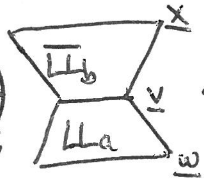
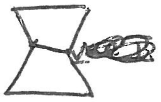
Last time: $\mathcal{D}$ (or BSBim) has a right R-basis given by double leaves

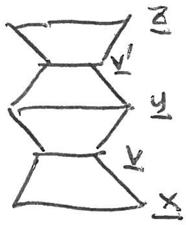
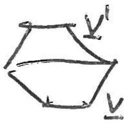
ttst

ask for 0,1 sequences

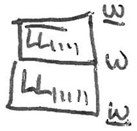
sts

$\mathrm{Hom}_{\mathcal{D}}(\underline{w}, \underline{x}) = \bigoplus_{\substack{v \in W \\ a,b}}$  $\cdot R$ for $v \in W$, a, b subexp for v

Moreover, $\mathcal{D}$ is filtered. For any ideal $I \subset W$, $\bigoplus_{v \in I}$  R, is an ideal in the cat. This is because

composing  factors thru  $= \sum$  for $u \leq v$, $u \leq v' \Rightarrow u \in I$

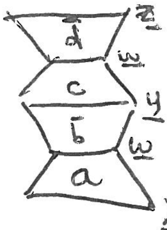
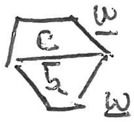
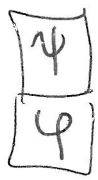
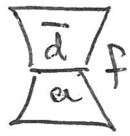
(argument continues)

Usual ideals: $\leq w$, $< w$ for $w \in W$. Let $\underline{w}$ a rex for w. Then $\mathrm{End}_{\mathcal{D}^{\leq w}/\mathcal{D}^{< w}}(\underline{w}) =$  $\cdot R$ $= \mathbb{1}_{\underline{w}} \cdot R$

(choice of rex moves irrelevant modulo lower terms).

This is FANTASTIC.

Gives us complete control over $\mathcal{D}^{\leq w}/\mathcal{D}^{< w}$!

ψ, $\varphi =$  $= ?$  $= \mathbb{1}_{\underline{w}} \cdot f + \text{lower terms}$, so  $=$  f in $\mathcal{D}^{\leq w}/\mathcal{D}^{< w}$

$f \in R$

this is a pairing on $M(\underline{y}, w) = \{\text{subexp of } \underline{y} \text{ for } w\}$ (or an R-span)

$\varphi_y(c, \bar{b}) \in R$.

The ass gr. is controlled by φ_y for all y. This kind of filtered category is known as an (object adapted) cellular category, has lovely properties. Let's use them!

Claim 1: For any rex $\underline{w}$ $\exists!$ indecomp $B_w \subset \underline{w}$ s.t. $B_w \not\subset \underline{y}$ for $y < w$. (in Kar($\mathcal{D}$))

Pf: $\mathbb{1}_{\underline{w}} = e_1 + e_2 + \dots + e_n$ ortho. idemp. If two had nonzero coeff of  $\in \mathbb{R}^{\times}$ degree 0

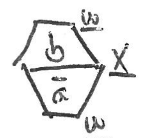
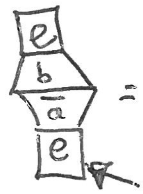
$e_i = \sum$  then $e_i e_j \in \mathbb{R}^{\times} \subset \mathrm{End}_{\mathcal{D}/\mathcal{D}^{<w}}(B_{\underline{w}}) \Rightarrow e_i e_j \neq 0$ ※.

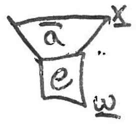
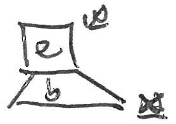
So ! has nonzero coeff (in fact, coeff 1). Call its image B_w.

Rmk: Any idempotent closed $\mathbb{R}$-linear additive cat w/ f.d. Hom spaces (in each degree) is Krull-Schmidt, things split into indecomps in unique way, and End(Indecomp) is a local ring.

Claim 2: Spose $\underline{x}$ any expression, $\exists\, a,b \in M(\underline{x},\underline{w})$ s.t. $\varphi_{\underline{x}}(b,\bar{a}) \in \mathbb{R}^{\times} \subset R$ ②
deg $a = d$ deg $b = -d$

Then $B_w(-d) \overset{\oplus}{\subset} \underline{x}$.

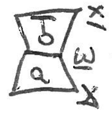
Pf:  $= \underset{\mathbb{R}^{\times}}{\overset{\cap}{K}}\, \mathbb{1}_{\underline{w}}$ + lower terms $\Rightarrow$  $= K\, \mathbb{1}_{\underline{w}}$ + lower terms.
idemp from claim 1.

Now  : $\underset{B_w}{\overset{\underline{x}}{\uparrow}}$  : $\underset{\underline{x}}{\overset{B_w}{\uparrow}}$ Composition is invertible in End(B_w)

Why? B_w indecomp $\Rightarrow$ End(B_w) is local. If it were in maxl ideal, would still be in maxl ideal in $\mathcal{D}/\mathcal{D}^{<w}$, but its invertible there.
(contained in maxl ideal)

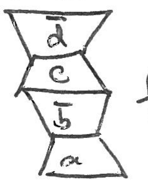
Thus $B_w(-d) \overset{\oplus}{\subset} \underline{x}$. ■ ⊛ Reversing, also get $B_w(+d) \overset{\oplus}{\subset} \underline{x}$!

SCT: These B_w do not depend on rex chosen. Get all indecomps up to shift.

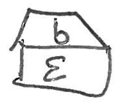
Pf: ■ Let $\underline{x}$ arbitrary, $\varepsilon \in$ End$(\underline{x})$ a primitive idempotent, $\varepsilon = \sum$  $\cdot f_{a,\bar{b}}$

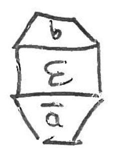
choose w maxl w/ $f_{a,\bar{b}} \neq 0$. (not nec unique). $\exists$TS $\mathrm{Im}(\varepsilon) \overset{\oplus}{\supset} B_w(-d)$
b/c indecomp.
(this shows rex independent too)

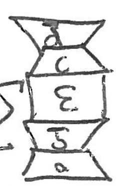
Work in $\mathcal{D}/\mathcal{D}^{<w}$ (rather $/\mathcal{D}^{\not\geq w}$)

$\varepsilon^2 = \sum$  $f_{a,\bar{b}}\, f_{c,\bar{d}} = \sum$  $\varphi(c,\bar{b})\, f_{a\bar{b}}\, f_{c\bar{d}}$

So $\sum_{b,c} \varphi(c,\bar{b})\, f_{a,\bar{b}}\, f_{c\bar{d}} = f_{a\bar{d}}$. If all $\varphi(c,\bar{b}) \in R_+$ (or 0) then so are all $f_{a\bar{d}}$, but no idempotent is in maximal ideal. So some $\varphi(c,\bar{b}) \in \mathbb{R}^{\times}$.

But this only says $B_w(\ ) \overset{\oplus}{\subset} \underline{x}$. Want $B_w \overset{\oplus}{\subset} \mathrm{Im}(\varepsilon)$, i.e., want  

with  $= \underset{\mathbb{R}^{\times}}{\overset{\cap}{K}}\, \mathbb{1}_{\underline{w}}$ + lower terms

but $\underset{\varepsilon}{\overset{\varepsilon^3}{\|}} = \sum$  use same argument. ■

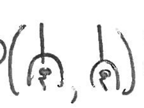
So how do we find B_w? $e = \mathbb{1}_{\underline{w}} - \sum$ other idemps.

find other idemps by finding nondegenerate parts of $\varphi_{\underline{y},\underline{w}}$!

__Ex:__ $\cancel{w} = sts$. All LL are strictly pos degree except LL_{111} and LL_{100}

Look at $LL_{\underline{x},s} = \{$  $\}$
001 , 100

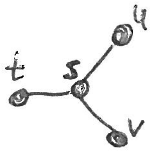
nothing to paragraph to reach degree 0, ignore

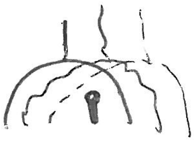
$\varphi($  $) = | \partial_s(\alpha_t) = | a_{s,t}$

so when $a_{s,t} \neq 0$, get an idempotent $\frac{1}{a_{s,t}}$ 

(over $\mathbb{R}$, this is just $m_{st} \neq 2$; over $\mathbb{F}_2$ in type B_2, interesting)

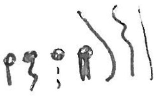
__Ex:__ $\underline{x} = tsut$. No other LL maps have degree ≤ 0, indecomposable.

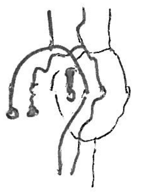
__Ex:__  $\underline{x} = tuv\ s\ tuv$ consider $\varphi_{\underline{x},\, tuv}$

8 maps: degree -2 

degree 0 

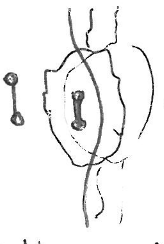
degree 2  etc etc. ← Kernel is at least 2D for φ/R_+

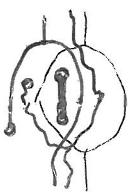
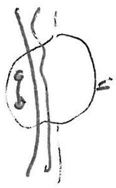
degree 4  ← clearly in kernel of φ/R_+

pair degree -2 against $+2$:  $=$  $\partial_v(\alpha_s) = -$ 

pairing $\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$

so $B_{tuv}(2) \oplus B_{tuv}(-2) \overset{\oplus}{\subset} B_{\underline{x}}$

pair degree 0:  $= 0$ since $\partial_u \partial_v(\alpha_s) = 0$

 $=$  $\cdot -1$

pairing is $\begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$ det $= -2$.

So $B_{tuv}^{\oplus 3} \subset B_{\underline{x}}$

(but in characteristic 2, $B_{tuv}^{\oplus 2} \subset B_{\underline{x}}$

so B_x is bigger!)