Aarhus Master Class

Tom Mrowka

MIT

August 1-5, 2011

X a compact riemannian (usually) 4-manifold X possibly with boundary.

$\mathrm{G} \subset \mathrm{Aut}(\mathrm{V})$ a compact Lie group.

$\mathfrak{g} \subset \mathrm{End}(\mathrm{V})$, Lie algebra of G.

$\pi : P \to X$ a principal G bundle. $E = P \times_{\mathrm{G}} V$ denote the associated V bundle. The vertical tangent space $VTP = \ker d\pi$. There is a canonical isomorphism of $\iota_p : VT_pP \to \mathfrak{g}$.

$\mathrm{ad}P = P \times_{\mathrm{ad}} \mathfrak{g} \subset \mathrm{End}(E)$ and $\mathrm{Ad}P = P \times_{\mathrm{Ad}} G \subset \mathrm{Aut}(E)$. Forms with values in $\mathrm{ad}P$ are a super lie algebra.

$$[\cdot \wedge \cdot] : \Lambda^i(T^*X) \otimes \mathrm{ad}P \otimes \Lambda^j(T^*X) \otimes \mathrm{ad}P \to \Lambda^{i+j}(T^*X) \otimes \mathrm{ad}P$$

so that

$$[a \wedge b] = (-1)^{|a|+|b|+1}[b \wedge a]$$

For us G will be SO_3 or SU_2.

A connection A in P can be viewed many ways.

- A system of parallel transport in P

The curvature of a connection measures the failure of H_A to be involutive. As a two form on P

$$F_A = dA + \frac{1}{2}[A \wedge A].$$

Can view $F_A \in C^\infty(X, \Lambda^2 \otimes \mathrm{ad}P)$.

A connection A in P can be viewed many ways.

- A system of parallel transport in P
- A G invariant subbundle $H_A \subset TP$ that is tranversal to the vertical tangent space $VTP = \ker(d\pi)$

The curvature of a connection measures the failure of H_A to be involutive. As a two form on P

$$F_A = dA + \frac{1}{2}[A \wedge A].$$

Can view $F_A \in C^\infty(X, \Lambda^2 \otimes \mathrm{ad}P)$.

A connection A in P can be viewed many ways.

- A system of parallel transport in P
- A G invariant subbundle $H_A \subset TP$ that is tranversal to the vertical tangent space $VTP = \ker(d\pi)$
- A $\mathfrak{g}$ valued one form A on P which is ad-equivariant and is ι_p^{-1} on the vertical tangent space.

The curvature of a connection measures the failure of H_A to be involutive. As a two form on P

$$F_A = dA + \frac{1}{2}[A \wedge A].$$

Can view $F_A \in C^\infty(X, \Lambda^2 \otimes \mathrm{ad}P)$.

A connection A in P can be viewed many ways.

- A system of parallel transport in P
- A G invariant subbundle $H_A \subset TP$ that is tranversal to the vertical tangent space $VTP = \ker(d\pi)$
- A $\mathfrak{g}$ valued one form A on P which is ad-equivariant and is ι_p^{-1} on the vertical tangent space.
- A covariant derivative

$$d_A : C^\infty(X; \mathrm{ad}P) \to C^\infty(X, T^*X \otimes \mathrm{ad}P).$$

The curvature of a connection measures the failure of H_A to be involutive. As a two form on P

$$F_A = dA + \frac{1}{2}[A \wedge A].$$

Can view $F_A \in C^\infty(X, \Lambda^2 \otimes \mathrm{ad}P)$.

The Yang-Mills functional. The Yang-Mills energy of A is

$$E(A) = -\int_X tr(F_A \wedge *F_A) = \int_X |F_A|^2 * 1.$$

(Recall the $tr(A^2)$ is a negative definite form on skew symmetric matrices.) The Euler-Lagrange equations for E is

$$d_A^* F_A = 0$$

the Yang-Mills Equations. Recall that the curvature always satisfies the Bianchi Identity.

$$d_A F_A = 0$$

Informally A is a non-linear harmonic form!

The Chern-Weil formula. Consider:

$$\int_X tr(F_A \wedge F_A).$$

If X is closed then this integral is depends only on P and not on A.

Proof: $A, A' = A + a$. Let $A_t = A + ta$. Then

$$F_{A_t} = F_A + t d_A a + \frac{t^2}{2}[a \wedge a].$$

$$\begin{aligned}\frac{d}{dt}\int_X tr(F_{A_t} \wedge F_{A_t}) &= 2\int_X tr(d_{A_t} a \wedge F_{A_t}) \\ &= 2\int_X d\, tr(a \wedge F_{A_t}) \\ &= 0.\end{aligned}$$

If $G = SO_3$ then we define the instanton number:

$$k = -\frac{1}{4}\langle p_1(P), [X]\rangle = -\frac{1}{8\pi^2}\int_X tr(F_A \wedge F_A) \in \frac{1}{4}\mathbb{Z}$$

Note if P lifts to an SU_2 bundle Q then $\mathrm{ad}Q \equiv \mathrm{ad}P$ and so

$$\begin{aligned} p_1(P) &= p_1(\mathrm{ad}P) = p_1(\mathrm{ad}Q) \\ &= -c_2(\mathrm{ad}Q \otimes \mathbb{C}) = -c_2(End(E)) = -4c_2(E) \end{aligned}$$

where E denotes the $\mathbb{C}^2$-bundle associated to Q. and similarly if P lifts to a U_2 bundle Q

$$p_1(P) = -4c_2(Q) + c_1^2(Q).$$

P is determined upto isomorphism by $p_1(P)$ and $w = w_2(P) \in H^2(X, \mathbb{Z}_2)$.

Spaces of connections and gauge transformations.

A a connection. $\mathcal{A}$ is the space of C^∞-connections. $\mathcal{A}$ is an affine space for the space

$$\Omega^1(X; \mathrm{ad}P) = C^\infty(X; T^*X \otimes \mathrm{ad}P).$$

$g : P \to P$ is an automorphism (gauge transformation) of P.

g is a section of the bundle $\mathrm{Ad}P$.

$\mathcal{G}$ the space of C^∞-sections of $\mathrm{Ad}P$ or group of gauge transformations.

$\nabla_A f$ or $d_A f$ The induced covariant derivative in any associated bundle, $F = P \times_\rho V$.

The action of $g \in \mathcal{G}$.

$$g \cdot A = A + g d_A g^{-1}.$$

$d_A g^{-1}$ means that covariant derivative of g^{-1} thought of as a section of $\mathrm{End}(E) = E^* \otimes E$ induced by A. Locally

We are interested in the studying connections up to isomorphism i.e.

$$\mathcal{B} = \mathcal{A}/\mathcal{G}.$$

We need to understand how bad the action is.

Lemma

$g \cdot A = A$ if and only if g is a parallel section of $\mathrm{End}(E)$.

Proof.

$$g \cdot A = A \Longrightarrow g d_A g^{-1} = -(d_A g) g^{-1} = 0.$$

and hence $d_A g = 0$. □

N.B When $G = SO_3$ the we can form the bundle $\mathrm{Ad}^1 P$ of "determinant 1" gauge transformations

$$\mathrm{Ad}^1 P = P \times_{SO_3} SU_2.$$

where $SO_3 = SU_2/Z(SU_2)$ acts on SU_2 by conjugation. The sections of this bundle $\mathcal{G}^1$ maps to $\mathcal{G}$.

Exercise

The map $\mathcal{G}^1 \to \mathcal{G}$ has kernel $\mathbb{Z}_2$ if X is connected. The cokernel isomorphic to $H^1(X; \mathbb{Z}_2)$.

We will often want to work with $\mathcal{A}/\mathcal{G}^1$ rather than $\mathcal{A}/\mathcal{G}$. To deal with later we'll choose submanifold Poincaré dual to $w_2(P)$.

Given a loop γ based at $x_o \in X$ the holonomy of A along γ the automorphism of $P|_{x_o}$ given by parallel transport along γ.

$$\mathrm{hol}_A(\gamma) \in \mathrm{Ad}P|_{x_o}.$$

The holonomy group of A at x_o

$$\mathrm{hol}_A(x_o) = \{\mathrm{hol}_A(\gamma) | \gamma \in \Omega_{x_o}(X)\}$$

$Stab_A$ is a subgroup of G and is the commutant of the holonomy group of A. Note that the center of G, $Z(G) \subset Stab_A$ for all A.

Definition

A is called reducible if $Stab_A \neq Z(G)$.

Exercise

Show that the possible subgroups of SO_3 that appear are $1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, SO_2, O_2$ and SO_3 while if we restrict to the determinant 1 gauge group, the possible subgroups of SU_2 that appear as $Stab(A)$ for some connection are $\mathbb{Z}_2, U_1$ and SU_2.

The quotient space $\mathcal{B} = \mathcal{A}/\mathcal{G}^1$ will have singularities if A contains reducible connections. Compare

$$\mathfrak{so}_3^n/SO_3$$

where SO_3 acts by the adjoint action. For each point in the quotient a neighborhood is modeled on a neighborhood of zero in one of the following three spaces.

We would like to show that $\mathcal{B}$ has similar local models.

Exercise

Show that the possible subgroups of SO_3 that appear are $1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, SO_2, O_2$ and SO_3 while if we restrict to the determinant 1 gauge group, the possible subgroups of SU_2 that appear as $Stab(A)$ for some connection are $\mathbb{Z}_2, U_1$ and SU_2.

The quotient space $\mathcal{B} = \mathcal{A}/\mathcal{G}^1$ will have singularities if A contains reducible connections. Compare

$$\mathfrak{so}_3^n / SO_3$$

where SO_3 acts by the adjoint action. For each point in the quotient a neighborhood is modeled on a neighborhood of zero in one of the following three spaces.

- $\mathfrak{so}_3^{n-1}$.

We would like to show that $\mathcal{B}$ has similar local models.

Exercise

Show that the possible subgroups of SO_3 that appear are $1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, SO_2, O_2$ and SO_3 while if we restrict to the determinant 1 gauge group, the possible subgroups of SU_2 that appear as $Stab(A)$ for some connection are $\mathbb{Z}_2, U_1$ and SU_2.

The quotient space $\mathcal{B} = \mathcal{A}/\mathcal{G}^1$ will have singularities if A contains reducible connections. Compare

$$\mathfrak{so}_3^n/SO_3$$

where SO_3 acts by the adjoint action. For each point in the quotient a neighborhood is modeled on a neighborhood of zero in one of the following three spaces.

- $\mathfrak{so}_3^{n-1}$.
- $(\mathbb{R}^2)^{n-1}/SO_2 \times \mathbb{R}^n$.

We would like to show that $\mathcal{B}$ has similar local models.

Exercise

Show that the possible subgroups of SO_3 that appear are $1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, SO_2, O_2$ and SO_3 while if we restrict to the determinant 1 gauge group, the possible subgroups of SU_2 that appear as $Stab(A)$ for some connection are $\mathbb{Z}_2, U_1$ and SU_2.

The quotient space $\mathcal{B} = \mathcal{A}/\mathcal{G}^1$ will have singularities if A contains reducible connections. Compare

$$\mathfrak{so}_3^n/SO_3$$

where SO_3 acts by the adjoint action. For each point in the quotient a neighborhood is modeled on a neighborhood of zero in one of the following three spaces.

- $\mathfrak{so}_3^{n-1}$.
- $(\mathbb{R}^2)^{n-1}/SO_2 \times \mathbb{R}^n$.
- The origin which has no better model than $\mathfrak{so}_3^n/SO_3$.

We would like to show that $\mathcal{B}$ has similar local models.

We will need to work with Sobolev spaces and Fredholm operators on them. There are the very basic facts. Let $L^p_k(T^n)$ be the completion of space of C^∞ functions with respect to the norm.

$$\|f\|^p_{L^p_k}(T^n) = \sum_{j=0}^{k} \int_{T^n} |\nabla^{(j)} f|^p dx^1 \wedge \dots dx^n.$$

We also will use Sobolev norms with fractional derivatives. These can be defined by interpolation. The L^2-version can be easily defined by Fourier transform. If

$$f(\mathbf{x}) = \sum_{\mathbf{n}} \hat{f}(\mathbf{n}) e^{i\mathbf{b}\cdot\mathbf{x}}$$

Then we

$$\|f\|^2_{L^2_s} = \sum_{\mathbf{n}} (1 + |\mathbf{n}|^2)^{s/2} |\hat{f}(\mathbf{n})|^2$$

Then we have the following properties.

$$L^p_k(T^n) \hookrightarrow L^q_l(T^n)$$

If $k - n/p \geq l - n/q$ and $k \geq l$. If both inequalities are strict the embedding is compact.

$$L^p_k(T^n) \hookrightarrow C^{l,\alpha}(T^n)$$

if $k - n/p \geq l + \alpha$ where $k \geq l \geq 0$ and $\alpha > 0$. If both inequalities are strict the embedding is compact.

Furthermore when $k - n/p$ and $l - n/q$ are both negative

$$L_k^p(T^n) \times L_l^q(T^n) \hookrightarrow L_m^r(T^n)$$

is continuous if

$$k - n/p + l - n/q \geq r - n/m \text{ and } k, l \geq r.$$

Suppose that $k - n/p$ is positive and $k, l \geq r$ then $L_l^q(T^n)$ is module over $L_k^p(T^n)$.

In particular for $n = 4$ we have

$$L^2_1 \hookrightarrow L^4,$$
$$L^2_2 \hookrightarrow L^4_1 \hookrightarrow L^p \text{ for all } p < \infty$$
$$L^2_2 \not\hookrightarrow C^0.$$

The following multiplications are continuous.

$$L^2_1 \times L^2_1 \mapsto L^2,$$
$$L^2_2 \times L^2_1 \mapsto L^p_1 \text{ or } L^2_s \text{ where } p < 2 \text{ or } s < 1.$$

Note that $\nabla \ln(\ln(r)) = \frac{1}{r\ln(r)}) = \frac{1}{r\ln(r)}$

$$\int_0^{1/2} (\frac{1}{r\ln(r)})^4 r^3 dr = \int_0^{1/2} (\frac{1}{\ln(r)^4})\frac{dr}{r} = -\frac{1}{4}\ln^{-3}(1/2) < \infty.$$

We need to complete our spaces of connections into Sobolev spaces. Suppose $G = U_m$ or SO_m and let E be and $\mathbb{C}^m$ or $\mathbb{R}^m$ bundle so that P is the frame bundle of E. Using this connection there is a preferred gauge-equivariant Sobolev norm

$$\|f\|^p_{\mathrm{L}^p_{k,A}} = \int_X \sum_{i=0}^{k} |\nabla^i_A f|^p *_g 1 \, .$$

Any easy consequence of the Sobolev multiplication theorems is that the $\mathrm{L}^p_{k,A}$-norm and the $\mathrm{L}^p_{k,A'}$ are equivalent norms provided that difference is in

$$A - A' \in \mathrm{L}^q_{l,A}(X, T^*X \otimes \mathrm{ad}P))$$

and the multiplication $\mathrm{L}^p_k \times \mathrm{L}^q_l \to \mathrm{L}^p_{k-1}$ is continuous, i.e. $l \geq n/q - 1$ and $l > k - 1$ and indeed there is a constant $C > 0$ depending on $\|A - A'\|_{\mathrm{L}^q_{l,A}}$ so that

$$\|e\|_{\mathrm{L}^p_{k,A'}} \leq C\|e\|_{\mathrm{L}^p_{k,A}} \, . \quad (1)$$

Fractional order Sobolev norms. Fix a smooth connection A_0 and consider the corresponding Laplace operator

$$\nabla^*_{A_0}\nabla_{A_0} + 1 : C^\infty(X; \mathrm{ad}P) \to C^\infty(X; \mathrm{ad}P)\,.$$

This operator has compact inverse and one can construct compact operators

$$(\nabla^*_{A_0}\nabla_{A_0} + 1)^{-s} : \mathrm{L}^p(X; \mathrm{ad}P) \to \mathrm{L}^p(X; \mathrm{ad}P)$$

for any $s > 0$. For smooth A_0 these operators are represented by a kernel with pole of order $\mathrm{dist}(x, y)^{-n+2s}$ along the diagonal. The space of L^p_s-sections of a bundle is

$$\{(\nabla^*_{A_0}\nabla_{A_0} + 1)^{-s/2}e | e \in \mathrm{L}^p(X; \mathrm{ad}P)\}.$$

This space has a norm using the connection A_0,

$$\|(\nabla^*_{A_0}\nabla_{A_0} + 1)^{-s/2}e\|_{\mathrm{L}^p_{s,A_0}} = \|e\|_{\mathrm{L}^p} \qquad (2)$$

For any pair of connections A, A' with $A' - A_0, A - A_0 \in \mathrm{L}^2_{s'}$ where $s' \geq n/q - 1$ and $s' > s - 1$ the above definition of the spaces $\mathrm{L}^2_{s',A}$ and $\mathrm{L}^2_{s',A'}$ and the corresponding norms $\|\cdot\|_{\mathrm{L}^2_{s',A}}$ and $\|\cdot\|_{\mathrm{L}^2_{s',A'}}$ still make sense. Using that the fractional order spaces are interpolation spaces for the integral norms we see that the estimate (**??**) holds.

When dependence of the norms on the choice of connection is not important for the discussion we will drop the connection from the notation for the norm.

Note that if X is a manifold with boundary we can also define $\acute{L}^p_{s,A}$ to be completion of smooth sections with support in the interior of X in the norm $\|\cdot\|_{\mathrm{L}^p_{s,A}}$. Then we have the following useful duality result for $1 \leq p < \infty$

$$(\acute{L}^p_{s,A})^* = L^q_{-s,A}.$$

For consider the space $\mathcal{A}^p_k$ of L^p_k connections. The gauge group that acts naturally on this space of connections is

$$\mathcal{G}^p_{k+1} = \{g \in L^p_{k+1}(X, \mathrm{ad}P) | g \in \mathrm{Ad}P \, a.e.\}.$$

When $k + 1 - n/p > 0$ this consists of continuous sections.

Lemma

For $G = U_m$ or SO_m and $k + 1 - n/p > 0$ is a Banach Lie group

Proof. Consider the map

$$m : L^p_{k+1}(X, \mathrm{End}(E)) \mapsto L^p_{k+1}(X, \mathrm{End}(E))$$

given by

$$m(A) = AA^*$$

Then $\mathcal{G}^p_{k+1} = m^{-1}(\mathbf{1})$. For $k + 1 - n/p > 0$ m is a smooth map since L^p_{k+1} is a Banach algebra. M is also easily seen to have surjective differential and the kernel of

Lemma

For $k+1-n/p>0$, The map $\mathcal{A}^p_k \times \mathcal{G}^p_{k+1} \to \mathcal{A}^p_k$ is a smooth map of Banach manifolds.

Proof: In a trivialization the map is given by

$$(a,g) \mapsto gag^{-1} + gdg^{-1}$$

By the assumptions L^p_{k+1} is a Banach algebra and L^p_k is module over L^p_{k+1}. The above map is a composition of a continuous linear maps and continuous multiplications so it is smooth. □

The space of connections, being an affine space is alway a Banach manifold.

Lemma

For $k+1-n/p \geq 0$, $\mathcal{B}_k^p = \mathcal{A}_k^p/\mathcal{G}_{k+1}^p$ is a Hausdorff topological space.

Proof: We proof this in the case that of L_2^2 gauge transformations. The general case is similar. We must show that $\{(A, g \cdot A)|A \in \mathcal{A}_1^2, g \in \mathcal{G}_2^2\} \subset \mathcal{A}_1^2 \times \mathcal{A}_1^2$ is closed. Fix $A_o \in \mathcal{A}_k^p$

$$(A_i, g_i \cdot A_i) \xrightarrow{i \to \infty} (A, B)$$

$$A_i = A_o + a_i, g_i \cdot A_i = A_o + b_i.$$

Then

$$g_i d_{A_o} g_i^{-1} + g_i a_i g_i^{-1} = b_i \Rightarrow$$

$$d_{A_o} g_i = g_i a_i - b_i g_i. \quad (3)$$

The sequences a_i,b_i converge in L^2_1 to a and b respectively. We need to show that $A_0 + a$ is L^2_2 gauge equivalent to $A_0 + b$. We see that $d_{A_o} g_i$ bounded in L^4 so g_i is bounded in L^4_1. Now differentiate Equation (**??**)

$$\nabla^2_{A_0} g_i = \nabla_{A_0} g_i a_i - b_i \nabla_{A_0} g_i + g_i \nabla_{A_0} a_i - \nabla_{A_0} b_i g_i$$

The right hand side is bounded in L^2 for each is either $L^2 \times L^\infty$ or $L^4 \times L^4$. Thus we can pass to subsequence where g_i converges L^2_2-weakly to g and strongly in L^2. The strong convergence implies again after passing to a subsequence a.e. convergence and so $gg^* = 1$ a.e. thus g is gauge transformation. We can take the limit in Equation (**??**) in L^r for $r < 2$ showing that g takes $A + a$ to $A + b$ as required. $\square$.

Orbifold structure for $\mathcal{A}/\mathcal{G}$. For concreteness consider we'll restrict to the L^2 case and consider the space, $\mathcal{A}_{3/2}$, of connections on a four-manifold X. The gauge group, $\mathcal{G}_{5/2}$. Slices. Take orthogonal complement of tangent space of gauge orbit.

$$T_A\mathcal{G} = d_A\Omega^0(X, \mathrm{ad}P).$$

Using $d\, tr(\sigma \wedge *a) = tr(d_A\sigma \wedge *a) + tr(\sigma \wedge d_A * a)$ we have

$$-\int_X tr(d_A\sigma \wedge *a) = \int_X tr(\sigma \wedge d_A * a) = -\int_X tr(\sigma \wedge *d_A^* a).$$

Thus we want

$$0 = \langle d_A\sigma, a\rangle = \langle \sigma, d_A^* a\rangle.$$

for all $\sigma \in \Omega^0(X; \mathrm{ad}P)$ so $d_A^* a = 0$.
$A \in \mathcal{A}_{3/2}$ we have:

$$\mathcal{S}_{s,A}(\epsilon) = \{A + a | a \in \mathrm{L}^2_{3/2,A}(X; T^*X \otimes \mathrm{ad}P),$$
$$d_A^* a = 0 \text{ and } \|a\|_{L^2_{s,A}} < \epsilon\}$$

To state the result in general we need to discuss how to deal with reducible connections. If A is reducible Stab_A, the stabilizer of A, is a Lie subgroup of SO_m. Stab_A acts on S_A and freely on $\mathcal{G}_{5/2}$ and it is straightforward to see that the quotient

$$(\mathcal{S}_{A,3/2}(\epsilon) \times \mathcal{G}_{5/2})/\mathrm{Stab}_A$$

is a smooth Hilbert manifold. The tangent space at the equivalence class of $(A + a, 1)$ is identified with the quotient of

$$\{\alpha | \alpha \in \mathrm{L}^2_{3/2,A}, d_A^*\alpha = 0\} \times \mathrm{L}^2_{5/2,A}(X, \mathrm{ad}P)$$

by the finite-dimensional (and hence closed) subspace

$$\{(-[a, \xi^0], \xi^0) | d_A\xi^0 = 0\}.$$

Proposition

For all $A \in \mathcal{A}_{3/2}$ there is an $\epsilon > 0$ so that the map

$$\mathfrak{m} : (\mathcal{S}_{3/2,A}(\epsilon) \times \mathcal{G}_{5/2})/\mathrm{Stab}_A \to \mathcal{A}_{3/2}$$

$$\mathfrak{m}([A + a, g]) = A + gag^{-1} - (d_A g)g^{-1}$$

is a $\mathcal{G}_{5/2}$-equivariant diffeomorphism onto its image.

The proof will use the following facts.

- The operator

$$d_A : L^p_{A,s}(X, \mathrm{ad} P) \to L^p_{A,s-1}(X, T^*X \otimes \mathrm{End}(E))$$

has finite dimensional kernel and closed range.

The proof will use the following facts.

- The operator

$$d_A : L^p_{A,s}(X, \mathrm{ad}P) \to L^p_{A,s-1}(X, T^*X \otimes \mathrm{End}(E))$$

  has finite dimensional kernel and closed range.
- There is a closed "Hodge" L^2 orthogonal decomposition.

$$L^p_{A,s}(X, \mathrm{ad}P) = \ker(d_A) \oplus d_A^*(L^p_{A,s+1}(X, T^*X \otimes \mathrm{ad}P))$$

The proof will use the following facts.

- The operator

$$d_A : L^p_{A,s}(X, \mathrm{ad} P) \to L^p_{A,s-1}(X, T^*X \otimes \mathrm{End}(E))$$

  has finite dimensional kernel and closed range.
- There is a closed "Hodge" L^2 orthogonal decomposition.

$$L^p_{A,s}(X, \mathrm{ad} P) = \ker(d_A) \oplus d_A^*(L^p_{A,s+1}(X, T^*X \otimes \mathrm{ad} P))$$

- A complement for the range of $d_A : L^p_{A,s}(X, \mathrm{ad} P) \to L^p_{A,s-1}(X, T^*X \otimes \mathrm{End}(E))$ is

$$\ker d_A^* : L^p_{A,s-1}(X, T^*X \otimes \mathrm{ad} P) \to L^p_{A,s-2}(X, \mathrm{ad} P)$$

The proof will use the following facts.

- The operator

$$d_A : L^p_{A,s}(X, \mathrm{ad}P) \to L^p_{A,s-1}(X, T^*X \otimes \mathrm{End}(E))$$

  has finite dimensional kernel and closed range.
- There is a closed "Hodge" L^2 orthogonal decomposition.

$$L^p_{A,s}(X, \mathrm{ad}P) = \ker(d_A) \oplus d_A^*(L^p_{A,s+1}(X, T^*X \otimes \mathrm{ad}P))$$

- A complement for the range of $d_A : L^p_{A,s}(X, \mathrm{ad}P) \to L^p_{A,s-1}(X, T^*X \otimes \mathrm{End}(E))$ is

$$\ker d_A^* : L^p_{A,s-1}(X, T^*X \otimes \mathrm{ad}P) \to L^p_{A,s-2}(X, \mathrm{ad}P)$$

- There is a constant $\lambda_1(A) > 0$ so that for all $\xi \perp \ker(d_A)$ we have $\int_X |d_A\xi|^2 \geq \lambda_1 \int_X |\xi|^2$.

Proof.
Write $\xi = \xi^0 + \xi^1$ for the Hodge decomposition where $d_A\xi^0 = 0$ and ξ^1 is L^2-orthogonal to elements of the kernel of d_A.
$\mathfrak{m}$ is a local diffeomorphism if the differential of $\mathfrak{m}$ is an isomorphism at $(A, 1)$ Since $\mathfrak{m}$ is $\mathcal{G}_{5/2}$-equivariant it suffices to check this at the equivalence classes of $(A, 1)$.
The differential is given by the map

$$\mathcal{D}_{(A,1)}\mathfrak{m}([\alpha, \xi]) = d_A\xi + \alpha$$

This map is surjective by the Hodge decomposition. If (α, ξ) is the kernel then $\alpha = 0$ and $\xi \in \ker d_A$ so $[\alpha, \xi] = 0$.

$\mathfrak{m}$ is injective. It suffices to show that if $g \cdot (A + a) = A + b$ is in the slice then $g \in \mathrm{Stab}_A$. The condition that $g \cdot (A + a) = A + b$ is equivalent to

$$d_A g = ga - bg. \tag{4}$$

Taking d_A^* of this equation gives, using the slice condition $d_A^* a = d_A^* b = 0$,

$$d_A^* d_A g = - * d_A g \wedge *a - *b \wedge *d_A g. \tag{5}$$

Use the Hodge decomposition $g = g^0 + g^1$ again and take the inner product of this equation with g^1.

$$
\begin{aligned}
\|d_A g^1\|^2_{\mathrm{L}^2} &\le (\|a\|_{\mathrm{L}^4} + \|b\|_{\mathrm{L}^4})\|d_A g^1\|_{\mathrm{L}^2}\|g^1\|_{\mathrm{L}^4} \\
&\le \kappa(\|a\|_{\mathrm{L}^4} + \|b\|_{\mathrm{L}^4})\|d_A g^1\|_{\mathrm{L}^2}\|g^1\|_{\mathrm{L}^2_{1,A}} \\
&\le \kappa(1 + \lambda_1^{-1}(A))^{1/2}(\|a\|_{\mathrm{L}^4} + \|b\|_{\mathrm{L}^4})\|d_A g^1\|^2_{\mathrm{L}^2}
\end{aligned}
$$

We have used

$$
\begin{aligned}
\|g^1\|^2_{\mathrm{L}^2_{1,A}} &= \|g^1\|^2_{L^2} + \|d_A g^1\|^2_{L^2} \\
&\le (\lambda_1(A)^{-1} + 1)\|d_A g^1\|^2_{L^2}.
\end{aligned}
$$

So for $\epsilon\kappa(1 + \lambda_1^{-1}(A))^{1/2} < 1/2$ we have $u^1 = 0$ and hence $u = u^0$ is in $\mathrm{Stab}(A)$ as required. □

Remark

Not that the proof of injectivity of $\mathfrak{m}$ only used L^4-smallness. exercise *Repeat this argument for $\mathcal{D}_{(1,A)}\mathfrak{m}$ so show that for all $A \in \mathcal{S}_{1,A}$ this differential is invertible. This then can be used to giva a sharper result that will be important in the proof of Uhlenbeck compactness.*

Remark

Not that the proof of injectivity of $\mathfrak{m}$ *only used* L^4*-smallness.* exercise *Repeat this argument for* $\mathcal{D}_{(1,A)}\mathfrak{m}$ *so show that for all* $A \in \mathcal{S}_{1,A}$ *this differential is invertible. This then can be used to giva a sharper result that will be important in the proof of Uhlenbeck compactness.*

Proposition (Big slices)

For all $A \in \mathcal{A}_{3/2}$ *then the map*

$$\mathfrak{m} : (\mathcal{S}_{1,A}(\epsilon) \times \mathcal{G}_{5/2}/\mathrm{Stab}_A) \to \mathcal{A}_{3/2}$$
$$\mathfrak{m}([A + a, g]) = A + gag^{-1} - (d_A g)g^{-1}$$

is a $\mathcal{G}_{5/2}$*-equivariant diffeomorphism onto its image. The image contains an* L^2_1 *ball about* A*.*

When we have a manifold with boundary the notion of slice needs to be modified.

$$\int_X tr(d_A\sigma \wedge *a) = \int_{\partial X} tr(\sigma \wedge *a) + \int_X tr(\sigma \wedge *d_A^* a).$$

So the condition of being orthogonal to the tangent space of the gauge group implies formally

$$d_A^* a = 0 \text{ and } *\, a|_{\partial X} = 0.$$

In this and the next lecture we prove Uhlenbeck's Compactness Theorem for Yang-Mills connections.

Lemma

Let $B =$ be a geodesic ball in a riemannian manifold. Let Γ be the connection arising from a trivialization $P = \times G$ for principal G bundle. There are positive constants C, ϵ_o so that for all $A = \Gamma + a$ with $a \in L^2_1(B, T^*B \times \mathfrak{g})$ and $\int_B tr(F_A \wedge *F_A) \leq \epsilon_o$ there is a map $g \in L^2_2(B, M_n(\mathbb{C}))$ with $g \in G$ a.e. so that if we set

$$b = gag^{-1} + gdg^{-1}$$

we have

In this and the next lecture we prove Uhlenbeck's Compactness Theorem for Yang-Mills connections.

Lemma

Let $B =$ be a geodesic ball in a riemannian manifold. Let Γ be the connection arising from a trivialization $P = \times G$ for principal G bundle. There are positive constants C, ϵ_o so that for all $A = \Gamma + a$ with $a \in L^2_1(B, T^*B \times \mathfrak{g})$ and $\int_B tr(F_A \wedge *F_A) \leq \epsilon_o$ there is a map $g \in L^2_2(B, M_n(\mathbb{C}))$ with $g \in G$ a.e. so that if we set

$$b = gag^{-1} + gdg^{-1}$$

we have

- $d^*b = 0$ *and* $*b|_{\partial B} = 0$ *and*

In this and the next lecture we prove Uhlenbeck's Compactness Theorem for Yang-Mills connections.

Lemma

Let $B =$ be a geodesic ball in a riemannian manifold. Let Γ be the connection arising from a trivialization $P = \times G$ for principal G bundle. There are positive constants C, ϵ_o so that for all $A = \Gamma + a$ with $a \in L^2_1(B, T^*B \times \mathfrak{g})$ and $\int_B tr(F_A \wedge *F_A) \leq \epsilon_o$ there is a map $g \in L^2_2(B, M_n(\mathbb{C}))$ with $g \in G$ a.e. so that if we set

$$b = gag^{-1} + gdg^{-1}$$

we have

- $d^*b = 0$ *and* $*b|_{\partial B} = 0$ *and*
- $\int_B (|\nabla_\Gamma b|^2 + |b|^2) * 1 \leq C \int_B |F_A|^2 * 1.$

Proof. $B = \exp_x(B_0(\delta))$. The first move in the proof is notice that we can replace the L^2-norm of the curvature by an equivalent norm. We exploit this possibility by changing the metric to the pullback metric from the metric from T_xB. Let $\tilde{*}$ denote the Hodge star for this metric. The proof of this is a version of the continuity argument. Let

$$V_\epsilon = \{a \in L^2_{\frac{3}{2}}(B, T^*B \otimes \mathfrak{g}) | - \int_B tr(F_A \wedge \tilde{*} F_A) \leq \epsilon\}.$$

We use a density argument to get the L^2_1-case. Let $W_\epsilon \subset V_\epsilon$ be the subset where there is a $C > 0$ so that conclusions of the lemma holds.

We will show that for ϵ small enough and C large enough that:

- V_ϵ is connected.

We will show that for ϵ small enough and C large enough that:

- V_ϵ is connected.
- W_ϵ is closed.

We will show that for ϵ small enough and C large enough that:

- V_ϵ is connected.
- W_ϵ is closed.
- W_ϵ is open in V_ϵ.

We will show that for ϵ small enough and C large enough that:

- V_ϵ is connected.
- W_ϵ is closed.
- W_ϵ is open in V_ϵ.

We will show that for ϵ small enough and C large enough that:

- V_ϵ is connected.
- W_ϵ is closed.
- W_ϵ is open in V_ϵ.

Thus $W_\epsilon = V_\epsilon$.

V_ϵ is connected. Let for $0 \leq t \leq 1$ let $\tau_t : B_x(\delta) \to B_x(t\delta)$ denote the scaling by t along geodesics through x. Consider the path of connection

$$A_t = \tau_t^*(A|_{B_x(t\delta)}).$$

We claim that for A an $L^2_{3/2}$-connection this path is continuous in the $L^2_{3/2}$-topology. For $t \neq 0$ this is the same point that translation is point-wise continuous on L^p spaces. At $t = 0$ continuity follows directly from the definitions.

Since the L^2-norm of the curvature is conformally invariant and dilation is a conformal map we have that

$$-\int_B (tr(F_{A_t} \wedge \tilde{*} F_{A_t})) = -\int_{B_x(t\delta)} (tr(F_A \wedge \tilde{*} F_A)) \leq \epsilon$$

and $\lim_{t \mapsto 0} -\int_B (tr(F_{A_t} \wedge \tilde{*} F_{A_t})) = 0$.

W_ϵ is closed in V_ϵ. Let $A_i = \Gamma + a_i$ be a sequence in W_ϵ which converges in the $L^2_{3/2}$-topology to $A = \Gamma + a \in V_\epsilon$. There is also then a sequence of gauge transformations g_i so that $g_i \cdot A_i = \Gamma + b_i$ and the b_i are in Coulomb gauge and satisfy the estimates, so it follows that b_i converge weakly in the L^2_1 topology to b . Now consider the sequence of gauge transformations g_i. They satisfy the equations:

$$dg_i = g_i a_i - b_i g_i.$$

Thus $dg_i \in L^4$ and so $g_i \in L^4_1$ (g_i is bounded). Taking the gradient of both sides gives:

$$\nabla dg_i = \nabla g_i a_i + g_i \nabla a_i - \nabla b_i g_i - b_i \nabla g_i$$

hence g_i is bounded in L^2_2. Thus after passing to a subsequence we may assume the g_i converge weakly in L^2_2 to g.

$$|g_i| \leq M \overset{subseq}{\Longrightarrow} g_i \xrightarrow{a.e} g.$$

Hence $g \in U_n$ almost everywhere.

$$\|a_i\|_{L^2_1}, \|b_i\|_{L^2_1}, \|g_i\|_{L^2_2} \leq M \overset{subseq}{\Longrightarrow} a_i \overset{L^4}{\rightharpoonup} a, b_i \overset{L^4}{\rightharpoonup} b, g_i \overset{L^4, L^2_1}{\rightarrow} g.$$

where a_i, b_i converge strongly in $L^{4-\epsilon}$ and g_i converges strongly in L^p for all p and in L^2_1 so that in the equation

$$dg_i = g_i a_i - b_i g_i.$$

boths sides converges strongly in L^2. Thus the we have

$$dg = ga - bg$$

so that $\Gamma + a$ is gauge equivalent to $\Gamma + b$. To see that $\Gamma + a \in W_\epsilon$ we need to see that b satisfies the estimate and that $b \in L^2_{3/2}$

$$\begin{aligned}\int_U |F_{\Gamma+b}|^2 &= \int_U \langle db + b\wedge b, db + b\wedge b\rangle \\ &\geq \int_U |db + d^*b|^2 - \|b\|^4_{L^4} \\ &\geq \int_U |\nabla b|^2 - C\epsilon\|b\|^2_{L^2_1} \\ &\geq ((1+\lambda_1)/2 - C\epsilon)\|b\|^2_{L^2_1}.\end{aligned}$$

We've used the identity

$$\int_U (|(d+d^*)b|^2 = \int_U |\nabla b|^2 + \int_{\partial U} *bd$$

Check that $b \in L^2_{3/2}$. Take d^* of this equation giving

$$\Delta g = - * (dg \wedge *a) + gd^* a + *(*b \wedge dg).$$

Rewrite as:

$$\Delta g - *(*b \wedge dg) = - * (dg \wedge *a) + gd^* a.$$

Note that we also have $*dg|_{\partial B} = *a|_{\partial b}$. The red terms both are in $L^2_{1/2}$ since $L^2_{3/2} \times L^2_1 \mapsto L^2_{1/2}$. Thinking of the blue term as function of g it is a bounded operator

$$L^2_{5/2} \to L^2_{1/2}$$

with operator norm controlled by $\|b\|_{L^2_1}$. The map $h \mapsto (\Delta h, *dh|_{\partial U})$ viewed a map

$$L^2_{5/2}(U) \to L^2_{1/2}(U) \times L^2_1(\partial U)$$

is surjective with finite dimensional kernel.

The equation has index 0 and is invertible for $b = 0$ so for b with small L^2_1-norm it follows that $g \in L^2_{5/2}$ is the unique solution.

The equation

$$b = gag^{-1} - g^{-1}dg$$

says that b is in $L^2_{3/2}$.

W_ϵ is open in V_ϵ Openness follows from the Big Slices Lemma, Lemma **??** which tells us that if A is gauge equivalent into the $S_{\Gamma,1}(\epsilon)$ then so is a neighborhood of A in the $L^2_{3/2}$-topology.
Finally suppose that A_i is a sequence of $L^2_{3/2}$ connections connections in V_ϵ converging to an L^2_1 connection A then following the proof of closedness we see that A can be put into a good gauge and this representative satisfies the estimates. □

The same circle of ideas proves the following lemma which will later be used in the proof of the Uhlenbeck compactness theorem.

Lemma (Curvature is Proper)

With ϵ_o as in the previous lemma. Let $SV_{(\epsilon_0/2)}$ be the set of L^2_1 connections $A = \Gamma + a$ with

$$\int_B |F_A|^2 * 1 \leq \epsilon_0/2$$

and

$$d^* a = 0 \text{ and } * a|_{\partial B} = 0.$$

Then the curvature map

$$F_\bullet : SV_{\epsilon_o/2} \to L^2(B, \Lambda^2(T^*B) \otimes \mathrm{ad}P)$$
$$A \to F_A$$

is proper

Proof. Suppose F_{A_i} is a Cauchy sequence.

$$\begin{aligned}\int_B |F_{A_i} - F_{A_j}|^2 &= \int_B |da_i + a_i \wedge a_i - da_j - a_j \wedge a_j|^2 * 1 \\ &\geq \int_B (|da_i - da_j|^2 - |a_i \wedge a_i - a_j \wedge a_j|^2) * 1 \\ &\geq \int_B |(d + d^*)(a_i - a_j)|^2 - (\|a_i\|_{L^4} + \|a_j\|_{L^4})\|a_i - a_j\|^2_{L^4} \\ &\geq \int_B |\nabla(a_i - a_j)|^2 - 2C\epsilon\|a_i - a_j\|^2_{L^2_1} \\ &\geq ((1+\lambda)/2 - 2C\epsilon)\|a_i - a_j\|^2_{L^2_1}.\end{aligned}$$

Thus A_i is also Cauchy. □

X an oriented riemannian n-manifold. The Hodge Star

$$* : \Lambda^k \to \Lambda^{n-k}$$

Behavior under conformal change:

$$g \mapsto e^{2\sigma} g \implies * \mapsto e^{(n-2k)\sigma} *$$

In particular if $n = 2k$ Hodge star is conformally invariant. Also note that

$$*^2 = (-1)^{k(n-k)}$$

In particular for $n = 4$, $k = 2$

$$* : \Lambda^2 \to \Lambda^2$$

has $*^2 = 1$.

Thus 2-forms have a conformally invariant decomposition.

$$\begin{array}{ccccc} \Lambda^2 & = & \Lambda^+ & \oplus & \Lambda^- \\ & & * = 1 & & * = -1 \end{array}$$

Explicitly: e^1, e^2, e^3, e^4 oriented orthonormal frame.

$$\omega_I = e^1 \wedge e^2 + e^3 \wedge e^4, \omega_J = e^1 \wedge e^3 - e^2 \wedge e^4, \omega_K = e^1 \wedge e^4 + e^2 \wedge e^3$$

span Λ^+, the Self-Dual two forms and

$$e^1 \wedge e^2 - e^3 \wedge e^4,\ e^1 \wedge e^3 + e^2 \wedge e^4,\ e^1 \wedge e^4 - e^2 \wedge e^3$$

span Λ^-, the Anti-Self-Dual two forms. Note that Λ^+ and Λ^- are pointwise orthogonal under both the riemmanian inner product and the wedge product.

Thus we can decompose the curvature of a connection:

$$F_A = F_A^+ + F_A^-.$$

$F_A^+ = 0 \quad (F_A^- = 0)$ are the Anti-Self-Dual (Self-Dual) Yang-Mills equation.
Note that the Bianchi identity implies

$$d_A F_A = 0$$

and hence if A is SD or ASD we have

$$d_A^* F_A = - * d_A * F_A = \pm * d_A F_A = 0.$$

Thus SD and ASD are critical points for E but more is true!

If X is a closed four-manifold recall:

$$8\pi^2 k = \int_X tr(F_A \wedge F_A), \quad E(A) = -\int_X tr(F_A \wedge *F_A) \geq 0.$$

Thus adding and subtracting these formulae gives:

$$E(A) + 8\pi^2 k = -2\int_X tr(F_A^- \wedge *F_A^-) \geq 0$$

and

$$E(A) - 8\pi^2 k = -2\int_X tr(F_A^+ \wedge *F_A^+) \geq 0$$

Thus

$$E(A) \geq 8\pi^2 |k|$$

with equality if and only if $F_A^+ = 0$ when $k \geq 0$ and $F_A^- = 0$ when $k \leq 0$.

The basic instanton. We'll use quaternionic notation. $\mathbb{H} = \mathbb{R} + \mathbb{R}I + \mathbb{R}J + \mathbb{R}K$ where $IJ = K$ + cyclic and $IJ = -JI$ + cyclic.

$$\mathbf{x} = x_0 + x_1 I + x_2 J + x_3 K$$

Conjugation

$$\bar{\mathbf{x}} = x_0 - x_1 I - x_2 J - x_3 K.$$

$$S^7 \subset \mathbb{H}^2 \text{ the unit sphere}$$

Two different $SU_2 = Sp(1) = \{\mathbf{x} \in \mathbb{H} | \bar{\mathbf{x}}\mathbf{x} = \mathbf{1}\}$ actions.

$$(\mathbf{x}_0, \mathbf{x}_1)q = (\mathbf{x}_0 q, \mathbf{x}_1 q) \text{ or}$$
$$(\mathbf{x}_0, \mathbf{x}_1)q = (\bar{q}\mathbf{x}_0, \bar{q}\mathbf{x}_1).$$

In either case the quotient is $S^4 = \mathbb{HP}^1$.

Two principal bundle with total space S^7.

$$P_{\pm} \to S^4$$

where P_+ has the right action and P_- has the left action. These are the unit sphere bundles of $\mathbb{H}$-bundles $S_{\pm} \to S^4$. For example

$$S_+ = \{([\mathbf{x}_0, \mathbf{x}_1], (\mathbf{v}_0, \mathbf{v}_1)) | (\mathbf{v}_0, \mathbf{v}_1) = \bar{h}(\mathbf{x}_0, \mathbf{x}_1), \quad h \in \mathbb{H}\}.$$

Give $P_\pm$ the connection $A_\pm$ which declares that the horizontal space at $p \in P_\pm$ is the orthogonal complement of the fiber $P_\pm \to S^4$. $Sp_2(= Spin_5)$ acts on S^7 isometrically preserving the connections $A_\pm$. The stabilizer of a point $p \in S^7$ is a copy of $Sp_1(= SU_2 = Spin_3)$ so that the curvature of the connection is an SU_2 equivariant map

$$\Lambda^2(T_p^* S^4) = \Lambda^+ \oplus \Lambda^-(T_p^* S^4) \to \mathrm{ad}P_\pm|_p.$$

Need to understand SU_2 action. $\mathrm{ad}P|_\pm$ is the vertical tangent space at p which is canonically identified with $\mathfrak{su}_2$ with the adjoint action.

Write $d\mathbf{x} = dx_0 + dx_1 I + dx_2 J + dx_3 K$. Note that $d\mathbf{x} \wedge d\bar{\mathbf{x}}$ is a purely imaginary 2-form ($im\,\mathbb{H}$-valued). Indeed

$$d\mathbf{x} \wedge d\bar{\mathbf{x}} = -2(\omega_I I + \omega_J J + \omega_K K).$$

In particular $d\mathbf{x} \wedge d\bar{\mathbf{x}}$ is self-dual.

$$d\bar{g}\mathbf{x}h \wedge d\overline{\bar{g}\mathbf{x}h} = \bar{g}d\mathbf{x} \wedge d\bar{\mathbf{x}}g.$$

Thus $d\mathbf{x} \wedge d\bar{\mathbf{x}}$ is invariant under the left action and equivariant under the right action). Thus A_+ is a self-dual connection.
Similarly $d\bar{\mathbf{x}} \wedge d\mathbf{x}$ is anti-self-dual and invariant under the left action and equivariant under the left action. Thus A_- is an anti-self-dual connection.

Excerise: Show that the connection one-form

$$a = \frac{\mathrm{Im}\bar{\mathbf{x}} \wedge d\mathbf{x}}{(1+|x|^2)} = \frac{\bar{\mathbf{x}} \wedge d\mathbf{x} - \mathbf{x} \wedge d\bar{\mathbf{x}}}{2(1+|x|^2)}$$

represents A_+ in a suitable trivialization of P_+.
We can construct from A_- other ASD-connections using conformal invariance. The dilatation $\tau_\lambda(\mathbf{x}) = \lambda\mathbf{x}$ induces a conformal diffeomorphism. Indeed the basic instanton is invariant under SO_5 acting on S^4 but the conformal group $SO_{5,1}$ acts so effectively there is a

$$SO_{5,1}/SO_5 = \mathbb{H}^5$$

worth of ASD-connections in P_+. Atiyah-Hitchin-Singer prove any ASD connections in P_+ is gauge equivalent to one of these.

This example exhibits the phenomenon of bubbling. As the the parameter $\lambda \mapsto \infty$ the connections

$$\tau_\lambda^*(a) = \frac{\mathrm{Im}\lambda\bar{\mathbf{x}} \wedge d\lambda\mathbf{x}}{(1 + |\lambda x|^2)}$$
$$= \frac{\mathrm{Im}\bar{\mathbf{x}} \wedge d\mathbf{x}}{(1/\lambda^2 + |x|^2)}$$

converge away from the origin to

$$\frac{\mathrm{Im}\bar{\mathbf{x}} \wedge d\mathbf{x}}{(|x|^2)}$$

which is gauge equivalent to zero! by the gauge transformation

$$g(\mathbf{x}) = \mathbf{x}/|\mathbf{x}|.$$

An SO_3 bundle on $P \to X$ has two characteristic classes

$$p_1(P) \in H^4(X;\mathbb{Z}) \equiv \mathbb{Z} \text{ and } w_2(P) \in H^2(X;\mathbb{Z}_2)$$

These are constrained by

$$\wp(w_2(P)) \equiv p_1(P) \pmod 4.$$

where

$$\wp : H^2(X;\mathbb{Z}_2) \to H^4(X;\mathbb{Z}_4)$$

is the Pontraygin square.

Given $P \to X$ and riemannian metric g let

$$M_{k,w} = \{A \in \mathcal{A}(P) | F_A^+ = 0\}/\mathcal{G}(P).$$

where $k = -\frac{1}{4}\langle p_1(P), [X]\rangle$ and $w = w_2(P) \in H^2(X; \mathbb{Z}_2)$.

Fredholm maps and Kuranshi description of the moduli space.

Definition

A smooth map of Hilbert manifolds $f : \mathcal{M} \to \mathcal{N}$ is called Fredholm if for all $x \in \mathcal{M}$

$$d_x f : T_x\mathcal{M} \to T_{f(x)}\mathcal{N}$$

is a Fredholm map.

Theorem

If $f : \mathcal{M} \to \mathcal{N}$ *is a Fredholm map and* $x \in \mathcal{M}$ *then there is*

So that $\phi(0) = x$, $d_0\phi : T_0U \to \ker(d_x f) \subset T_x\mathcal{M}$ *and so that*

$$\phi(\kappa^{-1}(0)) = f^{-1}(0) \cap \mathcal{W}.$$

Theorem

If $f : \mathcal{M} \to \mathcal{N}$ is a Fredholm map and $x \in \mathcal{M}$ then there is

- *a neighborhood* $0 \ni U \ker(d_x f) \subset T_x\mathcal{M}$ *and neighborhood* $x \ni \mathcal{W} \subset \mathcal{M}$.

So that $\phi(0) = x$, $d_0\phi : T_0U \to \ker(d_x f) \subset T_x\mathcal{M}$ *and so that*

$$\phi(\kappa^{-1}(0)) = f^{-1}(0) \cap \mathcal{W}.$$

Theorem

If $f : \mathcal{M} \to \mathcal{N}$ *is a Fredholm map and* $x \in \mathcal{M}$ *then there is*

- *a neighborhood* $0 \ni U \ker(d_x f) \subset T_x\mathcal{M}$ *and neighborhood* $x \ni \mathcal{W} \subset \mathcal{M}$*.*
- *a map* $\kappa : U \to \text{coker } d_x f$*.*

So that $\phi(0) = x$*,* $d_0\phi : T_0 U \to \ker(d_x f) \subset T_x\mathcal{M}$ *and so that*

$$\phi(\kappa^{-1}(0)) = f^{-1}(0) \cap \mathcal{W}.$$

Theorem

If $f : \mathcal{M} \to \mathcal{N}$ *is a Fredholm map and* $x \in \mathcal{M}$ *then there is*

- *a neighborhood* $0 \ni U \ker(d_x f) \subset T_x\mathcal{M}$ *and neighborhood* $x \ni \mathcal{W} \subset \mathcal{M}$.
- *a map* $\kappa : U \to$ *coker* $d_x f$.
- *an immersion* $\phi : U \to \mathcal{M}$.

So that $\phi(0) = x$, $d_0\phi : T_0 U \to \ker(d_x f) \subset T_x\mathcal{M}$ *and so that*

$$\phi(\kappa^{-1}(0)) = f^{-1}(0) \cap \mathcal{W}.$$

If A is an ASD connection then we have the deformation complex

$$0 \mapsto L^2_3(X; \mathrm{ad} P) \xrightarrow{d_A} L^2_2(X; \Lambda^1 \otimes \mathrm{ad} P) \xrightarrow{d_A^+} L^2_1(X; \Lambda^+ \otimes \mathrm{ad} P) \to 0.$$

This is an elliptic complex and the index or of the complex is

$$-8k + \frac{3}{2}(\chi(X) + \sigma(X))$$

We call the negative of the index

$$8k - \frac{3}{2}(\chi(X) + \sigma(X))$$

the formal dimension of the moduli space.

This index was computed by Atiyah-Hitchin-Singer using the index theorem. The result is.

Theorem

The map $A + a \mapsto F^+(A)$ is a Fredholm map viewed as a map

$$\phi : S_{A,2} \to L^2_1(X, \Lambda^+(X)).$$

The index of ϕ is

$$8k - \frac{3}{2}(\chi(X) + \sigma(X)) + \dim Stab(A)$$

Examples.

1. $X = S^4$, k, $M_{1,0}(S^4) = B^5$ and dimension formula gives

$$8k - \frac{3}{2}(2+0) = 8k - 3$$

Note that $k = 1$ case checks with the example.

2. X arbitrary, $k = 0$. A is a trivial connection. The complex becomes 3 copies of

$$0 \mapsto L^2_3(X) \xrightarrow{d} L^2_2(X;\Lambda^1) \xrightarrow{d^+} L^2_2(X;\Lambda^+) \to 0.$$

where the dimensions of the cohomology groups are $b^0(X), b^1(X), b^+(X)$ respectively.

$$\begin{aligned}
&-\frac{3}{2}(\chi(X) + \sigma(X)) \\
&= -\frac{3}{2}(2b^0(X) - 2b^1(X) + b^+(X) + b^-(X) + b^+(X) - b^-(X)) \\
&= -3(b^0(X) - b^1(X) + b^+(X)).
\end{aligned}$$

These calculations can be turned around to prove the dimension formula using *excision*.

The Chern-Simons Function. Given A connection in $P \to X$.

$$\int_X tr(F_A \wedge F_A) \pmod{8\pi^2\mathbb{Z}}$$

depends only on the gauge equivalence class of $A|_{\partial X}$. Indeed if A' is a connection in P' and there is a bundle iso $g : P|_{\partial X} \to P'_{\partial X}$ so that $gA = A'$ then

$$\int_X tr(F_A \wedge F_A) - \int_X tr(F_{A'} \wedge F_{A'}) = \int_{DX} tr(F_{A''} \wedge F_{A''}) \in 8\pi^2\mathbb{Z}.$$

DX is the double of X and A'' is a connection in a bundle $P'' \to DX$ obtained by patching together P with the connection A and P' with the connection A' using g.

Definition

Let B be a connection in $Q \to Y$

$$CS(B) = \int_X tr(F_A \wedge F_A) \pmod{8\pi^2\mathbb{Z}}$$

Another viewpoint on CS. Given a pair of connections B_0 and B_1 choose $B_t, \quad t \in [0,1]$. View this path as a connection A in $[0,1] \times Q \to [0,1] \times Y$.

$$CS_{B_0}(B_1) = \int_{[0,1]\times Y} tr(F_A \wedge F_A).$$

If B_0 comes from a trivialization and $B_1 = B_0 + b$ and we set $B_t = B_0 + tb$ so that:

$$F_A = d(tb) + \frac{t^2}{2}[b \wedge b] = dt \wedge b + tdb + \frac{t^2}{2}[b \wedge b].$$

Then

$$tr(F_A \wedge F_A) = dt \wedge tr(2b \wedge (tdb + t^2[b \wedge b]))$$

and so

$$\int_{[0,1]\times Y} tr\big(F_A \wedge F_A\big) = \int_{[0,1]\times Y} dt \wedge tr(2b \wedge tdb + t^2 b \wedge [b \wedge b]\big)$$
$$= \int_Y tr(b \wedge db + \frac{1}{3} b \wedge b \wedge b).$$

Exercise

If B_0 does not arise from a trivialization show that

$$CS_{B_0}(B_1) = \int_Y tr(2b \wedge F_{B_0} + b \wedge db + \frac{1}{3} b \wedge [b \wedge b])$$

Proposition

Suppose $G = SU_2$. Then $P = Y \times SU_2$ and g can be viewed as a map $g : Y \to SU_2$. We claim

$$CS(B_1) - CS(gB_1) = \deg(g)$$

Proof: In this case the difference is the Chern-Weil integral for a connection in the bunlde arising from the mapping torus of g. The $c_2(P)$ is the Euler class of the associated complex two plane bundle. There is a section of this mapping torus bundle with $\deg(g)$ transverse zeroes.

The Chern-Simons functions provides another local characterization of Anti-Self-Duality.

Proposition (Mean Value Property)

Suppose is connected smooth manifold with boundary. A in $P \to X$ the

$$-\int_X tr(F_A \wedge *F_A) \geq -CS(A|_{\partial X})$$

with equality if and only if A is ASD.

Proof:

$$-\int_X tr(F_A \wedge *F_A) + \int_X tr(F_A \wedge F_A) = -2 \int tr(F_A^+ \wedge *F_A^+).$$

□

Uhlenbeck's compactness theorem. Let U be a small geodesic ball in a Riemannian four-manifold. Since all bundles on a ball are trivial we write $M(U)$ for the moduli space on U. For $\epsilon > 0$ write

$$M^{\epsilon}(U) = \{[A] \in M(U) | \int_U |F_A|^2 * 1 \leq \epsilon\}.$$

Proposition

Let U be a geodesic ball in a Riemannian four-manifold. There is an ϵ_o so that for any proper subball $U' \subset U$ the set of connections the restriction map

$$M^{\epsilon_o}(U) \to M(U')$$

is a compact map.

Proof: Take a sequence of gauge equivalence classes of connections $[A_i]$ We can choose the representatives $A_i = \Gamma + a_i$ so that they are in Coulomb gauge and we have uniform L^2_1-estimates on a_i. Pass to a subsequence where a_i converge weakly in L^2_1. By Fubini's theorem for each i $a_i|_{\partial B_r}$ is in L^2_1 except for r in a set of measure zero. Since the sequence is countable for all i, $a_i|_{\partial B_r}$ a.e in r. Now fix such an r. Since the sequence is uniformly bounded we can pass to a subsequence where $a_i|_{\partial B_r}$ is uniformly bounded in L^2_1. Now we can pass to a subsequence where $a_i|_{\partial B_r}$ converges strongly in $L^2_{1/2}$. In particular $CS(A_i|_{\partial B_r})$ converges. We now longer have that $a_i|_{B_r}$ is in Coulomb gauge since it doesn't satisfy the boundary condition. However by the Big Slices Lemma we can put these a_i in to Coulomb gauge by a sequence of $L^2_{5/2}$-gauge transformations giving new connection one forms $\tilde{a}_i$.

We can assume the gauge transformations converge strongly in $L^2_2 \cap C^0$ in particular $\tilde{a}_i|_{\partial B_r}$ still converges in $L^2_{1/2}$ and $\tilde{a}_i$ still converges weakly in L^2_1 to $\tilde{a}$. Set $A = \Gamma + \tilde{a}$. Since A_i are ASD, the convergence of CS implies $\int_{B_r} |F_{A_i}|^2 * 1$ converges to $\int_{B_r} |F_A|^2 * 1$. This implies strong convergence of F_{A_i} and hence by the properness of the curvature map the strong convergence to $\tilde{a}_i$ as required.□
We can also deduce regularity of solution from a similar discussion.

Theorem

Let A be an L^2_1-connection. Suppose that $F^+_A = 0$. Then there is a an L^2_2-gauge transformations g so that gA is C^∞.

Theorem

Let G be a compact Lie group. Let $P \to X$ be principle G-bundle Let A_i be a sequence of ASD-connections with

$$\int_X |F_{A_i}|^2 \leq M$$

then after passing to a subsequence there are

so that for any open subset $U \Subset X \setminus \{x_1, \ldots, x_n\}$ $g_i \cdot A_i$ converges in the C^∞-topology to $A|_{X \setminus \{x_1, \ldots, x_n\}}$.

Theorem

Let G be a compact Lie group. Let $P \to X$ be principle G-bundle Let A_i be a sequence of ASD-connections with

$$\int_X |F_{A_i}|^2 \leq M$$

then after passing to a subsequence there are

- *a collection of finitely many points $x_1, \ldots, x_n \subset X$,*

so that for any open subset $U \Subset X \setminus \{x_1, \ldots, x_n\}$ $g_i \cdot A_i$ converges in the C^∞-topology to $A|_{X \setminus \{x_1, \ldots, x_n\}}$.

Theorem

Let G be a compact Lie group. Let $P \to X$ be principle G-bundle Let A_i be a sequence of ASD-connections with

$$\int_X |F_{A_i}|^2 \le M$$

then after passing to a subsequence there are

- *a collection of finitely many points $x_1, \ldots, x_n \subset X$,*
- *a bundle $P' \to X$.*

so that for any open subset $U \Subset X \setminus \{x_1, \ldots, x_n\}$ $g_i \cdot A_i$ converges in the C^∞-topology to $A|_{X \setminus \{x_1,\ldots,x_n\}}$.

Theorem

Let G be a compact Lie group. Let $P \to X$ be principle G-bundle Let A_i be a sequence of ASD-connections with

$$\int_X |F_{A_i}|^2 \leq M$$

then after passing to a subsequence there are

- *a collection of finitely many points $x_1, \ldots, x_n \subset X$,*
- *a bundle $P' \to X$.*
- *a sequence of bundle isomorphisms $g_i : P|_{X \setminus \{x_1, \ldots, x_n\}} \to P'|_{X \setminus \{x_1, \ldots, x_n\}}$,*

so that for any open subset $U \Subset X \setminus \{x_1, \ldots, x_n\}$ $g_i \cdot A_i$ converges in the C^∞-topology to $A|_{X \setminus \{x_1, \ldots, x_n\}}$.

Theorem

Let G be a compact Lie group. Let $P \to X$ be principle G-bundle Let A_i be a sequence of ASD-connections with

$$\int_X |F_{A_i}|^2 \leq M$$

then after passing to a subsequence there are

- *a collection of finitely many points $x_1, \ldots, x_n \subset X$,*
- *a bundle $P' \to X$.*
- *a sequence of bundle isomorphisms $g_i : P|_{X \setminus \{x_1, \ldots, x_n\}} \to P'|_{X \setminus \{x_1, \ldots, x_n\}}$,*
- *a connection A in a bundle $P' \to X$*

so that for any open subset $U \Subset X \setminus \{x_1, \ldots, x_n\}$ $g_i \cdot A_i$ converges in the C^∞-topology to $A|_{X \setminus \{x_1, \ldots, x_n\}}$.

Proof. We can pass to a subsequence where the sequence of curvature densities $|F_{A_i}|^2 * 1$ converges in the weak topology to a density ω. If we cover X by balls of radius 2^{-N} then for each N so that the balls of half the radius cover as well. Then for each n we can find $i(n)$ so that Then at most $2M/\epsilon_o$ of these ball have

$$\int_B |F_{A_i}|^2 \leq \epsilon_o/2$$

for all $i \geq i(N)$. We can pass to a further subsequence so that the centers of these bad balls converge to points $x_1, \ldots x_n$. We thus get an exhaustion of $X \setminus \{x_1, \ldots x_n\}$ by a sequence of balls B_j where

$$\int_{B_j} |F_{A_i}|^2 \leq \epsilon_o/2.$$

Finally by a diagonalization argument and the compact inclusion lemma we can pass to a subsequence where there are sequence g_i of gauge tranformations to that

Removeable singularites.

Theorem (Uhlenbeck)

Let $P \to X$ be a principle bundle. Suppose that A is finite energy ASD connection on $X \setminus \{x_1, \ldots, x_n\}$. Then there is a bundle $P' \to X$ and a connection A' in P so that $A|_{X\setminus\{x_1,\ldots,x_n\}}$ and $A'|_{X\setminus\{x_1,\ldots,x_n\}}$ are gauge equivalent.

Removeable singularites.

Theorem (Uhlenbeck)

Let $P \to X$ be a principle bundle. Suppose that A is finite energy ASD connection on $X \setminus \{x_1, \ldots, x_n\}$. Then there is a bundle $P' \to X$ and a connection A' in P so that $A|_{X\setminus\{x_1,\ldots,x_n\}}$ and $A'|_{X\setminus\{x_1,\ldots,x_n\}}$ are gauge equivalent.

Definition

A sequence of connection A_i in $P \to X$ converges upto bubbling to $A \in P' \to X$ if there is a collection of points $\{x_1, \ldots, x_n\}$ and a sequence of gauge transformations g_i so that for every compact subset $K \Subset X \setminus \{x_1, \ldots, x_n\}$ so that $A_i|_K \mapsto A|_K$ in the C^∞-topology.

Together with the compactness theorem this leads to the Uhlenbeck compactification.

Definition

The Uhlenbeck compactification $UM_k(X)$ of $M_k(X)$ is the closure of $M_k(X) \subset M_k(X) \cup X \times M_{k-1}(X) \cup Sym_2 X \times M_{k-1}(X) \cup \ldots \cup Sym_k(X) \times M_0(X)$ with respect to the upto bubbling topology.

Together with the compactness theorem this leads to the Uhlenbeck compactification.

Definition

The Uhlenbeck compactification $UM_k(X)$ of $M_k(X)$ is the closure of $M_k(X) \subset M_k(X) \cup X \times M_{k-1}(X) \cup Sym_2 X \times M_{k-1}(X) \cup \ldots \cup Sym_k(X) \times M_0(X)$ with respect to the upto bubbling topology.

Remark

The subset of $M_k(X) \cup X \times M_{k-1}(X) \cup Sym_2 X \times M_{k-2}(X) \cup \ldots \cup Sym_k(X) \times M_0(X)$ which actually appears in the closure is important to understand. Taubes' gluing work showed how to understand this subset in many situations.

We will want to deal with orbifolds, orbifold bundles and orbifold connections. We'll only deal with $\mathbb{Z}_2$-orbifolds. X a Hausdorff space together with the following data. A collection of triples (Γ_i, V_i, ϕ_i) Γ_i is either $\mathbb{Z}_2$ or the trivial group. $V_i \subset \mathbb{R}^n$ is a Γ_i invariant open subset and $\phi : V_i/\Gamma_i \to X$ is a homeomorphism onto an open subset U_i. The open subsets U_i are an open cover. Further for each inclusion $U_i \hookrightarrow U_j$ there is an injective immersion $\phi_{ij} : V_i \hookrightarrow V_j$ which intertwines the Γ_i and Γ_j actions. (Note that there is a unique injective homomorphism $\Gamma_i \hookrightarrow \Gamma_j$ and we'd have to be a little more careful here if the Γ_i had non-trivial automorphisms.)

First for each (Γ_i, V_i, ϕ_i) we have that Γ_i acts on $V_i \times G$ commuting with the G action. A orbifold principal bundle is the data of transition functions

$$\tau_{ij} : V_i \times G \to V_j \times G$$

which again intertwines the Γ_i and Γ_j actions.

We will consider only very special 4-dimensional $\mathbb{Z}_2$-orbifolds. We allow only one local model where the $\mathbb{Z}_2$ action is

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, -x_3, -x_4).$$

Let F be the fixed point $(x_1, x_2, 0, 0)$. In this case the quotient $\mathbb{R}^4/\mathbb{Z}_2$ has a natural smooth structure so we can think of X as smooth four manifold with a distinguished two dimensional submanifold S. We'll denote the orbifold by (X, S) though S might not globally be the fixed point set. S may or may not be orientable.

Suppose P is an orbifold SU_2-bundle and A is an orbifold connection in P. We get an bundle SU_2 on $X \setminus S$ with connectio. P may not extend over S. Suppose further that the $\mathbb{Z}_2$ action on P is non trivial. Then over F we get map F to $\mathrm{Aut}(SU_2) = SO_3$. Any element of order 2 in SO_3 conjugate to an inner automorphism by the element

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

These inner automorphism form an $\mathbb{RP}^2 \subset SO_3$.

We can trivialize P so that the automorphism is locally constant Consider an invariant connection:

$$\Gamma + \begin{bmatrix} it_1 & z_1 \\ -\bar{z}_1 & -it_1 \end{bmatrix} dx^1 + \ldots + \begin{bmatrix} it_4 & z_4 \\ -\bar{z}_4 & -it_4 \end{bmatrix} dx^4.$$

where t_1, t_2, z_3, z_4 are even under the involution while t_3, t_4, z_1, z_2 are odd. In particular along F the connection restricts to

$$\begin{bmatrix} it_1 & 0 \\ 0 & -it_1 \end{bmatrix} dx^1 + \begin{bmatrix} it_2 & 0 \\ 0 & -it_2 \end{bmatrix} dx^2$$

Examples. 1. Let $\mathbb{Z}_2$ act on S^4 by compactifying the standard action above. The fixed set is S^2. The action lifts to an action on the total space $P_\pm = S^7 \to S^4$ and leaves the standard instanton (anti-instanton) invariant.
$X = S^4/\mathbb{Z}_2$ is homemorphic to S^4. The moduli space of $\mathbb{Z}_2$ invariant instantons is

$$\mathbb{H}^3 = SO_{3,1}/SO_3.$$

The action of such a connection

$$\kappa = \frac{1}{8\pi^2}\int_X tr(F_A \wedge F_A) = \pm 1/2$$

So for we have

$$d = 3,\ \kappa = 1/2,\ S \cdot S = 0,\ \chi(S) = 2,\ \chi(X) + \sigma(X) = 2.$$

2. Recall that $\mathbb{CP}^2/conj = S^4$ (a folk observation made precise independently by Anrold, Kuiper, and Massey). Consider the map $f : \mathbb{C}^3 \to Sym_o(\mathbb{R}^3)$.

$$f : \mathbb{C}^3 \to \mathbf{z} \subset S^5 \to \Re(\mathbf{z}\bar{\mathbf{z}}^t - \frac{1}{3}|\mathbf{z}|^2).$$

This is the real part of the moment map for SU_3 acting on $\mathbb{C}^3$. The image moment map consists of traceless hermitian matrices with exact 2 eigenvalues. If $\mathbf{z} \in S^5$ then $\mathbf{z}$ is an eigenvalue $2/3$ while if $\mathbf{w}$ is orthogonal to $\mathbf{z}$ the matrix acts by $-1/3$. Thus the image of the moment map is exactly $\mathbb{CP}^2$. Each of these authors show that image of the real part is a copy of S^4. Call this orbifold $(S^4, \mathbb{RP}^2_-)$.

The Fubini-Study metric gives and ASD connection in $\mathrm{End}_0(T^*\mathbb{CP}^2)$.

$$p_1(T^*\mathbb{CP}^2) = c_1^2 - 4c_2 = (9-12)H^2 = -3H^2$$

so $k = 3/4$. The dimension of this moduli space is

$$6 - 3(1+1) = 0$$

and eventually one can show this connection is unique upto gauge. The involution on $\mathbb{CP}^2$ is an isometry so lifts uniquely to $T^*\mathbb{CP}^2$. Pushing this down to the
Note that

$$d = 0,\ \kappa = 3/8,\ S\cdot S = -2,\ \chi(S) = 1,\ \chi(X) + \sigma(X) = 2.$$

3. If we take the same example but reverse orientation we have $\overline{\mathbb{CP}}^2/\mathbb{Z}_2 = S^4$. On $\overline{\mathbb{CP}}^2$ there is a (unique upto gauge) reducible ASD connection in $\mathbb{R} \oplus E$ where E is tautological bundle thought of on $\overline{\mathbb{CP}}^2$. Now $p_1(\mathbb{R} \oplus E) = -c_2(\mathbb{C} \oplus E \oplus E^-1) = -1$ so $k = 1/4$. Thus this moduli has formal dimension

$$8(1/4) - 3(1) = -1.$$

The -1 is accounted for by the stabilizer of the connection. There are two $\mathbb{Z}_2$ actions on $\mathbb{R} \oplus E$ actions covering conjugation on $\overline{\mathbb{CP}}^2$, these are $\pm$ conjugation on E and -1 on $\mathbb{R}$. These however are gauge equivalent by multiplication by i in E. The resulting connection on S^4 has

$$d = 0,\ \kappa = 1/8,\ S \cdot S = 2,\ \chi(S) = 1,\ \chi(X) + \sigma(X) = 2.$$

4. Next $S^2 \times S^2$ with involution which is conjugation on each factor. The quotient is again S^4, the involution acts by -1 on $H^2(S^2 \times S^2)$, $S = S^1 \times S^1$. Let L be line bundle which is the tensor product of the tautological bundle on the first factor and its dual on the second $\pi_1^*(\mathcal{O}(-1)) \times \pi_2^*(\mathcal{O}(1))$. With the product metric and both factors of the same area the curvature of L is ASD. The bundle $\mathbb{R} \oplus L$ then has a unique reducible *ASD* connection. $p_1(\mathbb{R} \oplus L) = c_1^2(L) = -2$ so the dimension of the moduli space is

$$4 - 6 = -2$$

Since the connection is reducible the deformation complex splits into the trivial part which has cohomology

$$H^0 = H^2 = \mathbb{R}, and H^1 = 0$$

account for all of the index. The L part of the complex has trivial cohomology. The involution acts by -1 on H^0 and 1 on H^2, thus corresponding orbifold connection on S^4, T^2

Using excision one can see that the dimension the of the moduli space is an affine function of $\kappa, \chi(X) + \sigma(X), S \cdot S$, and $\chi(S)$ whether or on S is orientable:

$$d = 8\kappa - \frac{3}{2}(\chi(X) + \sigma) + aS \cdot S + b\chi(F) + c$$

From the examples we see that $a = 1/2, b = 1, c = 0$ so we have

Theorem

The formal dimension of the moduli space $M_\kappa(X, F)$ is

$$d = 8\kappa - \frac{3}{2}(\chi(X) + \sigma) + \frac{1}{2}S \cdot S + \chi(S).$$

Compactness theorem for orbifold connections. Since locally orbifold connections are just invariant connections there is nothing more to prove although when curvature concentrates along F there is only a drop κ drops in units of $1/2$ rather than 1.
To describe the Uhlenbeck compactification on the pair (X, S) consider weighted collections of points

$$Sym_*(X,S) = \{\sum_i n_i x_i | n_i \in \mathbb{Z} \text{ if } x_i \in X \backslash S \text{ and } n_i \in \frac{1}{2}\mathbb{Z} \text{ if } x_i \in S\}$$

Then for $\kappa \in \frac{1}{2}\mathbb{Z}$ set

$$Sym_\kappa(X,S) = \{\sum_i n_i x_i \in Sym_*(X,S) | \sum_i n_i = \kappa\}.$$

Thus

$$
\begin{aligned}
Sym_{\frac{1}{2}}(X, S) &= S \\
Sym_1(X, S) &= X \cup_F Sym_2(F) \\
Sym_{\frac{3}{2}}(X, S) &= (X \times F) \cup_{F \times F} Sym_3(F) \\
Sym_2(X, S) &= (Sym_2(X) \cup Sym_2(F) \times X \cup Sym_4(F))/\sim
\end{aligned}
$$

etc.

Thus we have a compactification which now looks like

Definition

The Uhlenbeck compactification $UM_\kappa(X,S)$ of $M_\kappa(X,S)$ is the closure of

$$M_\kappa(X,S) \subset \bigcup_{\kappa' \in \frac{1}{2}\mathbb{Z}} Sym_{\kappa'}(X,S) \times M_{\kappa-\kappa'}(X,S).$$

with respect to the upto bubbling topology.

Floer's Instanton Homology

Let Y be a 3-manifold $P = Y \times SU_2$. Γ the trivial connection and $B = \Gamma + b$.

$$CS(B) = \int_Y \mathrm{tr}(b \wedge db + \frac{1}{3} b \wedge [b \wedge b])$$

Floer's great insight was that using the analytic tools developed by Uhlenbeck, Taubes and Donaldson one could carry over the Morse complex construction to CS viewed as a function on $\mathcal{B} = \mathcal{A}/\mathcal{G}$.

To compute the directional derivative $\frac{d}{ds}CS(B+sc)|_{s=0}$ introduce the one parameter family of $4d$-connections on $[0,1]\times Y$ with t being the coordinate in $[0,1]$

$$A_s = B + st\, c$$

Then

$$F_{A_s} = F_B + d_B^{(4)}(stc) + \frac{s^2t^2}{2}[c\wedge c].$$

so that

$$\frac{d}{ds}F_{A_s}|_{s=0} = dt\wedge d_B c$$

$$\begin{aligned}\frac{d}{ds}CS(B+sc)|_{s=0} &= \frac{d}{ds}\int_{[0,1]\times Y} tr(F_{A_s}\wedge F_{A_s})|_{s=0}\\ &= 2\int_0^1\int_Y dt\wedge tr(c\wedge F_B)\\ &= 2\int_Y tr(c\wedge F_B)\end{aligned}$$

To compute the directional derivative $\frac{d}{ds}CS(B+sc)|_{s=0}$ introduce the one parameter family of $4d$-connections on $[0,1]\times Y$ with t being the coordinate in $[0,1]$

$$A_s = B + st\,c$$

Then

$$F_{A_s} = F_B + d_B^{(4)}(stc) + \frac{s^2t^2}{2}[c\wedge c].$$

so that

$$\frac{d}{ds}F_{A_s}|_{s=0} = dt\wedge d_B c$$

$$\begin{aligned}\frac{d}{ds}CS(B+sc)|_{s=0} &= \frac{d}{ds}\int_{[0,1]\times Y} tr(F_{A_s}\wedge F_{A_s})|_{s=0}\\ &= 2\int_0^1\int_Y dt\wedge tr(c\wedge F_B)\\ &= 2\int_Y tr(c\wedge F_B)\end{aligned}$$

Thus critical points of CS are flat connections $F_{\ldots} = 0$

The gauge group $\mathcal{G} = \{g : Y \to SU_2\}$ acts on $\Omega^1(Y) \otimes su_2$ by

$$g \cdot b = gbg^{-1} + gdg^{-1}.$$

Under this action we have

$$CS(g \cdot B) = CS(B) + \deg(g) \quad \text{and} \quad F_{g \cdot B} = gF_B g^{-1}$$

Thus the set of critical points of CS is preserved by the $\mathcal{G}$ action and

$$\{a | F_B = 0\} = Hom(\pi_1(Y), SU_2)/\mathrm{conj} = R(Y)$$

where the identification is by the holonomy representation.

Examples:

- S^3. Only the trivial representation ρ_{triv}. $R(Y) = pt$. $\mathrm{Stab}(\rho_{triv}) = SU_2$.

Examples:

- S^3. Only the trivial representation ρ_{triv}. $R(Y) = pt$. $\mathrm{Stab}(\rho_{triv}) = SU_2$.
- Poincare sphere, $SU_2/$Binary Icosohedral group. Has three representations upto conjugacy ρ_{triv} and ρ_{def} and $\bar{\rho}_{def}$

Examples:

- S^3. Only the trivial representation ρ_{triv}. $R(Y) = pt$. $\mathrm{Stab}(\rho_{triv}) = SU_2$.
- Poincare sphere, $SU_2/$Binary Icosohedral group. Has three representations upto conjugacy ρ_{triv} and ρ_{def} and $\bar{\rho}_{def}$
- $S^1 \times S^2$. $R(Y) = SU_2/\mathrm{conj}$ an interval.

Examples:

- S^3. Only the trivial representation ρ_{triv}. $R(Y) = pt$. $\mathrm{Stab}(\rho_{triv}) = SU_2$.
- Poincare sphere, $SU_2/$Binary Icosohedral group. Has three representations upto conjugacy ρ_{triv} and ρ_{def} and $\bar{\rho}_{def}$
- $S^1 \times S^2$. $R(Y) = SU_2/\mathrm{conj}$ an interval.
- T^3. $R(Y) = S^1 \times S^1 \times S^1/\{\mathbb{Z}_2\}$.

Examples:

- S^3. Only the trivial representation ρ_{triv}. $R(Y) = pt$. $\mathrm{Stab}(\rho_{triv}) = SU_2$.
- Poincare sphere, $SU_2/$Binary Icosohedral group. Has three representations upto conjugacy ρ_{triv} and ρ_{def} and $\bar{\rho}_{def}$
- $S^1 \times S^2$. $R(Y) = SU_2/\mathrm{conj}$ an interval.
- T^3. $R(Y) = S^1 \times S^1 \times S^1/\{\mathbb{Z}_2\}$.
- (Fintushuel-Stern) $S(p, q, r)$ a Seifert fibered homology sphere with three exceptional fibers. The representation space is a finite set of isolated points can be idenitie

Examples:

- S^3. Only the trivial representation ρ_{triv}. $R(Y) = pt$. $\mathrm{Stab}(\rho_{triv}) = SU_2$.
- Poincare sphere, $SU_2/$Binary Icosohedral group. Has three representations upto conjugacy ρ_{triv} and ρ_{def} and $\bar{\rho}_{def}$
- $S^1 \times S^2$. $R(Y) = SU_2/\mathrm{conj}$ an interval.
- T^3. $R(Y) = S^1 \times S^1 \times S^1/\{\mathbb{Z}_2\}$.
- (Fintushuel-Stern) $S(p, q, r)$ a Seifert fibered homology sphere with three exceptional fibers. The representation space is a finite set of isolated points can be idenitie
- (Fintushuel-Stern) S a Seifert fibered homology sphere with four exceptional fibers. The representation space is a union of S^2's

Given a riemannian metric on Y we can formally compute the gradient of CS and we have

$$2\int_Y \mathrm{tr}(F_B \wedge c) = 2\int_Y \mathrm{tr}(*F_B \wedge *c) = -2\langle *F_B, c\rangle.$$

The downward gradient flow for $\mathcal{L} = -\frac{1}{2}CS$ is

$$\frac{B}{dt} = -\nabla_B \mathcal{L} = \frac{1}{2}\nabla_B CS = - * F_B.$$

A major miracle occurs here. If we view a solution $B(t)$ as a connection A on $\mathbb{R} \times Y$ the connection satisfies the Anti-Self-Dual Yang-Mills equation

$$F_A = - *_4 F_A.$$

As a $4d$-connection

$$F_A = dt \wedge \frac{dB}{dt} + F_B$$

and

$$*F_A = dt \wedge *_3 F_B + *_3 \frac{dB}{dt}$$

This later equations is invariant not just under $\mathcal{G}_3 = \{g : Y \to SU_2\}$ but under $\mathcal{G}_4 = \{h : \mathbb{R} \times Y \to SU_2\}$.

Problems.

- gradient flow of CS not well defined Hessian of CS, $*d_B$ has infinitely many positive and negative eigenvalues.

Problems.

- gradient flow of CS not well defined Hessian of CS, $*d_B$ has infinitely many positive and negative eigenvalues.
- $\mathcal{B}$ is singular.

Problems.

- gradient flow of CS not well defined Hessian of CS, $*d_B$ has infinitely many positive and negative eigenvalues.
- $\mathcal{B}$ is singular.
- CS is not single valued (cf Novikov homology).

Problems.

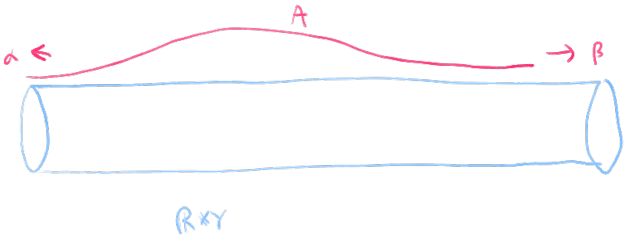
- gradient flow of CS not well defined Hessian of CS, $*d_B$ has infinitely many positive and negative eigenvalues.
- $\mathcal{B}$ is singular.
- CS is not single valued (cf Novikov homology).
- Compactification of moduli spaces (including gluing).

Problems.

- gradient flow of CS not well defined Hessian of CS, $*d_B$ has infinitely many positive and negative eigenvalues.
- $\mathcal{B}$ is singular.
- CS is not single valued (cf Novikov homology).
- Compactification of moduli spaces (including gluing).
- Construct pertubations to achieve Morse-Smale Condition that do no destroy compactness properties.

Given a pair of critical points α, β set

$$M(\alpha, \beta) = \{A| - \int_Z tr(F_A \wedge *F_A) < \infty,\ F_A = - * F_A,$$
$$\lim_{t \mapsto -\infty} [A|_{t \times Y}] = \alpha,\ \lim_{t \mapsto \infty} [A|_{t \times Y}] = \beta\}/\mathcal{G}_4.$$



Note that A is ASD if and only if

$$-\int_Z tr(F_A \wedge *F_A) = \text{"} - CS(\alpha) + CS(\beta)\text{"}$$

where the difference is computed with respect the path $B(t) = A|_{t\times Y}$.

Fix $Q \to Y$ an SO_3 or SU_2 bundle. Given $B_\pm$ flat connections with trivial stabilizer choose a path $B(t)$ with $t \in \mathbb{R}$ so that for $t < -1$ $B(t) = B_-$ and $t > 1$ $B(t) = B_+$. Let A be the corresponding $4d$-connection on and

$\mathbb{R} \times Q = P \to Z = \mathbb{R} \times Y$. Then we make the following definitions.

$$\mathcal{A}(\alpha, \beta) = A + L^2_2(Z; T^*Z \otimes \mathrm{ad}P)$$

$$\mathcal{G} = g \in 1 + L^2_3(Z, \mathrm{End}(E)) | gg^* = 1.$$

Since gA is locally $gag^{-1} + gag^{-1}$ $\mathcal{G}$ acts on $\mathcal{A}(\alpha, \beta)$

$$\mathcal{B}(\alpha, \beta) = \mathcal{A}(\alpha, \beta)/\mathcal{G}.$$

Remark

If stabilizer of end points is not $Z(G)$ this is not a good definition.

Remark

If stabilizer of end points is not $Z(G)$ this is not a good definition.

Remark

The definition depends on a homotopy class $\gamma \in \pi_1(\mathcal{B}_Q; \alpha, \beta)$ so write $\mathcal{A}_\gamma(\alpha, \beta)$ or $\mathcal{B}_\gamma(\alpha, \beta)$ when necessary.

For $A \in \mathcal{A}(\alpha, \beta)$

$$F_A^+ \in L_1^2(Z, \Lambda^+(Z) \otimes \mathrm{ad} P)$$

Set

$$M_\gamma(\alpha, \beta) = \{[A] \in \mathcal{B}(\alpha, \beta) | F_A^+ = 0\}$$

We'd like to see that $M_\gamma(\alpha, \beta)$ has a Kuranishi local structure and that

$$M(\alpha, \beta) = \bigcup_{\gamma \in \pi_1(\mathcal{B}_Q, \alpha, \beta)} M_\gamma(\alpha, \beta)$$

Write $A = B + b + cdt$ where B is a pull back connection. Then

$$F_A = F_B + dt \wedge \dot{b} + d_B b - dt \wedge d_B c + \frac{1}{2}[b \wedge b] + dt \wedge [c \wedge b]$$

$$0 = F_A^+ = dt \wedge (\dot{b} + *F_B + d_B c + *d_B b + *\frac{1}{2}[b \wedge b] + [c \wedge b])$$
$$+ *\dot{b} + F_B + *d_B c + d_B b + \frac{1}{2}[b \wedge b] + *[c \wedge b].$$

The slice condition

$$\begin{aligned} 0 = -\mathbf{d}_B^*(b + cdt) &= *_4 \mathbf{d}_B *_4 (b + cdt) \\ &= *\mathbf{d}_B dt \wedge *b + *c \\ &= *_4(-dt \wedge (d_B * b + dt \wedge *\dot{c} + d_B * c) \\ &= \dot{c} + d_B^* b + dt \wedge d_B^* c. \end{aligned}$$

Linearized equations at B.

$$0 = \begin{bmatrix} \dot{b} \\ \dot{c} \end{bmatrix} + \begin{bmatrix} *d_B & d_B \\ d_B^* & 0 \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix}$$

This of the form $\frac{d}{dt} + D$ where D is a first order self-adjoint elliptic operator.

Theorem

If D is an invertible first order self-adjoint elliptic operator acting on a vector bundle $F \to Y$ then

$$\frac{d}{dt} + D : L^2_k(Z;F) \to L^2_{k-1}(Z,F)$$

is invertible.

Sketch of proof in the case $k = 1$. Note that

$$
\begin{aligned}
\|(\frac{d}{dt} + D)u\|^2_{L^2(Z)} \int_Z \langle(\frac{d}{dt} + D)u, (\frac{d}{dt} + D)u\rangle * 1 \\
&= \int_Z |\frac{du}{dt}|^2 + |Du|^2 + 2\langle\frac{du}{dt}, Du\rangle * 1 \\
&= \int_Z |\frac{du}{dt}|^2 + |Du|^2 + \frac{d}{dt}\langle u, Du\rangle * 1 \\
&= \int_Z |\frac{du}{dt}|^2 + |Du|^2 * 1 \\
&\geq C\|u\|^2_{L^2_1}.
\end{aligned}
$$

Thus $\frac{d}{dt} + D : L^2_k(Z; F) \to L^2_{k-1}(Z, F)$ is injective with closed range. Exercise: Show the range is dense. Generalize this to the case of general k. $\square$.

This can be used to prove in general that provided the Extend Hessian of $\mathcal{L}$

$$EH_B = \begin{bmatrix} *d_B & d_B \\ d_B^* & 0 \end{bmatrix}$$

is invertible for $B = B_{\pm}$ the linearized ASD equations and gauge fixing are Fredholm so the package we use to analyze the moduli space on closed manifolds carries over to this case. EH_B plays the role of the Hessian in finite dimensional (and no group action) Morse theory.

What does invertibility mean? If

$$\begin{bmatrix} b \\ c \end{bmatrix}$$

is in the kernel of EH_B and B is flat the b is a harmonic representative of $H^1(Y, \mathrm{ad}_B)$ and $c \in H^0(Y, \mathrm{ad}_B)$. In other words the B must be

- Irreducible

What does invertibility mean? If

$$\begin{bmatrix} b \\ c \end{bmatrix}$$

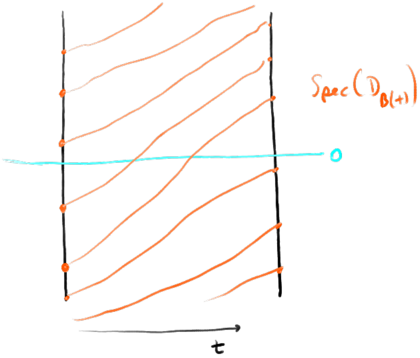
is in the kernel of EH_B and B is flat the b is a harmonic representative of $H^1(Y, \mathrm{ad}_B)$ and $c \in H^0(Y, \mathrm{ad}_B)$. In other words the B must be

- Irreducible
- Infinitesimally isolated (non-degenerate critical point.)

Given $A \in \mathcal{A}_\gamma(\alpha, \beta)$ write $A = B + c dt$ where $B = B(t)$ is path in $\mathcal{A}_Q$ and we can ask what is the index of

$$\frac{d}{dt} + D_B.$$

We now have a family of self-adjoint operators $D_{B(t)}$. Such a family has a spectral flow. Below we have a spectral



flow of $+2$.

Note the self-adjoint Fredholm operators $\mathcal{SF}$ has three components

$$\mathcal{SF} = \mathcal{SF}_- \cup \mathcal{SF}_o \cup \mathcal{SF}_+$$

where $\mathcal{SF}_\pm$ are the essentially positive and negative operators. $\mathcal{SF}_\pm$ are contractible while

$$\Omega\mathcal{SF}_o \equiv \mathbb{Z} \times BO(\text{ or } \mathbb{Z} \times BU \text{ for complex ops}).$$

Theorem (Atiyah-Patodi-Singer..)

$$Ind(\frac{d}{dt} + D_t) = sf(D_t)$$

Idea of proof. Index is homotopy invariant so homotope the family so that D_t all have the same eigenvectors. Then consider

$$T_{\pm} = \frac{d}{dt} + \pm \tanh(t) : L^2_1(\mathbb{R}) \to L^2(\mathbb{R}).$$

The kernel of T_+ is spanned by $sech(t)$ while for T_- the kernel is spanned by $\cosh(t)$. Since $T_{\pm}^* = -T_{\mp}$ when

$$\mathrm{Ind}(T_{\pm}) = \pm 1 = sf(\pm \tanh(t)).$$

$\square$.

Note the spectral flow is the what remains of the Morse index. $\mu(\alpha)$ should be the number of negative eigenvalue of the Hessian. Infinite in this case. However in the finite dimensional situation

$$\dim(M(x,y)) = \mu(y) - \mu(x) = sf(Hess_{\gamma(t)} f)$$

and we can make sense in this case of the difference

$$\mu(\beta) - \mu(\alpha) = sf(EH_{\gamma(t)})$$

If $\gamma : S^1 \to \mathcal{B}_Q$ is a closed loop i.e. or $\tilde{\gamma} : [0,1] \to \mathcal{A}_Q$ and so that there is a gauge transformation with $g\tilde{\gamma}(0) = \tilde{\gamma}(1)$ then *sf* may be non-zero! Indeed let A be the connection in $S^1 \times Q$ with

$$(D_\gamma(t)) = \mathrm{Ind}(D_A) = 8k - \frac{3}{2}(0+0) = 8k = 8\deg(g).$$

N.B. if Q is an SO_3 bundle $k \in \frac{1}{2}\mathbb{Z}$.