The situation in characteristic p

Let W be a finite or affine Weyl group with Cartan matrix $(\langle \alpha_s, \alpha_t^\vee \rangle)_{s,t \in S}$.

We can define $\mathcal{D}$ over $\mathbb{Z}$ (only input was Cartan matrix) and hence over any complete local ring $\Bbbk$. One always has an isomorphism

$$[\mathrm{Kar}(\mathcal{D})] \xrightarrow[\sim]{ch} H$$

Remark: It is important that $\Bbbk$ be complete local to apply Krull-Schmidt type arguments. Hence one can't work over $\mathbb{Z}$.

Consider the basis $\{ch(B_x)\} \subset H$.

1) idempotent lifting $\Rightarrow$ (B_x indecomposable $\Leftrightarrow$ $B_x \otimes_{\Bbbk} \Bbbk/m$ is indecomposable) ← m: maximal ideal in $\Bbbk$.

2) B_x absolutely indecomposable $\Rightarrow$ if k is a field then the k-canonical basis only depends on char k.

Hence $\{ch(B_x)\}$ only depends on the characteristic p of the residue field.

Set ${}^p\underline{H}_x := ch(B_x)$. We call $\{{}^p\underline{H}_x\}$ the $\underline{\text{p-canonical basis}}$.

0) The main theorem of this course (Soergel's conjecture) is ${}^0\underline{H}_x = \underline{H}_x$.

1) ${}^p\underline{H}_x {}^p\underline{H}_y = \sum {}^p\mu_{xy}^z \, {}^p\underline{H}_z$ with ${}^p\mu_{xy}^z \in \mathbb{N}[v^{\pm 1}]$. $\overline{{}^p\mu_{xy}^z} = {}^p\mu_{xy}^z$.

(obvious by classification of indecomposables).

0) ${}^p\underline{H}_x = \sum {}^pm_{yx} H_y$ then ${}^pm_{yx} \in \mathbb{N}[v^{\pm 1}]$ self-dual.

(Consider $\Bbbk = \mathbb{Z}_p$ p-adic integers. Then $B_x \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a self-dual object in $\mathcal{D}_{\mathbb{Q}_p}$ hence $B_x \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong B_{x,\mathbb{Q}_p} \oplus \bigoplus_{y<x} {}^pV_{yx} \otimes B_{y,\mathbb{Q}_p}$. ← ${}^pV_{yx}$: self-dual $\mathbb{Q}_p$ space.)

20th century maths: $[\Delta(w\cdot 0), L(x\cdot 0)]$.

21st century maths: calculate ${}^p m_{yx}$!

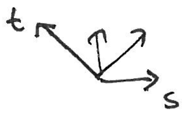
Apart from silly observations (i.e. ${}^p m_{y,x} \neq 0 \Rightarrow \mathcal{L}(y) \supset \mathcal{L}(x)$, $\mathcal{R}(y) \supset \mathcal{R}(x)$ etc.)

<u>nothing</u> is known about ${}^p m_{y,x}$.

Say $\overset{p}{\sim}$ equivalence relation generated by $x \underset{p}{\sim} y \Leftrightarrow {}^p m_{yx} \neq 0$.

Equivalence classes "p-linkage classes".

<u>Examples</u>: B_2:



	$\alpha_s^\vee$	$\alpha_t^\vee$
α_s	2	-1
α_t	-2	2

Intersection form at sts:

$= \partial_s(\alpha_t) = -2$.

Hence $B_s B_t B_s$ is indecomposable if 2 is not invertible in $\Bbbk$.

<u>Exercise</u>: ${}^p\underline{H}_x = \underline{H}_x$ unless $p=2$ and $x = sts$ when

${}^2\underline{H}_{sts} = \underline{H}_{sts} + \underline{H}_s$. Hence 2-linkage classes
$\{x\}$ $x \neq s, sts$.
$\{s, sts\}$.

G_2: $p=2$ Cartan matrix becomes symmetric!

.ststst

p-can basis is stable under $s \leftrightarrow t$.

${}^2\underline{H}_{stst} = \underline{H}_{stst} + \underline{H}_{st}$.

${}^2\underline{H}_{ststs} = \underline{H}_{ststs} + \underline{H}_s$

<u>Verbal</u>: reminds one of tilting modules over sl_2.

A_n: ${}^p\underline{H}_x = \underline{H}_x$ for $\forall x, p$ with $n \leq 6$.

In A_7, 38/40320 elts satisfy ${}^2\underline{H}_x \neq \underline{H}_x$. (See exercise sheets for an example).

There exist p-linkage classes with 2,3 elts.

<u>Polo</u>: $\{{}^p\underline{H}_x\} \neq \{\underline{H}_x\}$ in A_{4p-1}!

I hope this course has convinced you that this basis is in principle computable.

Survey: $B_{x, \mathbb{F}_p} \longleftrightarrow$ indecomposable Braden-MacPherson sheaf (Fiebig) $\longleftrightarrow$ ~~indecompos~~ parity sheaf $\mathcal{E}(x, \mathbb{F}_p)$.

$\rightsquigarrow$ Janzen's conjecture

$\rightsquigarrow$ Lusztig's modular conjecture

$\rightsquigarrow$ characters of tilting modules for algebraic groups.

(three of the most important conjectures in rep theory)

In particular ${}^p\underline{H}_x = \underline{H}_x$ for x in some finite explicit subset (of order $\frac{|W|}{\text{index of connection}}$) in affine Weyl group $\Rightarrow$ (Fiebig) Lusztig's modular conjecture.

Remark: 1) I do have software to calculate $\{{}^p\underline{H}_x\}$ but it needs optimizing. Should be possible to check Lusztig's conjecture in rank ≤ 4. (Verbal: serious evidence is missing...)

2) Lusztig's conjecture assumes $p \geq h$. For $p < h$ one still expects connections to modular reps.

Conjecture (Riche-W) "Modular Koszul duality". $G/B \longleftrightarrow G^\vee/B^\vee$.

$$D = \left([IC(\hat{w}, \mathbb{O}) \underset{\mathbb{O}}{\otimes} \mathbb{F}_p : IC(\hat{v}; \mathbb{F}_p)]\right)_{\hat{v}, \hat{w} \in \hat{W}} \quad \leftarrow \text{decomposition matrix.}$$

$W \overset{\sim}{\longleftrightarrow} W^\vee$

$x \longleftrightarrow x^\vee$.

~~$D^{-1} = (d^{-1}_{v,w})_{v,w \in W}$~~

~~${}^pm_{w_0w, w_0x} = $~~

$$\left({}^pm_{w_0y, w_0x}(1)\right)_{x,y \in W} = D^{-1}.$$

$\rightsquigarrow$ would show that χ of stalks are in principle computable.

passage to dual group is essential!