$$

Conjecture 3 (Functional dilogarithm identities) [Gliozzi-Tateo 95]

For any family of positive real numbers $\{Y_{aa'}(u) \mid a \in I, a' \in I', u \in \mathbb{Z}\}$ satisfying (Y),

$$\frac{6}{\pi^2} \sum_{(i,i')\in I\times I'} \sum_{0\le u<2(h+h')} L\left(\frac{Y_{ii'}(u)}{1+Y_{ii'}(u)}\right) = 2hrr', \quad r = \operatorname{rank} X.$$

Conj. 3 $\Longrightarrow$ Conj. 1; set $X' = A_{\ell-1}$, take a constant solution $Y_{ii'} = Y_{ii'}(u)$.
The simplest case $X = X' = A_1$: Euler's identity.
The next simplest case $X = A_2$, $X' = A_1$: Abel's identity.

Main result

Known results on Conjectures 2 and 3:

Who and When	Periodicity	Funct.Dilog.Id.	Method
Gliozzi-Tateo 95	(A_r, A_1)	(A_r, A_1)	explicit solution
Frenkel-Szenes 95	(A_r, A_1)	(A_r, A_1)	explicit solution constancy condition (1)
Fomin-Zelevinsky 00∼			cluster algebra
Fomin-Zelevinsky 03	(any, A_1)		cluster algebra-like setting (2) Coxeter transformation
Chapoton 05		(any, A_1)	(1) + (2) evaluation at $0/\infty$ limit
Szenes 06 Volkov 06	$(A_r, A_{r'})$		flat connection on graph explicit solution
Fomin-Zelevinsky 07			cluster algebra / coefficients (3) F-polynomials (4)
Keller 08	(any, any)		(3)+(4) cluster category Auslander-Reiten theory

Using these ideas, methods, and results, we obtain the following theorem.

Theorem [N 09]

Conjecture 3 is true for any X and X'.

Outline

Constancy condition (1)

$\mathcal{I}$: any open or closed interval in $\mathbb{R}$
$\mathcal{C} = \mathcal{C}(\mathcal{I}) := \{f \mid f : \mathcal{I} \to \mathbb{R}_+, \text{differentiable}\}$, a multiplicative abelian group
$\mathcal{C} \otimes_{\mathbb{Z}} \mathcal{C}$: the additive abelian group with generator $f \otimes g$ ($f, g \in \mathcal{C}$) and relation

$$(fg) \otimes h = f \otimes h + g \otimes h, \quad f \otimes (gh) = f \otimes g + f \otimes h$$

$$(\Longrightarrow 1 \otimes h = h \otimes 1 = 0, \quad f^{-1} \otimes h = f \otimes h^{-1} = -f \otimes h)$$

$S^2\mathcal{C}$: subgroup of $\mathcal{C} \otimes_{\mathbb{Z}} \mathcal{C}$ generated by $f \otimes g + g \otimes f$ ($f, g \in \mathcal{C}$)
$\bigwedge^2 \mathcal{C} := \mathcal{C} \otimes_{\mathbb{Z}} \mathcal{C}/S^2\mathcal{C}$ (write $f \otimes g$ as $f \wedge g$)

Theorem [Frenkel-Szenes 95]

Let $f_1(t), \dots, f_n(t)$ be differentiable functions from $\mathcal{I}$ to $(0,1)$. Suppose that they satisfy the following relation in $\bigwedge^2 \mathcal{C}$:

$$\sum_{i=1}^{n} f_i \wedge (1 - f_i) = 0 \quad \text{(constancy condition)}$$

Then, the dilogarithm sum $\sum_{i=1}^{n} L(f_i(t))$ is constant with respect to $t \in \mathcal{I}$.

Proof. The proof of [FS95] is surprisingly simple.

$$dL(x) = -\frac{1}{2}\left\{\frac{\log(1-x)}{x} + \frac{\log x}{1-x}\right\} dx$$
$$= -\frac{1}{2}\{\log(1-x)d\log x - \log x d\log(1-x)\}. \quad \text{(proof continued)}$$

Constancy condition (2)

$$d\sum_{i=1}^{n} L(f_i(t)) = -\frac{1}{2}\sum_{i=1}^{n}\{\log(1-f_i(t))d\log f_i(t) - \log f_i(t) d\log(1-f_i(t))\}\,.$$

By assumption,

$$\sum_{i=1}^{n} f_i \otimes (1-f_i) = \sum_{i=1}^{k} g_i \otimes h_i + h_i \otimes g_i \quad \text{for some } g_i, h_i \in \mathcal{C}.$$

For any $t, s \in \mathcal{I}$, we have an additive group homomorphism $\mathcal{C} \otimes_{\mathbb{Z}} \mathcal{C} \to \mathbb{R}$, $f \otimes g \mapsto \log f(t) \log g(s)$. Therefore,

$$\sum_{i=1}^{n} \log f_i(t) \log(1-f_i(s)) = \sum_{i=1}^{k} \{\log g_i(t) \log h_i(s) + \log h_i(t) \log g_i(s)\}\,.$$

Taking the derivative for t and s and setting $s = t$,

$$\sum_{i=1}^{n} d\log f_i(t) \cdot \log(1-f_i(t)) = \sum_{i=1}^{k} \{d\log g_i(t) \cdot \log h_i(t) + d\log h_i(t) \cdot \log g_i(t)\}\,,$$

$$\sum_{i=1}^{n} \log f_i(t) \cdot d\log(1-f_i(t)) = \sum_{i=1}^{k} \{\log g_i(t) \cdot d\log h_i(t) + \log h_i(t) \cdot d\log g_i(t)\}\,.$$

Therefore, we have $d\sum_{i=1}^{n} L(f_i(t)) = 0$. □

Q: How to find a set of functions $\{f_i\}$ satisfying the constancy condition?
A: Periodic cluster algebras give such functions.

Outline

Cluster algebra with coefficients

triplet (B, x, y) (initial seed)
B: skew symmetric matrix $B = (B_{ij})_{i,j\in I}$ (mutation matrix)
x: I-tuple of formal variables $x = (x_i)_{i\in I}$ (cluster)
y: I-tuple of formal variables $y = (y_i)_{i\in I}$ (coefficient tuple)
Mutation of (B, x, y) at $k \in I$: $(B', x', y') = \mu_k(B, x, y)$

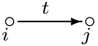
$$B'_{ij} = \begin{cases} -B_{ij} & i = k \text{ or } j = k, \\ B_{ij} + \frac{1}{2}(|B_{ik}|B_{kj} + B_{ik}|B_{kj}|) & \text{otherwise.} \end{cases}$$

$$y'_i = \begin{cases} {y_k}^{-1} & i = k, \\ y_i \left(\dfrac{1}{1 \oplus {y_k}^{-1}} \right)^{B_{ki}} & i \neq k,\ B_{ki} \geq 0, \\ y_i(1 \oplus y_k)^{-B_{ki}} & i \neq k,\ B_{ki} \leq 0. \end{cases}$$

$$x'_i = \begin{cases} \dfrac{y_k \prod_{j:B_{jk}>0} x_j^{B_{jk}} + \prod_{j:B_{jk}<0} x_j^{-B_{jk}}}{(1 \oplus y_k)x_k} & i = k, \\ x_i & i \neq k. \end{cases}$$

The mutation is involutive, i.e., $\mu_k(B', x', y') = (B, x, y)$.
Repeat mutation and collect all the seeds (B'', x'', y'').
The cluster algebra $\mathcal{A}(B, x, y)$ is a ring generated by all x''_i (cluster variables).

Skew symmetric matrix B $\leftrightarrow$ quiver Q (with no loop and 2-cycle)

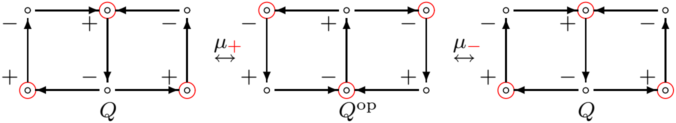
$B_{ij} = t > 0$ $\leftrightarrow$ 

Formulation of Y-system by cluster algebra

★ Roughly speaking,

	cluster algebra
Y-system	coefficients y_i
T-system	cluster variables x_i

★ Example. Y-system for $(X, X') = (A_3, A_2)$.

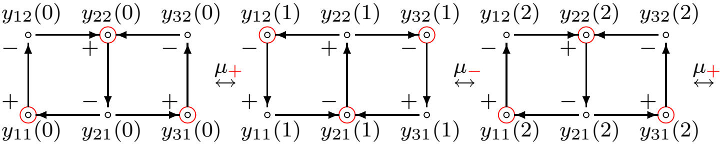


○: forward mutation points

Set $y(0) := y$, and repeat the mutations μ_+ and μ_- alternatively:

$$\cdots \overset{\mu_+}{\leftrightarrow} (Q^{\mathrm{op}}, y(-1)) \overset{\mu_-}{\leftrightarrow} (Q, y(0)) \overset{\mu_+}{\leftrightarrow} (Q^{\mathrm{op}}, y(1)) \overset{\mu_-}{\leftrightarrow} (Q, y(2)) \overset{\mu_+}{\leftrightarrow} \cdots$$

Then, $\{y_{ii'}(u)$'s at ∘$\}$ satisfy the Y-system. e.g., $y_{11}(0)y_{11}(2) = \dfrac{1 + y_{21}(1)}{1 + y_{12}(1)^{-1}}$.



$y_{11}(1) = y_{11}(0)^{-1}$ $\qquad$ $y_{11}(2) = y_{11}(1)\dfrac{1 + y_{21}(1)}{1 + y_{12}^{-1}(1)}$

Outline of proof

We want to show the identity.

$$\frac{6}{\pi^2} \sum_{(i,i')\in I\times I'} \sum_{0\le u<2(h+h')} L\left(\frac{Y_{ii'}(u)}{1+Y_{ii'}(u)}\right) = 2hrr', \quad r = \operatorname{rank} X. \qquad \text{(FDI)}$$

Step 1. Formulate the Y-system (Y) by cluster algebra with coefficients $\mathcal{A}(Q,x,y)$, where Q is some quiver. [Keller 08]

- $\{y_{ii'}(u)\text{'s at } \circ\}$ satisfy the Y-system.
- $y_{ii'}(u)$'s are subtraction-free rational functions of initial coefficients $y_{ii'}$ with either positive or negative degree. [Derksen-Weyman-Zelevinsky 10]
- Periodicity: $y_{ii'}(u+2(h+h')) = y_{ii'}(u)$ [Keller 08]

Step 2. Show the consistency condition of LHS of (FDI). [Frenkel-Szenes 95]

$$\sum_{\substack{(i,i')\in I\times I'\\ 0\le u<2(h+h')}} \frac{y_{ii'}(u)}{1+y_{ii'}(u)} \wedge \frac{1}{1+y_{ii'}(u)} = 0.$$

This is 'automatic' for periodic cluster algebras (using F-polynomials, etc.).

Step 3. Evaluate the LHS of (FDI) in the '$0/\infty$ limit'. [Chapoton 05]

$$\frac{6}{\pi^2} L\left(\frac{Y}{1+Y}\right) = \begin{cases} 0 & Y \to 0 \\ 1 & Y \to +\infty. \end{cases}$$

Take the limit $y_{ii'} \to 0$. Then, each $Y_{ii'}(u)$ goes either 0 or $+\infty$. Therefore,
LHS of (FDI) $= 2 \times \#\{y_{ii'}(u)\text{'s at } \circ \text{ in } 0 \le u < 2(h+h') \text{ with negative degree}\}$

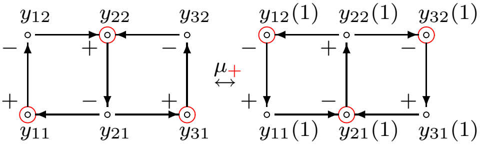
Step 4. Count the number above using the tropical Y-system.

Step 4: Tropical Y-system (1)

Tropical Y-system: Replace '+ ' by tropical '$\oplus$' for Laurent polynomials (monomials) of initial coefficient tuple $y = y(0)$.

$$\prod_{i,i'} y_{ii'}^{a_{ii'}} \oplus \prod_{i,i'} y_{ii'}^{b_{ii'}} = \prod_{i,i'} y_{ii'}^{\min(a_{ii'}, b_{ii'})}.$$

Example (continued) $(X, X') = (A_3, A_2)$



$(Q, y(0))$ $\quad$ $(Q^{\mathrm{op}}, y(1))$

$$y_{11}(1) = y_{11}^{-1},\ y_{22}(1) = y_{22}^{-1},\ y_{31}(1) = y_{31}^{-1},$$

$$y_{12}(1) = y_{12}\frac{1 \oplus y_{22}}{1 \oplus y_{11}^{-1}} = y_{11}y_{12},$$

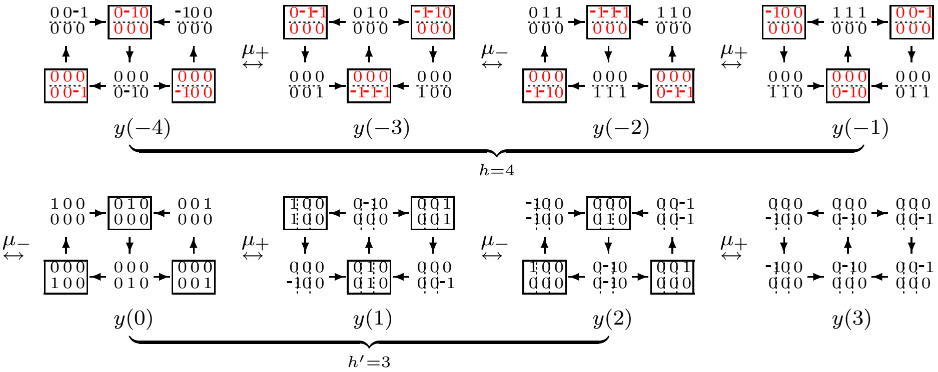
$$y_{21}(1) = y_{21}\frac{(1 \oplus y_{11})(1 \oplus y_{31})}{1 \oplus y_{22}^{-1}} = y_{21}y_{22},$$

$$y_{32}(1) = y_{32}\frac{1 \oplus y_{22}}{1 \oplus y_{31}^{-1}} = y_{31}y_{32},$$

Step 4: Tropical Y-system (2)

Example (continued) $(X, X') = (A_3, A_2)$. $h = 4$, $h' = 3$, $r = 3$, $r' = 2$.
Fact: There is the half periodicity $y_{ii'}(u + h + h') = y_{4-i,3-i'}(u)$. ($h + h' = 7$).
Let us show

$$\#\{y_{ii'}(u)\text{'s at } \circ \text{ with negative degree in } 0 \le u < h + h'\} = \frac{1}{2}hrr' = \frac{1}{2} \cdot 4 \cdot 3 \cdot 2 = 12.$$

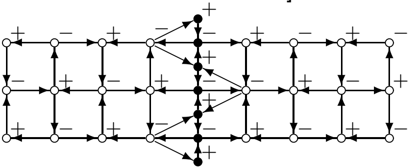


negative in $-h \le u < 0$, positive in $0 \le u < h'$ ‘factorization property’

Outline

Periodic cluster algebras

One can associate a dilogarithm identity for any periodic cluster algebra. [Nakanishi 10]
Example 1. The Y-system for nonsimply laced WZW models. $X = B_5$, $\ell = 4$.
[Inoue-Iyama-Keller-Kuniba-Nakanishi 10]

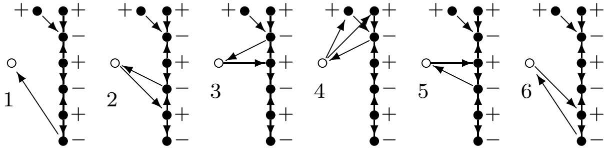
$Q =$ 

$\mu^\bullet_- \mu^\circ_- \mu^\bullet_+ \mu^\bullet_- \mu^\circ_+ \mu^\bullet_+ (Q) = Q$

(Q, x, y): initial seed

Periodicity: $(\mu^\bullet_- \mu^\circ_- \mu^\bullet_+ \mu^\bullet_- \mu^\circ_+ \mu^\bullet_+)^{13}(Q, x, y) = (Q, x, y)$, $13 = 9 + 4 = h^\vee(B_5) + \ell$.

Example 2. The Y-system for sine-Gordon models. $n = 7$. [Nakanishi-Tateo 10]



Q_1 Q_2 Q_3 Q_4 Q_5 Q_6

All the vertices $\bullet$ in the same position in the quivers $Q_1, \dots, Q_6$ are identified.
Periodicity: $(\mu^\bullet_- \mu_6 \mu^\bullet_+ \cdots \mu^\bullet_- \mu_2 \mu^\bullet_+ \mu^\bullet_- \mu_1 \mu^\bullet_+)^{13}(Q, x, y) = (Q, x, y)$.
$13 = (12 + 2 + 10 + 2)/2 = (h(D_7) + 2 + h(D_6) + 2)/2$.

And these examples should be a tip of iceberg.

Local constancy condition (for experts)

★ Let $\mathbf{i} = (i_1, \dots, i_m)$ be any period of a seed (B, x, y). Let $\mathbf{i} = \mathbf{i}(0) \mid \cdots \mid \mathbf{i}(\Omega - 1)$ be any decomposition of $\mathbf{i}$, where each $\mathbf{i}(u)$ consists of 'compatible mutations'.
The sequence of seeds $(B(u), y(u))$ $(u \in \mathbb{Z})$ and the forward mutation points are defined as before.

★ We define the 'fundamental regions' S_+ of forward mutation points by

$$S_+ = \{\text{forward mutation point } (i, u) \mid 0 \leq u < \Omega\}.$$

Proposition (Constancy condition [N 10])

$$\sum_{(i,u) \in S_+} y_i(u) \wedge (1 + y_i(u)) = 0.$$

★ Local verion of constancy condition (following the idea of [Fock-Goncharov 09]).
Let (B', x', y') and (B'', x'', y'') be any seeds such that $(B'', x'', y'') = \mu_k(B', x', y')$.
For each seed (B', x', y'), we set

$$W' = \frac{1}{2} \sum_{i \in I} F'_i \wedge (y'_i [y'_i]_{\mathbf{T}}) = \sum_{i \in I} F'_i \wedge y'_i + \frac{1}{2} \sum_{i \in I} B'_{ij} F'_i \wedge F'_j$$

Proposition (Local constancy condition [N 10]; cf. [Fock-Goncharov 09])

$$W'' - W' = y'_k \wedge (1 + y'_k).$$

(Local constancy condition) + (Periodicity) $\Longrightarrow$ (Constancy condition)

Points of further interest

Points of further interest:

- polylogarithm
- quantization (Kashaev, Fock-Goncharov, Kontsevich-Soibelman, Cecotti-Neitzke-Vafa, ...)
- 2d and 3d hyperbolic geometry (Kashaev, Fock-Goncharov, Gekhtman-Shapiro-Veinstein, Fomin-Shapiro-Thurston, Bridgeman, ...)

"The function is so shy that cluster algebraic nature is hidden under the mask of integral." — Anonymous (2010)