Quantization and super-A-polynomials

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Trefoil knot, 3_1

$$P_n(\mathbf{3_1}; a, q, t) = \sum_{k=0}^{n-1} a^{n-1} t^{2k} q^{n(k-1)+1} \frac{(q^{n-1}, q^{-1})_k (-atq^{-1}, q)_k}{(q,q)_k}$$

Quantum super-A-polynomial:

$$\widehat{\mathbf{A}}^{\text{super}}(\hat{\mathbf{x}}, \hat{\mathbf{y}}; \mathbf{a}, \mathbf{q}, \mathbf{t}) = \mathbf{a}_0 + \mathbf{a}_1 \hat{\mathbf{y}} + \mathbf{a}_2 \hat{\mathbf{y}}^2$$

$$a_0 = \frac{a^2 t^4 (\hat{x} - 1)\hat{x}^3 (1 + aqt^3 \hat{x}^2)}{q(1 + at^3 \hat{x})(1 + at^3 q^{-1} \hat{x}^2)}$$

$$a_1 = -\frac{a(1 + at^3 \hat{x}^2)(q - q^2 t^2 \hat{x} + t^2 (q^2 + q^3 + (1 + q^2)at)\hat{x}^2 + aq^2 t^5 \hat{x}^3 + a^2 q t^6 \hat{x}^4)}{q^2 (1 + at^3 \hat{x})(1 + at^3 q^{-1} \hat{x}^2)}$$

$$a_2 = 1$$

Classical super-A-polynomial from $q \to 1$ limit (no factorization!):

$$A^{\text{super}}(x, y; a, t) = a^2 t^4 (x - 1) x^3 + (1 + at^3 x) y^2 +$$

$$-a(1 - t^2 x + t^2 (2 + 2at) x^2 + at^5 x^3 + a^2 t^6 x^4) y$$

Trefoil knot, 3_1

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Quantum super-A-polynomial:

$$\widehat{\mathbf{A}}^{\text{super}}(\hat{\mathbf{x}}, \hat{\mathbf{y}}; \mathbf{a}, \mathbf{q}, \mathbf{t}) = \mathbf{a}_0 + \mathbf{a}_1 \hat{\mathbf{y}} + \mathbf{a}_2 \hat{\mathbf{y}}^2$$

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$$-a\left(1 - t^2 x + t^2 (2 + 2at)x^2 + at^5 x^3 + a^2 t^6 x^4\right) y$$

Note: $A^{\text{super}}(x, y; 1, -1) = (1 - x)(y - 1)(y + x^3)$

Trefoil knot – asymptotics

$$P_n(\mathbf{3_1};a,q,t) \;\sim\; \int dz\; e^{\frac{1}{\hbar}\left(\widetilde{\mathcal{W}}(\mathbf{3_1};z,x)+\mathcal{O}(\hbar)\right)}$$

$$\widetilde{\mathcal{W}}(\mathbf{3_1};z,x) = -\frac{\pi^2}{6} + \big(\log z + \log a\big)\log x + 2(\log t)(\log z)$$
$$+\mathrm{Li}_2(xz^{-1}) - \mathrm{Li}_2(x) + \mathrm{Li}_2(-at) - \mathrm{Li}_2(-atz) + \mathrm{Li}_2(z)$$

Saddle point: $\left.\frac{\partial\widetilde{\mathcal{W}}(\mathbf{3_1};z,x)}{\partial z}\right|_{z=z_0} = 0, \qquad y = \exp\left(x\frac{\partial\widetilde{\mathcal{W}}(\mathbf{4_1};z_0,x)}{\partial x}\right)$

Eliminating z_0 gives the same $A^{\mathrm{super}}(x,y;a,t)$ as before!

$$P_n(\mathbf{3_1}; a, q, t) \sim \int dz\, e^{\frac{1}{\hbar}\left(\widetilde{\mathcal{W}}(\mathbf{3_1};z,x)+\mathcal{O}(\hbar)\right)}$$

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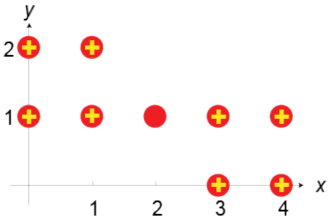


$$\begin{pmatrix} 0 & -a & 1 \\ 0 & a\,t^2 & a\,t^3 \\ 0 & -2\,a\,t^2 - 2\,a^2\,t^3 & 0 \\ -a^2\,t^4 & -a^2\,t^5 & 0 \\ a^2\,t^4 & -a^3\,t^6 & 0 \end{pmatrix}$$

Super-A-polynomial for $(2,5)$ torus knot

$$
\begin{pmatrix}
0 & 0 & -1 & 1 \\
0 & 0 & t^2 - t^3 & 2t^3 \\
0 & 0 & -2t^2 - 2t^3 + t^5 & t^6 \\
0 & 0 & 2t^4 - 2t^6 & 0 \\
0 & 0 & -3t^4 - 4t^5 - t^6 + 2t^7 + 2t^8 & 0 \\
0 & -2t^6 & -4t^7 - 3t^8 - t^9 & 0 \\
0 & 2t^6 + t^8 - t^9 & -2t^8 - t^{10} + t^{11} & 0 \\
0 & -4t^8 - 3t^9 - t^{10} & -2t^{11} & 0 \\
0 & 3t^8 + 4t^9 + t^{10} - 2t^{11} - 2t^{12} & 0 & 0 \\
0 & 2t^{11} - 2t^{13} & 0 & 0 \\
-t^{12} & 2t^{12} + 2t^{13} - t^{15} & 0 & 0 \\
2t^{12} & t^{15} - t^{16} & 0 & 0 \\
-t^{12} & t^{16} & 0 & 0
\end{pmatrix}
$$

Super-A-polynomial for $(2,7)$ torus knot

$$
\begin{pmatrix}
0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & -t^{2}+2t^{3} & -3t^{3} \\
0 & 0 & 0 & 2t^{2}+2t^{3}-2t^{5}+t^{6} & -3t^{6} \\
0 & 0 & 0 & -2t^{4}+2t^{5}+4t^{6}-t^{8} & -t^{9} \\
0 & 0 & 0 & 3t^{4}+4t^{5}+t^{6}-4t^{7}-2t^{8}+2t^{9} & 0 \\
0 & 0 & 0 & -3t^{6}+2t^{7}+7t^{8}+2t^{9}-2t^{10}-2t^{11} & 0 \\
0 & 0 & 0 & 4t^{6}+6t^{7}+2t^{8}-6t^{9}-5t^{10}+2t^{11}+t^{12} & 0 \\
0 & 0 & 3t^{8} & 9t^{9}+10t^{10}+4t^{11}-3t^{12}-4t^{13}-t^{14} & 0 \\
0 & 0 & -3t^{8}-2t^{10}+4t^{11} & 3t^{10}+6t^{12}+2t^{13}+2t^{14} & 0 \\
0 & 0 & 8t^{10}+4t^{11}+2t^{12}-2t^{13}+t^{14} & 6t^{13}+t^{15}-2t^{16} & 0 \\
0 & 0 & -6t^{10}-8t^{11}-5t^{12}+6t^{13}+8t^{14}+2t^{15} & 3t^{16} & 0 \\
0 & 0 & 9t^{12}+8t^{13}+t^{14}-t^{15}-2t^{16}+t^{17} & 0 & 0 \\
0 & 0 & -6t^{12}-12t^{13}-10t^{14}+8t^{15}+16t^{16}+8t^{17}+4t^{18} & 0 & 0 \\
0 & 0 & -9t^{15}-8t^{16}-t^{17}+t^{18}+2t^{19}-t^{20} & 0 & 0 \\
0 & 3t^{16} & -6t^{16}-8t^{17}-5t^{18}+6t^{19}+8t^{20}+2t^{21} & 0 & 0 \\
0 & -6t^{16}-t^{18}+2t^{19} & -8t^{19}-4t^{20}-2t^{21}+2t^{22}-t^{23} & 0 & 0 \\
0 & 3t^{16}+6t^{18}+2t^{19}+2t^{20} & -3t^{20}-2t^{22}+4t^{23} & 0 & 0 \\
0 & -9t^{18}-10t^{19}-4t^{20}+3t^{21}+4t^{22}+t^{23} & -3t^{23} & 0 & 0 \\
0 & 4t^{18}+6t^{19}+2t^{20}-6t^{21}-5t^{22}+2t^{23}+t^{24} & 0 & 0 & 0 \\
0 & 3t^{21}-2t^{22}-7t^{23}-2t^{24}+2t^{25}+2t^{26} & 0 & 0 & 0 \\
0 & 3t^{22}+4t^{23}+t^{24}-4t^{25}-2t^{26}+2t^{27} & 0 & 0 & 0 \\
t^{24} & 2t^{25}-2t^{26}-4t^{27}+t^{29} & 0 & 0 & 0 \\
-3t^{24} & 2t^{26}+2t^{27}-2t^{29}+t^{30} & 0 & 0 & 0 \\
3t^{24} & t^{29}-2t^{30} & 0 & 0 & 0 \\
-t^{24} & t^{30} & 0 & 0 & 0
\end{pmatrix}
$$

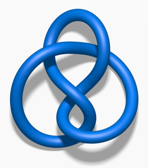
Super-A-polynomial for $(2,9)$ torus knot

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & t^{2}-3t^{3} & 4t^{3} \\
0 & 0 & 0 & 0 & -2t^{2}-2t^{3}+3t^{5}-3t^{6} & 6t^{6} \\
0 & 0 & 0 & 0 & 2t^{4}-4t^{5}-6t^{6}+3t^{8}-t^{9} & 4t^{9} \\
0 & 0 & 0 & 0 & -3t^{4}-4t^{5}-t^{6}+6t^{7}-6t^{9}+t^{11} & t^{12} \\
0 & 0 & 0 & 0 & 3t^{6}-5t^{7}-11t^{8}-3t^{9}+6t^{10}+4t^{11}-2t^{12} & 0 \\
0 & 0 & 0 & 0 & -4t^{6}-6t^{7}-2t^{8}+9t^{9}+3t^{10}-9t^{11}-3t^{12}+2t^{13}+2t^{14} & 0 \\
0 & 0 & 0 & 0 & 4t^{8}-6t^{9}-16t^{10}-6t^{11}+9t^{12}+9t^{13}-t^{14}-t^{15} & 0 \\
0 & 0 & 0 & 0 & -5t^{8}-8t^{9}-3t^{10}+12t^{11}+6t^{12}-12t^{13}-6t^{14}+3t^{15}+4t^{16}+t^{17} & 0 \\
0 & 0 & 0 & -4t^{10} & -16t^{11}-21t^{12}-9t^{13}+12t^{14}+14t^{15}-2t^{17} & 0 \\
0 & 0 & 0 & 4t^{10}+3t^{12}-9t^{13} & -4t^{12}-18t^{14}-15t^{15}-9t^{16}+4t^{17}+6t^{18}+2t^{19} & 0 \\
0 & 0 & 0 & -12t^{12}-3t^{13}-3t^{14}+6t^{15}-6t^{16} & -12t^{15}-8t^{17}+t^{18}-3t^{19} & 0 \\
0 & 0 & 0 & 9t^{12}+12t^{13}+9t^{14}-18t^{15}-18t^{16}-6t^{17}+3t^{18}-t^{19} & -12t^{18}-t^{20}+3t^{21} & 0 \\
0 & 0 & 0 & -18t^{14}-12t^{15}+6t^{16}+12t^{17}-11t^{19}-3t^{20} & -4t^{21} & 0 \\
0 & 0 & 0 & 12t^{14}+24t^{15}+24t^{16}-21t^{17}-45t^{18}-21t^{19}-3t^{20}+6t^{21} & 0 & 0 \\
0 & 0 & 0 & -16t^{16}-9t^{17}+15t^{18}+29t^{19}+9t^{20}-18t^{21}-12t^{22}-6t^{23} & 0 & 0 \\
0 & 0 & 0 & 10t^{16}+24t^{17}+27t^{18}-20t^{19}-51t^{20}-36t^{21}-12t^{22}+9t^{23}+6t^{24}+3t^{25} & 0 & 0 \\
0 & 0 & 0 & 24t^{19}+36t^{20}+30t^{21}-27t^{23}-21t^{24}-15t^{25}-3t^{26} & 0 & 0 \\
0 & 0 & -6t^{20} & 12t^{20}+18t^{21}+24t^{22}-3t^{23}-18t^{24}-4t^{25}+3t^{26}+6t^{27}+2t^{28} & 0 & 0 \\
0 & 0 & 12t^{20}+3t^{22}-9t^{23} & 27t^{23}+27t^{24}+16t^{25}-18t^{26}-18t^{27}-8t^{28} & 0 & 0 \\
0 & 0 & -6t^{20}-18t^{22}-6t^{24}+3t^{25}-3t^{26} & 6t^{24}+18t^{26}+6t^{28}-3t^{29}+3t^{30} & 0 & 0 \\
0 & 0 & 27t^{22}+27t^{23}+16t^{24}-18t^{25}-18t^{26}-8t^{27} & 12t^{27}+3t^{29}-9t^{30} & 0 & 0 \\
0 & 0 & -12t^{22}-18t^{23}-24t^{24}+3t^{25}+18t^{26}+4t^{27}-3t^{28}-6t^{29}-2t^{30} & 6t^{30} & 0 & 0 \\
0 & 0 & 24t^{24}+36t^{25}+30t^{26}-27t^{28}-21t^{29}-15t^{30}-3t^{31} & 0 & 0 & 0 \\
0 & 0 & -10t^{24}-24t^{25}-27t^{26}+20t^{27}+51t^{28}+36t^{29}+12t^{30}-9t^{31}-6t^{32}-3t^{33} & 0 & 0 & 0 \\
0 & 0 & -16t^{27}-9t^{28}+15t^{29}+29t^{30}+9t^{31}-18t^{32}-12t^{33}-6t^{34} & 0 & 0 & 0 \\
0 & 0 & -12t^{28}-24t^{29}-24t^{30}+21t^{31}+45t^{32}+21t^{33}+3t^{34}-6t^{35} & 0 & 0 & 0 \\
0 & -4t^{30} & -18t^{31}-12t^{32}+6t^{33}+12t^{34}-11t^{36}-3t^{37} & 0 & 0 & 0 \\
0 & 12t^{30}+t^{32}-3t^{33} & -9t^{32}-12t^{33}-9t^{34}+18t^{35}+18t^{36}+6t^{37}-3t^{38}+t^{39} & 0 & 0 & 0 \\
0 & -12t^{30}-8t^{32}+t^{33}-3t^{34} & -12t^{35}-3t^{36}-3t^{37}+6t^{38}-6t^{39} & 0 & 0 & 0 \\
0 & 4t^{30}+18t^{32}+15t^{33}+9t^{34}-4t^{35}-6t^{36}-2t^{37} & -4t^{36}-3t^{38}+9t^{39} & 0 & 0 & 0 \\
0 & -16t^{32}-21t^{33}-9t^{34}+12t^{35}+14t^{36}-2t^{38} & -4t^{39} & 0 & 0 & 0 \\
0 & 5t^{32}+8t^{33}+3t^{34}-12t^{35}-6t^{36}+12t^{37}+6t^{38}-3t^{39}-4t^{40}-t^{41} & 0 & 0 & 0 & 0 \\
0 & 4t^{35}-6t^{36}-16t^{37}-6t^{38}+9t^{39}+9t^{40}-t^{41}-t^{42} & 0 & 0 & 0 & 0 \\
0 & 4t^{36}+6t^{37}+2t^{38}-9t^{39}-3t^{40}+9t^{41}+3t^{42}-2t^{43}-2t^{44} & 0 & 0 & 0 & 0 \\
0 & 3t^{39}-5t^{40}-11t^{41}-3t^{42}+6t^{43}+4t^{44}-2t^{45} & 0 & 0 & 0 & 0 \\
-t^{40} & 3t^{40}+4t^{41}+t^{42}-6t^{43}+6t^{45}-t^{47} & 0 & 0 & 0 & 0 \\
4t^{40} & 2t^{43}-4t^{44}-6t^{45}+3t^{47}-t^{48} & 0 & 0 & 0 & 0 \\
-6t^{40} & 2t^{44}+2t^{45}-3t^{47}+3t^{48} & 0 & 0 & 0 & 0 \\
4t^{40} & t^{47}-3t^{48} & 0 & 0 & 0 & 0 \\
-t^{40} & t^{48} & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Superpolynomial for figure-8 knot

$$P_n(\mathbf{4_1}; a, q, t) =$$
$$= \sum_{k=0}^{\infty} (-1)^k a^{-k} t^{-2k} q^{-k(k-3)/2} \frac{(-atq^{-1}, q)_k}{(q,q)_k} (q^{1-n}, q)_k (-at^3 q^{n-1}, q)_k$$

n	$P_n(\mathbf{4_1}; a, q, t)$
1	1
2	$a^{-1}t^{-2} + t^{-1}q^{-1} + 1 + qt + at^2$
3	$a^{-2}q^{-2}t^{-4} + (a^{-1}q^{-3} + a^{-1}q^{-2})t^{-3} + (q^{-3} + a^{-1}q^{-1} + a^{-1})t^{-2} +$ $+(q^{-2} + q^{-1} + a^{-1} + a^{-1}q)t^{-1} + (q^{-1} + 3 + q) + (q^2 + q + a + aq^{-1})t +$ $+(q^3 + aq + a)t^2 + (aq^3 + aq^2)t^3 + a^2q^2t^4$



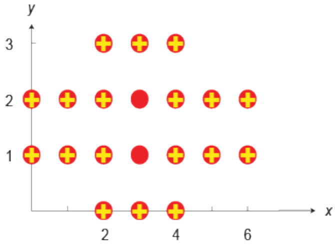
Super-A-polynomial for figure-8 knot

We find quantum curve:

$$\widehat{A}^{\text{super}}(\hat{x}, \hat{y}; a, q, t) = a_0 + a_1 \hat{y} + a_2 \hat{y}^2 + a_3 \hat{y}^3$$

Classicial limit and asymptotics:

$$A^{\text{super}}(x, y; a, t) \;=\; a^2 t^5 (x-1)^2 x^2 + a t^2 x^2 (1 + a t^3 x)^2 y^3 +$$
$$+ a t (x-1)(1 + t(1-t)x + 2 a t^3 (t+1) x^2 - 2 a t^4 (t+1) x^3 + a^2 t^6 (1-t) x^4 - a^2 t^8 x^5) y$$
$$-(1 + a t^3 x)(1 + a t (1-t) x + 2 a t^2 (t+1) x^2 + 2 a^2 t^4 (t+1) x^3 + a^2 t^5 (t-1) x^4 + a^3 t^7 x^5) y^2$$



Quantizability of $A(x, y)$

A-polynomials have very intricate structure!

$\hbar \log Z = S_0(u) + \ldots = \int \log y \frac{dx}{x} + \ldots$ must be well defined, irrespective of integration cycle, which implies:

$$\oint_\gamma \Big(\log |x| d(\arg y) - \log |y| d(\arg x) \Big) = 0$$

$$\frac{1}{4\pi^2} \oint_\gamma \Big(\log |x| d \log |y| + (\arg y) d(\arg x) \Big) \in \mathbb{Q}$$

Necessary condition from Newton polygon:

- write $A(x, y) = \sum_{i,j} a_{i,j} x^i y^j$
- construct face polynomials $f(z) = \sum_k a_k z^k$, for a_k along all walls
- find roots of all face polynomials

$A(x, y)$ is quantizable requires that all these roots are roots of unity

Quantizability of super-A-polynomials

How to reconcile quantizability constraints and seemingly aribitrary values of a and t ?!

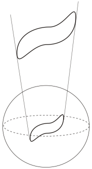
face	face polynomial for (2,2p+1) torus knot
first column	$-(at^2)^{p(p+1)}(z-1)^p$
last column	$(-1)^p(z+at^3)^p$
first row	za^p-1
last row	$-(at^2)^{p(p+1)}\big(z-(at^2)^p\big)$
lower diagonal	$(-1)^p\big(at^3\big)^p\big(z-a^{p+1}t^{2p+1}\big)^p$
upper diagonal	$(-1)^{p+1}a^p\big(z+a^pt^{2p+2}\big)^p$

Quantizability of A^{super} requires that a and t are roots of unity!

Consistent with other examples, as well as $a=q^N$!

Brane system and topological strings

space-time	:	$\mathbb{R} \times \mathbb{R}^4 \times T^*\mathbf{S}^3$
		$\cup \qquad \cup$
N M5-branes	:	$\mathbb{R} \times \mathbb{R}^2 \times \mathbf{S}^3$
$\lvert R \rvert$ M5-branes	:	$\mathbb{R} \times \mathbb{R}^2 \times L_K$



After geometric transition we obtain resolved conifold, with **framed** brane amplitude $\psi^{\mathrm{ref}}(x)$ computed by refined topological vertex, which satisfies **(in)homogeneous** difference equation (Iqbal-Kozcaz-Vafa, Fuji-Gukov-P.S.):

$$\Big(1 - \frac{q_1}{q_2}\hat{y} + \frac{q_1}{\sqrt{q_2}}\hat{x}(-\hat{y})^{f} + Qq_1^{1/2}\hat{x}(-\hat{y})^{f+1}\Big)\psi^{\mathrm{ref}}(x) \;=\; 1 - \frac{q_1}{q_2}$$

$\psi^{\mathrm{ref}}(x)$ can be interpreted as **unknot** superpolynomial in "Macdonald" basis (Iqbal-Kozcaz, 2011)

Dual 3d, N=2 theory associated to the knot complement

chiral field ϕ $\longleftrightarrow$ twisted superpotential $\Delta\widetilde{\mathcal{W}} = \mathrm{Li}_2\Big((-t)^{n_F} \prod_i (x_i)^{n_i}\Big)$

Therefore the **spectrum** of the theory can be read off from $\widetilde{\mathcal{W}}$:

$$\widetilde{\mathcal{W}}(\mathbf{3_1}; z, x) = \mathrm{Li}_2(xz^{-1}) - \mathrm{Li}_2(x) + \mathrm{Li}_2(-at) - \mathrm{Li}_2(-atz) + \mathrm{Li}_2(z) + \dots$$

$\mathbf{3_1}$**knot**	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	parameter
$U(1)_{\text{gauge}}$	-1	0	0	-1	1	z
$U(1)_F$	0	0	1	-1	0	$-t$
$U(1)_Q$	0	0	1	-1	0	a
$U(1)_L$	1	-1	0	0	0	x

- **SUSY vacua:** extremize with respect to dynamical fields: $\frac{\partial\widetilde{\mathcal{W}}}{\partial z_i} = 0$
- therefore super-A-polynomial describes **SUSY vacua** of $T_{M=\mathbb{S}^3\backslash K}$

Theories dual to $(\mathbf{2p+1})_1$ knots

Superpolynomial for theory A, from refined Chern-Simons theory:

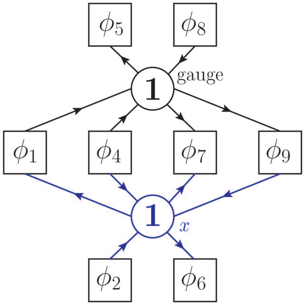
$$P^{S^r} = \sum_{\ell=0}^{r} \frac{(qt^2;q)_\ell(-at^3;q)_{r+\ell}(-aq^{-1}t;q)_{r-\ell}(q;q)_r}{(q;q)_\ell(q^2t^2;q)_{r+\ell}(q;q)_{r-\ell}(-at^3;q)_r} \frac{(1-q^{2\ell+1}t^2)}{(1-qt^2)} \times(-1)^r a^{-\frac{r}{2}} q^{\frac{3r}{2}-\ell} t^{-rp-\ell+\frac{r}{2}} \left[(-1)^\ell a^{\frac{r}{2}} q^{\frac{r^2-\ell(\ell+1)}{2}} t^{\frac{3r}{2}-\ell}\right]^{2p+1}$$

The same superpolynomial, but for theory B, from differentials ($k_0 \equiv r$):

$$P^{S^r} = a^{pr}q^{-pr} \sum_{0\le k_p\le\ldots\le k_2\le k_1\le r} \begin{bmatrix} r \\ k_1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \cdots \begin{bmatrix} k_{p-1} \\ k_p \end{bmatrix} \times \times\ q^{(2r+1)(k_1+k_2+\ldots+k_p)-\sum_{i=1}^{p} k_{i-1}k_i} t^{2(k_1+k_2+\ldots+k_p)} \prod_{i=1}^{k_1}(1+aq^{i-2}t)$$

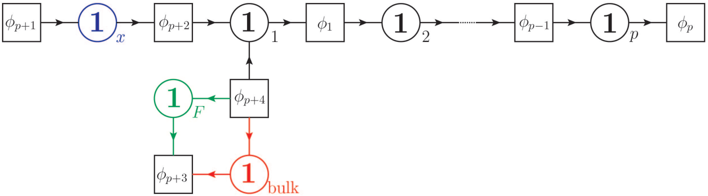
Theory A for $(\mathbf{2p+1})_\mathbf{1}$ knots, from refined Chern-Simons

	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9	parameter
$U(1)_{\text{gauge}}$	-1	0	0	-1	1	0	1	-1	1	z
$U(1)_F$	0	0	1	-3	0	3	2	-2	-1	$-t$
$U(1)_Q$	0	0	1	-1	0	1	0	0	-1	a
$U(1)_x$	1	-1	0	-1	0	1	1	0	-1	x

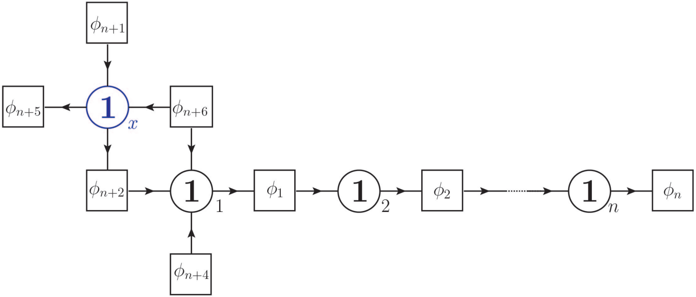


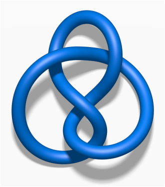
Theory B for $(\mathbf{2p+1})_1$ knots, from differentials

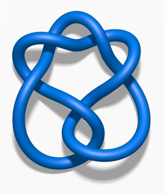
	ϕ_1	ϕ_2	$\cdots$	ϕ_{p-1}	ϕ_p	ϕ_{p+1}	ϕ_{p+2}	ϕ_{p+3}	ϕ_{p+4}	
$U(1)_{\text{gauge},1}$	1	0	$\cdots$	0	0	0	-1	0	-1	z_1
$U(1)_{\text{gauge},2}$	-1	1	$\cdots$	$\vdots$	$\vdots$	0	0	0	0	z_2
$\vdots$	0	-1	$\ddots$	0	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$		1	0	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$U(1)_{\text{gauge},p}$	0	0	$\cdots$	-1	1	0	0	0	0	z_p
$U(1)_F$	0	0	$\cdots$	0	0	0	0	1	-1	$-t$
$U(1)_Q$	0	0	$\cdots$	0	0	0	0	1	-1	a
$U(1)_x$	0	0	$\cdots$	0	0	-1	1	0	0	x

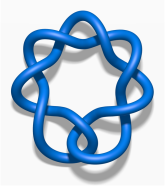


Quiver for T_K theory for twist knots, $K = (2n+2)_1$









Summary

Discussed today...

Homological knot invariants governed by classical and quantum **super-A-polynomial**, $\widehat{A}^{\text{super}}(\hat{x}, \hat{y}; a, q, t)$

- which encodes color dependence of knot superpolynomials...
- ...including their t-deformation and Q-deformation
- ...and also encodes information about a dual 3d theory

To be done...

- Find A^{super} for other knots
- Understand the structure and properties of A^{super}
- Consider different gauge groups, spacetimes, representations, etc.
- $\widehat{A}^{\mathrm{super}}$ from gluing? topological recursion?
- implications for dual 3d N=2 theories?
- ...and many more...

Thank you!

MARIE CURIE ACTIONS

FNP Foundation for Polish Science