Lecture 2.1: The classical approach to Soergel bimodules

<u>Def:</u> "classical" = known to more than two people for more than two years.

Recall: $\mathfrak{h}$ is a realization of W (e.g. geometric rep)

$R = S(\mathfrak{h}^*)$ = polynomial functions on $\mathfrak{h} \circlearrowleft W$.

<u>Standard bimodules:</u> For any $x \in W$ consider $R_x \in R\text{-Bim}$ defined as follows

$R_x \cong R$ as left R-module, $m \cdot r = x(r)m$ for $m \in R_x$, $r \in R$.

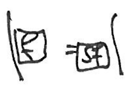
R_x can be viewed as a completely porous wall!

Obviously $R_x \otimes_R R_y = R_x R_y \in R\text{-Bim}$. Also

$$\mathrm{Hom}(R_x, R_y) = \begin{cases} R & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

StdBim = full $\oplus$, (n) subcat of R-Bim generated by $\{R_x \mid x \in W\}$.

Then StdBim $\cong$ 2-groupoid of W over R.

<u>Remark:</u> One can draw StdBim as in Ben's lecture. Only difference now is that $\mathrm{End}(R_x) = R$, hence one has polynomials in boxes.  .

<u>Soergel bimodules:</u> $\mathbb{S}\mathrm{Bim} \subset R\text{-Bim}$ is the full $\otimes, \oplus, (n)$ Karoubian subcat generated by $B_s := R \otimes_{R^s} R(1)$.

Hence objects of $\mathbb{S}\mathrm{Bim}$ are isomorphic to sums of summands of $BS(\underline{w}) = R \otimes_{R^s} \ldots \otimes_{R^s} R\,(m)$.

<u>Verbal:</u> associated to each ~~direct sum~~ simple reflection one has a Frobenius object, one looks at the full subcategory generated by these!

<u>Examples:</u> a) If $W = S_2$ we have seen $B_s B_s \cong B_s(1) \oplus B_s(-1)$. Hence

$$\text{ind. Soergel bimodules}/_{\cong,(n)} = \{R, B_s\}.$$

b) if $W = S_3$, $R, B_s, B_t, B_{st} := B_s B_t, B_{ts} := B_t B_s$ are indecomposable. (Ben's lecture)

<u>Exercise:</u> $B_s B_t B_s \cong B_{sts} \oplus B_s$. $B_t B_s B_t \cong B_{sts} \oplus B_t$. Hence (give the argument verbally)

$$\text{ind. Soergel bimod.} = \{B_{id} = R, B_s, B_t, B_{st}, B_{ts}, B_{sts}\}.$$

$\mathbb{S}Bim$ is <u>not</u> abelian. For example one has exact sequences

$$(\Delta) \qquad \begin{array}{ccccc} 1 & \longmapsto & \alpha_s \otimes 1 - 1 \otimes \alpha_s & & \\ R_s(-1) & \hookrightarrow & B_s & \twoheadrightarrow & R \\ & & f \otimes g & \underset{\text{counit}}{\longmapsto} & fg \end{array}$$

$$(\nabla) \qquad \begin{array}{ccccc} 1 & \longmapsto & \frac{1}{2}(\alpha_s \otimes 1 + 1 \otimes \alpha_s) & & \\ R(-1) & \xrightarrow[\text{unit}]{} & B_s & \twoheadrightarrow & R_s(1) \\ & & f \otimes g & \longmapsto & f s(g) \end{array}$$

If we tensor these exact sequences together we see that any Bott-Samelson bimodule has a filtration $0 \subset B^1 \subset \dots \subset B^m = BS(\underline{w})$ s.t. $B^m/B^{m-1} \cong \bigoplus R_x(?)$'s.

In general the order in which summands R_x appear has nothing to do with the Bruhat order.

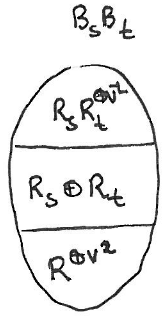
Fix an enumeration $w_0, w_1, \dots$ of W s.t. $w_i \leq w_j \Rightarrow i \leq j$.

<u>Def:</u> A <u>standard filtration</u> on a Soergel bimodule is a filtration $0 = B^0 \subset \dots \subset B^m = B$ s.t. $B^i/B^{i-1} \cong \bigoplus R_{x_i}^{\oplus h_{x_i}}$ for some $h_{x_i} \in \mathbb{N}[v^{\pm 1}]$.

<u>Notation:</u> $p = \sum a_i v^i \in \mathbb{N}[v^{\pm 1}]$, $M^{\oplus p} = \bigoplus M(-i)^{\oplus a_i}$.

<u>Soergel:</u> any Soergel bimodule <u>admits</u> a <u>unique</u> standard filtration.
tricky — exercise! (it's a support filtration)

$\rightsquigarrow$ $ch : \mathbb{S}Bim \longrightarrow \mathcal{H} : \quad B \longmapsto \sum_{x \in W} v^{+\ell(x)} h_x \, H_x$.



<u>Eg:</u> $ch(B_s) = \underline{H}_s$. ~~$B_sB_t$ ... $R^{\oplus v^2}$~~

$\Rightarrow ch(B_sB_t) = v^2 H_{id} + vH_s + vH_t + H_{st} = \underline{H}_{st}$.

<u>Localization:</u> Let Q denote the fraction field of R.

<u>Lemma:</u> $BS(\underline{w}) \otimes_R Q = R \otimes_{R^s} R \otimes_{R^t} \dots \otimes_{R^u} R \otimes_R Q \cong Q \otimes_{Q^s} Q \otimes_{Q^t} \dots \otimes_{Q^u} Q$

Hence $BS(\underline{w}) \otimes_R Q$ is actually a Q-bimodule.

<u>Proof:</u> Consider the inclusion $R \otimes_{R^s} R \otimes_R Q \cong R \otimes_{R^s} Q \overset{i}{\hookrightarrow} Q \otimes_{Q^s} Q$. Enough to show i is an iso.

We have $\frac{1}{f} \otimes 1 = s(f) \otimes \frac{1}{f s(f)} \in \operatorname{im} i$. Hence i is an isomorphism. $\square$

It follows that localization gives a monoidal functor

$$\mathbb{S}\mathrm{Bim} \longrightarrow Q\text{-}\mathrm{Bim}.$$

Also, both sequences split (Δ) and (∇). (In fact they split each other!)

In fact, the standard filtration on any Soergel bimodule splits after localisation.

<u>Remark:</u> Localization categorifies the specialization $v \mapsto 1$.

$$\begin{tikzcd} \mathbb{S}\mathrm{Bim} \arrow[r, "ch"] \arrow[d] & \mathcal{H} \arrow[d] \\ \mathbb{S}\mathrm{Bim}_Q \arrow[r, "ch"] & \mathbb{Z}W \end{tikzcd}$$

<u>Soergel's categorification theorem:</u>

$[\mathbb{S}\mathrm{Bim}]$ Split Grothendieck group of $\mathbb{S}\mathrm{Bim}$: generated by symbols $[M]$ for $M \in \mathbb{S}\mathrm{Bim}$. Subject to $[M] = [M'] + [M'']$ if $M \cong M' \oplus M''$.

Ring: $[M][N] = [M \otimes N] = [M \otimes_R N]$.

$\mathbb{Z}[v^{\pm 1}]$ algebra: $v[M] = [M(-1)]$.

<u>Thm</u> a) $\exists!$ homomorphism $\mathbb{H} \xrightarrow{c} [\mathbb{S}\mathrm{Bim}] : \underline{H}_s \mapsto [B_s]$.

b) One has a bijection:

$$\left\{ \begin{matrix} \text{indecomposable} \\ \text{Soergel bimodules} \end{matrix} \right\} / (n), \cong \quad \longleftrightarrow \quad W$$

$$B_w \longleftrightarrow w$$

Moreover B_w is the unique summand of $BS(\underline{w})$ which does not occur as a summand $BS(\underline{y})$ for any $\underline{y}$ with $\ell(\underline{y}) < \ell(\underline{w})$.

c) c is an isomorphism with inverse ch. For any $B, B' \in \mathbb{S}\mathrm{Bim}$ $\mathrm{Hom}(B, B')$ is a free left (or right) R-module of graded rank

$$\mathrm{gr\,rank}\ \mathrm{Hom}_{\mathbb{S}\mathrm{Bim}}(B, B') = (ch(B), ch(B'))$$

standard pairing.

<u>Soergel's conjecture:</u> $\boxed{ch(B_x) = \underline{H}_x}$ $: S(x)$ (implies KL positivity + KL conjecture).

Assume Soergel's conjecture. Check: $(\underline{H}_x, \underline{H}_y) \in \begin{cases} 1 + v^2\mathbb{Z}[v] & \text{if } x = y \\ v\mathbb{Z}[v] & \text{if } x \neq y. \end{cases}$

"assymptotic orthogonality"

Hence if Soergel's conjecture implies that

$$\mathrm{Hom}^0(B_x, B_y) = \begin{cases} \mathbb{R} & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

degree 0 maps.

<u>Exercise:</u> This is if and only if.