LECTURE 1.3   SOERGEL BIMODULES

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Fairly easy to define + play with – it amounts to the in-depth study of the <u>reflection repn</u> $\mathfrak{h}$

Fix (W,S). We'll define $\mathfrak{h}^*$ using the <u>symmetric (generalized) Cartan matrix</u> A of (W,S)

$$A=\begin{pmatrix} 2 & a_{st} & & \\ a_{ts} & 2 & & \\ & & 2 & \\ & & & 2 \end{pmatrix} \quad \text{with } a_{s,t}=a_{t,s}=-2\cos\frac{\pi}{m_{st}} \text{ for } m_{st}<\infty$$

(for $m_{s,t}=\infty$, $a_{s,t}=a_{t,s}=\pm 2$ is an option, as is anything suitably generic see exercises

Let $\mathfrak{h}^*$ be the $|S|$-dim v.s$/\mathbb{R}$ spanned by $\langle \alpha_s\rangle_{s\in S}$, called <u>simple roots</u>.

$W \circlearrowright \mathfrak{h}^*$ by $s(\alpha_t)=\alpha_t-a_{s,t}\alpha_s$ so $t=s$: $s(\alpha_s)=-\alpha_s$
$t\neq s$: $s(\alpha_t)=\alpha_t+2\cos\frac{\pi}{m_{st}}\alpha_s$

<u>Rmk</u>: How exactly $\mathfrak{h}^*$ is defined is a distracting technical point – in fact, many other versions of $\mathfrak{h}^*$ will do (non-symmetric matrices, have $\mathfrak{h}$, coroots, etc.) extension to $>|S|$ dim, etc.). This will suffice for our purposes.

From $\mathfrak{h}^*$, we will extract a bunch of interesting graded commutative rings.

<u>Def</u>: Let $R=\mathrm{Sym}(\mathfrak{h}^*)=\mathbb{R}[\alpha_s]$ (a poly ring), graded w/ $\deg\mathfrak{h}^*=2$. $W\circlearrowright R$.

For $I\subset S$, $W_I\subset W$, consider $R^I=R^{W_I}$ invariants. (Defined for all I, but we're really only interested when W_I is finite, i.e. I is <u>finitary</u>, for reasons you'll see.)

<u>Ex</u>: $W=S_n \circlearrowright R=\mathbb{R}[x_1,\dots,x_n]/\sum x_i=0$ ← as in Remark, can ignore this.

$W_I=S_3\times S_2\times S_1\times\dots$ $R^I=\mathbb{R}[x_1+x_2+x_3,\ x_1x_2+x_1x_3+x_2x_3,\ x_1x_2x_3,\ x_4+x_5,\ x_4x_5,\ x_6,\dots]/\sum x_i=0$ is still a poly ring

<u>Key Ex</u>: $R^s\subset R$ for fixed $s\in S$. We have $\alpha_s^2\in R^s$. Since $\alpha_t\longleftrightarrow\alpha_t+2\cos\frac{\pi}{m_{st}}\alpha_s$ we see $\alpha_t+\cos\frac{\pi}{m_{st}}\alpha_s\in R^s$. In fact, $R^s=\mathbb{R}[\alpha_s^2,\ \alpha_t+\cos\frac{\pi}{m_{st}}\alpha_s]$.

Usual $\mathbb{Z}/2\mathbb{Z}$ theory: $R=\underset{s(f)=f}{R^s}\oplus\underset{s(f)=-f}{R^{-s}}$, but here $R^{-s}=R^s\cdot\alpha_s$. Any $f=g+h\alpha_s$, $g,h\in R^s$

So in fact, R is free over R^s w/ basis $\{1,\alpha_s\}$, $R\cong R^s\oplus R^s(-2)$ as graded R^s-mod

Easy way to find coeffs g,h: Demazure operator $\partial_s(f)=\dfrac{f-s(f)}{\alpha_s}$ ← $\in R^{-s}=R^s\cdot\alpha_s$.

about ∂_s: • $\partial_s: R\to R^s$, deg -2 • R^s-linear, and kills R^s. $\partial^2=0$, exact. • On $\underset{\to\mathbb{R}}{\mathfrak{h}^*}$, get s column of A

- $f = g + h\alpha_s$
  $h = \frac{1}{2}\partial_s(f)$
  $g = \frac{1}{2}\partial_s(f\alpha_s)$

- Twisted Leibniz rule
  $\partial_s(fg) = \partial_s(f)g + s(f)\partial_s(g)$

- The pairing $(f,g) \mapsto \partial_s(fg)$ is perfect. I.e. the bases $\{1, \frac{\alpha_s}{2}\}$ $\{\frac{\alpha_s}{2}, 1\}$ are dual, $\partial_s(a_i b_j) = \delta_{ij}$.

$\Rightarrow R^s \subset R$ is a graded Frob. ext. More soon

What can we do with this ring ext?

Def: Let $B_s \equiv R \otimes_{R^s} R(1)$ be an R-bimodule. Pairs of polys w/ sliding. Visualize as a porous wall $\sum f \,|\, g$ that only s-symmetric polys can osmose thru.
← we'll make this notation more rigorous soon.

(1) means $1\otimes 1$ lies in degree -1

If $f = g + h\alpha_s$, $g, h \in R^s$ then $1 \otimes f = g \otimes 1 + h \otimes \alpha_s$. As left R-mod, B_s has basis $\begin{Bmatrix} 1\otimes 1 \\ 1\otimes\alpha_s \end{Bmatrix}$

Def: A Bott-Samelson bimod is $BS(\underline{w}) = B_s \otimes_R B_t \otimes_R \cdots \otimes_R B_u = R \otimes_{R^s} R \otimes_{R^t} R \otimes \cdots \otimes_{R^u} R(d)$
$\underline{w} = st\cdots u$
$f \,|\, g \} \quad \} h$

Exercises: a) Generalizing argument above, $BS(\underline{w})$ has basis as left R-mod given by $\{1 \otimes \alpha_s^{\varepsilon_s} \otimes \alpha_t^{\varepsilon_t} \otimes \cdots \otimes \alpha_u^{\varepsilon_u}\}$ for $\varepsilon_i \in \{0,1\}$

b) $B_s \otimes B_t$ can slide anything out of middle, since $\mathfrak{h}^* \subset (\mathfrak{h}^*)^s + (\mathfrak{h}^*)^t$ (except $a_{st} = \pm 2$)
as an R-bimodule, generated by $1\otimes 1\otimes 1$

So $\exists$ surjective map $R \otimes_{R^{s,t}} R(2) \to B_s \otimes B_t$.
$f \otimes g \longmapsto f \otimes 1 \otimes g$

Inverse map? When $m_{st} = 2$, yes! When $m_{st} \neq 2$, no! How to slide out $1 \otimes f \otimes 1$?

c) $B_s \otimes B_s = R \otimes_{R^s} R \otimes_{R^s} R(2) \underset{R\text{-bimod}}{\cong} R \otimes_{R^s} (R^s \oplus R^s(-2)) \otimes_{R^s} R(2) = R \otimes_{R^s} R(0) \oplus R \otimes_{R^s} R(2)$
$= B_s(-1) \oplus B_s(+1)$

categorifies $H_s H_s = vH_s + v^{-1}H_s$.

Exercise: Make this decomp explicit. $f \otimes g \otimes h \mapsto \partial_s(g) f \otimes h$ is one proj map.

d) $\underbrace{B_s \otimes B_t \otimes \cdots}_{m_{st}}$ has $R \otimes_{R^{s,t}} R(m)$ as a summand!

Def: A Soergel bimodule is a $(\oplus, \otimes, (n))$ of a summand of a Bott-Samelson bimod.
Forms a full monoidal subcat of R-bimodules.

The theory of Frob. extensions:

Def: A (commutative) ring ext $A \subset B$ is Frob if $\exists\ \partial : B \to A$, A-linear and if B is free finite rank/A w/ (homog) dual bases $\{a_i\}$ $\{b_j\}$ s.t. $\partial(a_i b_j) = \delta_{ij}$. (trace map)

Frobenius extensions $\rightsquigarrow$ "Frobenius reciprocity" holds

The bimodule ${}_B B_A$ gives functor ${}_B B_A \otimes_A \bullet : A\text{-mod} \to B\text{-mod}$ "Induction"

${}_A B_B$ —— " —— ${}_A B_B \otimes_B \bullet : B\text{-mod} \to A\text{-mod}$ "Restriction"

For any ring ext, Ind $\dashv$ Res, i.e. $\mathrm{Hom}_B(\mathrm{Ind}\,M, N) \cong \mathrm{Hom}_A(M, \mathrm{Res}\,N)$

determined by unit+counit of adjunction, counit $\mathrm{Hom}_B(\mathrm{Ind\,Res}\,M, M) \cong \mathrm{Hom}(\mathrm{Res}\,M, \mathrm{Res}\,M) \ni 1_M$

get natl trans $\mathrm{Ind\,Res} \to \mathbb{1}_{B\text{-Mod}}$

${}_B B \otimes_A B_B \to {}_B B_B$ just multiplication,

Unit: Similarly, get map ${}_A A_A \to {}_A B(l)_A$, inclusion.

For Frobenius ext, also get Res $\dashv$ Ind. Determined by ${}_A B_A \xrightarrow{\partial} A$

and $B \xrightarrow{\Delta} B \otimes_A B$ indep of dual bases
$1 \mapsto \sum a_i \otimes b_i$

Said another way $B \otimes_A B$ is a Frob Alg object in B-bimod.

Our example: $A = R^s$ $B = R$ $R \otimes_{R^s} R(1) = B_s$.

Rmk: $\exists$ graded version of all this business, Ind, Res biadjoint up to shift by l
$\deg \partial = -2l$ then call ext degree l. Next lecture real defn

Thm: If I is finitary, R^I is a poly ring, and $R^I \subset R$ is a Frob ext. of degree $l(w_I)$

To get the trace $R \xrightarrow{\partial_I} R^I$: Claim: ∂_s, ∂_t satisfy braid relation.
$\Rightarrow \partial_w$ defined for any $w \in W$. (Symmetric subtlety)

$\partial_I = \partial_{w_I}$. Clearly $\mathrm{Im}(\partial_{w_I}) \subset \bigcap_{s \in I} \mathrm{Im}(\partial_s) = R^I$
b/c can choose red exp with s on left.

Rmk: For $I \subseteq J$ finitary, $R^J \subset R^I$ is Frob of degree $l(w_J) - l(w_I)$, $\partial = \partial_{w_J w_I^{-1}}$

Def: Singular Soergel Bimodules are $\oplus, \otimes, (n), \subseteq$ of Ind_I^J Res_I^J for finitary I

More precisely, 2-cat: Ob: I finitary
1-mor: $R^I \otimes \ldots \otimes R^K \otimes_{R^J} R^I$ 2-mor: Bimodule maps.
and summands, etc

If time

Ex: $S = \{s,t\}$ $m_{s,t} = m$.

$$z = \alpha_s^2 - a_{s,t}\alpha_s\alpha_t + \alpha_t^2 = \alpha_s^2 + \alpha_t(\alpha_t - a_{st}\alpha_s) \in R^{s,t}$$

a) If $m = \infty$, $a_{st} \neq \pm 2$ (i.e. z is not a square) then $R^{s,t} = \mathbb{R}[z]$

b) If $m = \infty$, $a_{st} = \mp 2$ then $R^{s,t} = \mathbb{R}[\alpha_s \pm \alpha_t]$

either way, wrong transcendence degree, $R^{s,t} \subset R$ is NOT a finite extension.

c) $m < \infty$. Can define <u>pos roots</u> Φ^+ : ~~Note that~~ When m odd $\underbrace{\ldots st}_{m-1}(\alpha_s) = \alpha_t$, the collection

$$\{\underbrace{\ldots st}_{k}(\alpha_s)\}_{0 \le k \le m-1} = \{\underbrace{\ldots ts}_{k}(\alpha_t)\}_{0 \le k \le m-1} = \Phi^+, \text{ full } W \text{ orbit is } \pm\Phi^+.$$

$\mathbb{L} = \prod \Phi^+$ deg m, $s(\mathbb{L}) = t(\mathbb{L}) = -\mathbb{L}$ b/c permutes roots except one, $R^{-s,-t} = R^{s,t} \cdot \mathbb{L}$

$\partial_{w_0}(\mathbb{L}) = 2m$.

Similarly, let $Z = \prod (\Phi^+)^\perp$, $Z \in R^{s,t}$ and $R^{s,t} = \mathbb{R}[z, Z]$

$\mathbb{L} = \mu(\Delta(1)) = \sum a$

More exercises on roots, Demazure operators, etc on exercise sheet.

Finding dual bases explicitly is very annoying. Want root-theoretic description.