LECTURE 5.1 HECKE ALGEBRAS W/ UNEQUAL PARAMETERS + FOLDING

There's a generalization of $\mathbb{H}$, by first generalizing the length function ℓ on W, to get a weighted Coxeter gp (W, L). Instead of requiring $L(s)=1 \ \forall s \in S$, instead $L(s) \in \mathbb{Z}$. However, we want $L(w)$ well-defined, so L must satisfy the braid relation, i.e. $L(\underbrace{st\ldots}_{M}) = L(\underbrace{ts\ldots}_{M}) \implies L(s)=L(t)$ when m_{st} is odd.

Now define $\mathbb{H}_L$ w/ generators H_s as before, but replace the old quadratic relation $(H_s - v)(H_s + v^{-1}) = 0$ with a new one $(H_s - v^{L(s)})(H_s + v^{-L(s)}) = 0$.

The basic theory of the Hecke algebra extends to $\mathbb{H}_L$: • basis given by W

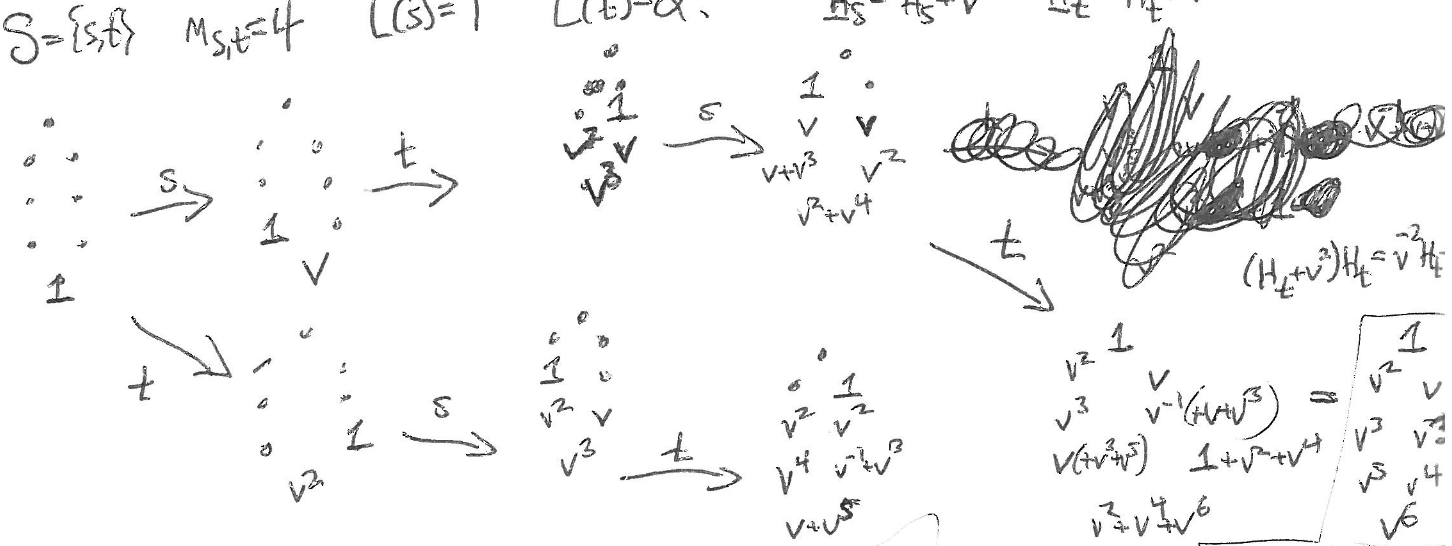
- bar involution $\overline{H_s} = H_s^{-1}$, $\overline{v} = v^{-1}$
- $\exists!$ KL basis $\underline{H}_w$ s.t. $\overline{\underline{H}_w} = \underline{H}_w$ and $\underline{H}_w = H_w \bmod v\mathbb{Z}[v]\langle H_x \rangle$

  The KL polys live in even degree (after shifting by $v^{L(w)-L(x)}$)

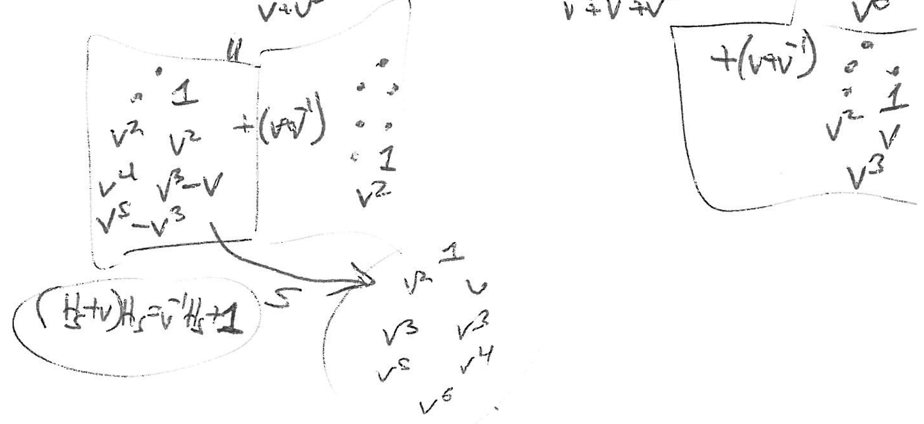
(Though some facts get weird when $L(s)<0$, $L(t)>0$.)

However, behavior of KL basis is quite different!

Ex: $S=\{s,t\}$ $m_{s,t}=4$ $L(s)=1$ $L(t)=2$. $\underline{H}_s = H_s + v$ $\underline{H}_t = H_t + v^2$

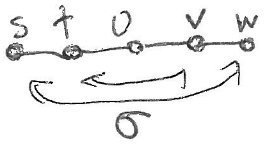


However much of this I wish to do will be decided



Where do weighted Cox gps come from? One nice set of examples: folding

Ex 1:



σ aut. of A_{2k+1}. Then $W^\sigma = \langle sw, tv, u \rangle$ is type B_{k+1}

$(W, \ell) =$

with induced length, i.e.

$L(sw) = 2 \quad L(tv) = 2 \quad L(u) = 1$

Ex 2:

$L = 1 \quad 1 \quad 1 \quad 1 \quad 2$

D_{k+1} gives B_k w/ lengths $2, 1, 1, 1, 1$

However, you don't get $\mathbb{H}_{(W^\sigma, S/\sigma, L)} \subset \mathbb{H}_{(W,S,\ell)}$!

① Clearly $\sigma \circlearrowleft \mathbb{H}_{(W,S,\ell)}$ too, but fixed subalgebra has basis of form

$\{H_x\}_{x \in W^\sigma} \cup \{H_y + H_{\sigma(y)}\}_{y \notin W^\sigma}$

while $\mathbb{H}_{(W^\sigma, S/\sigma, L)}$ should have size $\{H_x\}_{x \in W^\sigma}$.

Ex 1: ② Look at $H_s H_w \in \mathbb{H}_{(W,S,\ell)}$.

$$(H_s H_w)^2 = H_s^2 H_w^2 = \left((v^{-1} - v) H_s + 1\right)\left((v^{-1} - v) H_w + 1\right) = (v^{-2} + v^2 - 2) H_s H_w + \ldots$$

slightly better:

$$(\underline{H}_s \underline{H}_w)^2 = (\underline{H}_s^2)(\underline{H}_w^2) = (v + v^{-1}) \underline{H}_s (v + v^{-1}) \underline{H}_w = (v^2 + 2 + v^{-2}) \underline{H}_s \underline{H}_w$$

too bad!!
two bad!!!

Lusztig observed that you can categorify $\mathbb{H}_{(W^\sigma, S/\sigma, L)}$ "inside" the categorification of $\mathbb{H}_{(W,S,\ell)}$!!

Here's how to do it w/ Soergel Bimodules

Ex 1: $\sigma \circlearrowleft \mathbb{S}\mathrm{Bim}$

$B_s \leftrightarrow B_w$

change color in diagrams

On cat'l level, wrong to just look at "fixed parts" M st. $\sigma M \cong M$

Should look at equivariant objects

$$\mathbb{S}\mathrm{Bim}_\sigma := \mathrm{Ob} = \Big(\underset{\mathbb{S}\mathrm{Bim}}{\overset{\in}{M}},\ \varphi : M \xrightarrow{\sim} \sigma M\Big) \text{ st. } \begin{tikzcd} M \arrow[r, "\varphi"] \arrow[rr, bend right, "1"'] & \sigma M \arrow[r, "\sigma\varphi"] \arrow[d, phantom, "\#"] & M \\ & {} & \end{tikzcd}$$

(so clearly $M \cong \sigma M$)

$$\mathrm{Mor}\big((M,\varphi), (N,\psi)\big) = \left\{ \begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "f"'] \arrow[dr, phantom, "\#"] & \sigma M \arrow[d, "\sigma f"] \\ N \arrow[r, "\psi"'] & \sigma N \end{tikzcd} \right\}$$

Inherits monoidal structure.

Example objects: $(B_u, 1) \quad (B_t B_v, 1) \quad (B_t \oplus B_v, \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix})$ st. $ab = 1$

$(B_u, -1) \quad (B_t B_v, -1)$ all isomorphic

not isomorphic

$$\begin{tikzcd} B_u \arrow[r, "1"] \arrow[d, "f"'] & B_u \arrow[d, "\sigma f"] \\ B_u \arrow[r, "-1"'] & B_u \end{tikzcd}$$

$\sigma f = f$ on scalars

There's an action of $\mathbb{Z}/2\mathbb{Z}$ on objects: $(M,\varphi) \leftrightarrow (M,-\varphi)$

$(B_w,1) \updownarrow (B_w,-1)$ $\qquad (B_s\oplus B_w, (\ ))\circlearrowleft$

The Groth gp of this cat will be too big – 2 copies of W^σ, one of the rest

But you can "weight" or "twist" the grothendieck gp by a character of $\mathbb{Z}/2\mathbb{Z}$!

Def: $[SBim_\sigma]_{triv} = [SBim_\sigma] \big/ [(M,\varphi)] = [(M,-\varphi)]$ basis same as $(\mathbb{H}_{(W,S,\ell)})^\sigma$

$[SBim_\sigma]_{sgn} = [SBim_\sigma] \big/ [(M,\varphi)] = -[(M,-\varphi)]$ basis just W^σ.

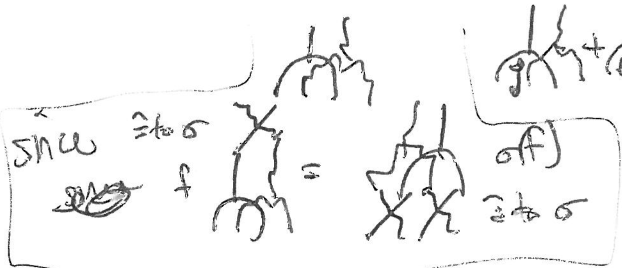
(Why bases? Suppose $End(M) = \Bbbk$ for an indecomp. Then $\underset{\sigma M \cong M}{(M,\pm 1)}$ and $\underset{\sigma M \ncong M}{(M\oplus\sigma M, (\begin{smallmatrix}0&a\\b&0\end{smallmatrix}))}$ are indecomps in $SBim_\sigma$.)

Thm: Assume the Soergel Conjecture holds. Then $[SBim_\sigma]_{triv}$ categorifies $(\mathbb{H}_{(W,S,\ell)})^\sigma$ and $[SBim_\sigma]_{sgn}$ categorifies $\mathbb{H}_{(W^\sigma, S/\sigma, L)}$

Ex: (twisted) $(\underline{H}_s\underline{H}_w)^2 = (v^2+2+v^{-2})\underline{H}_s\underline{H}_w$

$$B_sB_wB_sB_w \cong B_sB_w(2)\oplus B_sB_w\oplus B_sB_w\oplus B_sB_w(-2)$$

but $(B_sB_wB_sB_w,\pm1) \cong (B_sB_w(2),\pm1)\oplus(B_sB_w,+1)\oplus(B_sB_w,-1)\oplus(B_sB_w(-2),\pm1)$



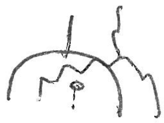
since f = of

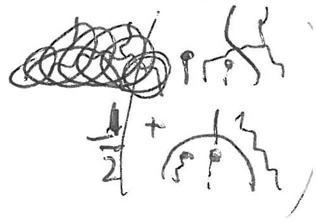
they cancel in Groth gp!!

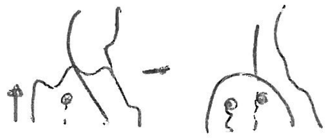
To get KL polys for [ ]$_{sgn}$ can count light leaves, like in usual way, but weigh by the trace of σ!

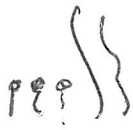
Ex: $\mathrm{Hom}^{LL}(B_tB_vB_uB_tB_v, B_tB_v)$

we know $B_tB_v(v+v^{-1}) \overset{\oplus}{\subset} B_tB_vB_uB_tB_v$

projection  $+1$

inclusion 

remaining maps are  trace -1

 trace $+1$

so get KL poly v^3-v.

Any ideas on how to prove in this talk?