Associative deformations of singular spaces

Victor Palamodov

Tel Aviv University

Aarhus 13-17 December 2010

Introduction

- Basic language for the theory of deformation: flatness, pull back, infinitesimals, versality - is proposed in Grothendieck's papers of 1960-61 (after seminal papers of Kodaira-Spencer 1956-58).

Introduction

- Basic language for the theory of deformation: flatness, pull back, infinitesimals, versality - is proposed in Grothendieck's papers of 1960-61 (after seminal papers of Kodaira-Spencer 1956-58).
- Similarity of deformation-quantizations and deformations in the commutative geometry was stressed by Sternheimer.

Introduction

- Basic language for the theory of deformation: flatness, pull back, infinitesimals, versality - is proposed in Grothendieck's papers of 1960-61 (after seminal papers of Kodaira-Spencer 1956-58).
- Similarity of deformation-quantizations and deformations in the commutative geometry was stressed by Sternheimer.
- The unifying concept is the notion of *associative deformation* of a complex analytic space.

Outlook

- Analytic algebras, Analytic spaces,

Outlook

- Analytic algebras, Analytic spaces,
- Resolution, Deformation complex,

Outlook

- Analytic algebras, Analytic spaces,
- Resolution, Deformation complex,
- Associative cohomology, Master equation,

Outlook

- Analytic algebras, Analytic spaces,
- Resolution, Deformation complex,
- Associative cohomology, Master equation,
- Versal deformation, Examples

Analytic algebras

- Let $\mathcal{O}_n = \mathbb{C}\{z_1, ..., z_n\}$ be the algebra of convergent power series of n complex variables with coeffcients in $\mathbb{C}$.

Analytic algebras

- Let $\mathcal{O}_n = \mathbb{C}\{z_1, ..., z_n\}$ be the algebra of convergent power series of n complex variables with coeffcients in $\mathbb{C}$.
- A regular analytic algebra is a $\mathbb{C}$-algebra isomorphic to $\mathcal{O}_n$ for some n. An analytic algebra $\mathcal{A}$ is a quotient $\mathcal{A} = \mathcal{O}_n/\mathcal{I}$. It is a local algebra with the maximal ideal $\mathfrak{m}(\mathcal{A}) = \mathfrak{m}(\mathcal{O}_n)/\mathcal{I}$.

Analytic algebras

- Let $\mathcal{O}_n = \mathbb{C}\{z_1, ..., z_n\}$ be the algebra of convergent power series of n complex variables with coeffcients in $\mathbb{C}$.
- A regular analytic algebra is a $\mathbb{C}$-algebra isomorphic to $\mathcal{O}_n$ for some n. An analytic algebra $\mathcal{A}$ is a quotient $\mathcal{A} = \mathcal{O}_n / \mathcal{I}$. It is a local algebra with the maximal ideal $\mathfrak{m}(\mathcal{A}) = \mathfrak{m}(\mathcal{O}_n) / \mathcal{I}$.
- The algebra $\mathcal{A}$ defines the analytic germ $S(\mathcal{A})$ in $\mathbb{C}^n$ given by

$$f_1(z) = ... = f_m(z) = 0$$

where $(f_1, ..., f_m)$ are generators of the ideal $\mathcal{I}$

Analytic algebras

- Let $\mathcal{O}_n = \mathbb{C}\{z_1, ..., z_n\}$ be the algebra of convergent power series of n complex variables with coeffcients in $\mathbb{C}$.
- A regular analytic algebra is a $\mathbb{C}$-algebra isomorphic to $\mathcal{O}_n$ for some n. An analytic algebra $\mathcal{A}$ is a quotient $\mathcal{A} = \mathcal{O}_n/\mathcal{I}$. It is a local algebra with the maximal ideal $\mathfrak{m}(\mathcal{A}) = \mathfrak{m}(\mathcal{O}_n)/\mathcal{I}$.
- The algebra $\mathcal{A}$ defines the analytic germ $S(\mathcal{A})$ in $\mathbb{C}^n$ given by

$$f_1(z) = ... = f_m(z) = 0$$

  where $(f_1, ..., f_m)$ are generators of the ideal $\mathcal{I}$
- $\mathcal{A}$ is called Artin algebra, if $\mathfrak{m}^k(\mathcal{A}) = 0$ for some $k > 0$, the germ $S(\mathcal{A})$ is a infinitesimal extension of the point $(\bullet, \mathbb{C})$.

Analytic algebras (cont.)

- A morphism of germs $(R, \mathcal{O}_R) \to (S, \mathcal{O}_S)$ is a continuous map $f : (R, \bullet) \to (S, \bullet)$ together with a homomorphism $\mathbb{C}$-algebras : $\varphi : \mathcal{O}_S \to \mathcal{O}_R$ such that $\varphi a(r) = a(f(r)), r \in R$.

Analytic algebras (cont.)

- A morphism of germs $(R, \mathcal{O}_R) \to (S, \mathcal{O}_S)$ is a continuous map $f : (R, \bullet) \to (S, \bullet)$ together with a homomorphism $\mathbb{C}$-algebras : $\varphi : \mathcal{O}_S \to \mathcal{O}_R$ such that $\varphi a(r) = a(f(r)), r \in R$.
- The morphism (f, φ) is called *flat*, if $\mathcal{O}_R$ is flat as $\mathcal{O}_S$-module via φ. The equation $\mathrm{Tor}_1^{\mathcal{O}_S}(\mathcal{O}_R, \mathbb{C}) = 0$ is sufficient for flatness..

Analytic algebras (cont.)

- A morphism of germs $(R, \mathcal{O}_R) \to (S, \mathcal{O}_S)$ is a continuous map $f : (R, \bullet) \to (S, \bullet)$ together with a homomorphism $\mathbb{C}$-algebras : $\varphi : \mathcal{O}_S \to \mathcal{O}_R$ such that $\varphi a(r) = a(f(r))$, $r \in R$.
- The morphism (f, φ) is called *flat*, if $\mathcal{O}_R$ is flat as $\mathcal{O}_S$-module via φ. The equation $\mathrm{Tor}_1^{\mathcal{O}_S}(\mathcal{O}_R, \mathbb{C}) = 0$ is sufficient for flatness..
- If φ is a surjection the morphism (f, φ) is called imbedding and $\mathcal{O}_S$ is extension of $\mathcal{O}_R$ by means of the ideal $J = \mathrm{Ker}\varphi$.

Analytic algebras (cont.)

- A morphism of germs $(R, \mathcal{O}_R) \to (S, \mathcal{O}_S)$ is a continuous map $f : (R, \bullet) \to (S, \bullet)$ together with a homomorphism $\mathbb{C}$-algebras : $\varphi : \mathcal{O}_S \to \mathcal{O}_R$ such that $\varphi a(r) = a(f(r)), r \in R$.
- The morphism (f, φ) is called *flat*, if $\mathcal{O}_R$ is flat as $\mathcal{O}_S$-module via φ. The equation $\mathrm{Tor}_1^{\mathcal{O}_S}(\mathcal{O}_R, \mathbb{C}) = 0$ is sufficient for flatness..
- If φ is a surjection the morphism (f, φ) is called imbedding and $\mathcal{O}_S$ is extension of $\mathcal{O}_R$ by means of the ideal $J = \mathrm{Ker}\varphi$.
- The morphism (f, φ) is called *small* extension, if $\mathfrak{m}(\mathcal{O}_S) J = 0$.

Analytic algebras (cont.)

- A morphism of germs $(R, \mathcal{O}_R) \to (S, \mathcal{O}_S)$ is a continuous map $f : (R, \bullet) \to (S, \bullet)$ together with a homomorphism $\mathbb{C}$-algebras : $\varphi : \mathcal{O}_S \to \mathcal{O}_R$ such that $\varphi a(r) = a(f(r)), r \in R$.
- The morphism (f, φ) is called *flat*, if $\mathcal{O}_R$ is flat as $\mathcal{O}_S$-module via φ. The equation $\mathrm{Tor}_1^{\mathcal{O}_S}(\mathcal{O}_R, \mathbb{C}) = 0$ is sufficient for flatness..
- If φ is a surjection the morphism (f, φ) is called imbedding and $\mathcal{O}_S$ is extension of $\mathcal{O}_R$ by means of the ideal $J = \mathrm{Ker}\varphi$.
- The morphism (f, φ) is called *small* extension, if $\mathfrak{m}(\mathcal{O}_S) J = 0$.
- An algebra is called small if it is a small extension of the simple point $(\bullet, \mathbb{C})$.

Complex analytic spaces

- For an open set $U \subset \mathbb{C}^n$ the sheaf of germs of holomorphic functions in U is denoted $\mathcal{O}_U$.

Complex analytic spaces

- For an open set $U \subset \mathbb{C}^n$ the sheaf of germs of holomorphic functions in U is denoted $\mathcal{O}_U$.
- For any sections $f_1, ..., f_m$ of $\mathcal{O}_U$ the analytic set $X_f = \{z \in U;\ f_1(z) = ... = f_m(z) = 0\}$ supplied with the quotient sheaf $\mathcal{O}_U / (f_1, ..., f_m)$ is a model c.a.s..

Complex analytic spaces

- For an open set $U \subset \mathbb{C}^n$ the sheaf of germs of holomorphic functions in U is denoted $\mathcal{O}_U$.
- For any sections $f_1, ..., f_m$ of $\mathcal{O}_U$ the analytic set $X_f = \{z \in U;\ f_1(z) = ... = f_m(z) = 0\}$ supplied with the quotient sheaf $\mathcal{O}_U / (f_1, ..., f_m)$ is a model c.a.s..
- A complex analytic space is a topological space with a sheaf of local $\mathbb{C}$-algebras that is locally isomorphic to a model space X_f.

Complex analytic spaces

- For an open set $U \subset \mathbb{C}^n$ the sheaf of germs of holomorphic functions in U is denoted $\mathcal{O}_U$.
- For any sections $f_1, ..., f_m$ of $\mathcal{O}_U$ the analytic set $X_f = \{z \in U;\ f_1(z) = ... = f_m(z) = 0\}$ supplied with the quotient sheaf $\mathcal{O}_U / (f_1, ..., f_m)$ is a model c.a.s..
- A complex analytic space is a topological space with a sheaf of local $\mathbb{C}$-algebras that is locally isomorphic to a model space X_f.
- A morphism of complex spaces $(X, \mathcal{O}_x) \to (Y, \mathcal{O}_y)$ is a pair (f, φ), where $f : X \to Y$ is a continuous map and $\varphi : f^*(\mathcal{O}_Y) \to \mathcal{O}_X$ is a homomorphism of sheaves of $\mathbb{C}$-algebras on X.

Complex analytic spaces

- For an open set $U \subset \mathbb{C}^n$ the sheaf of germs of holomorphic functions in U is denoted $\mathcal{O}_U$.
- For any sections $f_1, ..., f_m$ of $\mathcal{O}_U$ the analytic set $X_f = \{z \in U;\ f_1(z) = ... = f_m(z) = 0\}$ supplied with the quotient sheaf $\mathcal{O}_U / (f_1, ..., f_m)$ is a model c.a.s..
- A complex analytic space is a topological space with a sheaf of local $\mathbb{C}$-algebras that is locally isomorphic to a model space X_f.
- A morphism of complex spaces $(X, \mathcal{O}_x) \to (Y, \mathcal{O}_y)$ is a pair (f, φ), where $f : X \to Y$ is a continuous map and $\varphi : f^*(\mathcal{O}_Y) \to \mathcal{O}_X$ is a homomorphism of sheaves of $\mathbb{C}$-algebras on X.
- The cartesian diagram

$$\begin{array}{ccc} X & \to & \bullet \\ \downarrow & & \downarrow \\ \mathcal{X} & \overset{f}{\to} & S \end{array}$$

is called (commutative) deformation of X over the germ $(S, \bullet)$ if f is flat at each $x \in X, f(x) = \bullet$.

Deformation

- Let $(X, \mathcal{O}_X)$ be a c.a.s. and $(S, \mathcal{O}_S)$ be an Artin germ. An *associative deformation* of X with the base S is called any $\mathcal{O}_S$-sheaf $\mathcal{A}$ on X endowed with an associative $\mathcal{O}_S$-bilinear operation $\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$

Deformation

- Let $(X, \mathcal{O}_X)$ be a c.a.s. and $(S, \mathcal{O}_S)$ be an Artin germ. An *associative deformation* of X with the base S is called any $\mathcal{O}_S$-sheaf $\mathcal{A}$ on X endowed with an associative $\mathcal{O}_S$-bilinear operation $\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$
- and a isomorphism $\alpha : \mathcal{A} \otimes \mathbb{C} \cong \mathcal{O}_X$ of $\mathbb{C}$-sheaves on X such that

Deformation

- Let $(X, \mathcal{O}_X)$ be a c.a.s. and $(S, \mathcal{O}_S)$ be an Artin germ. An *associative deformation* of X with the base S is called any $\mathcal{O}_S$-sheaf $\mathcal{A}$ on X endowed with an associative $\mathcal{O}_S$-bilinear operation $\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$
- and a isomorphism $\alpha : \mathcal{A} \otimes \mathbb{C} \cong \mathcal{O}_X$ of $\mathbb{C}$-sheaves on X such that
- (i) $\mathcal{O}_S$-sheaf $\mathcal{A}$ is flat,

Deformation

- Let $(X, \mathcal{O}_X)$ be a c.a.s. and $(S, \mathcal{O}_S)$ be an Artin germ. An *associative deformation* of X with the base S is called any $\mathcal{O}_S$-sheaf $\mathcal{A}$ on X endowed with an associative $\mathcal{O}_S$-bilinear operation $\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$
- and a isomorphism $\alpha : \mathcal{A} \otimes \mathbb{C} \cong \mathcal{O}_X$ of $\mathbb{C}$-sheaves on X such that
- (i) $\mathcal{O}_S$-sheaf $\mathcal{A}$ is flat,
- (ii) the diagram is commutative:

$$\begin{array}{cccc} \mu \otimes \mathbb{C} : & \mathcal{A} \otimes_{O_S} \mathbb{C} \times \mathcal{A} \otimes_{O_S} \mathbb{C} & \to & \mathcal{A} \otimes_{O_S} \mathbb{C} \\ & \alpha \times \alpha \downarrow & & \alpha \downarrow \\ & \mathcal{O}_X \times \mathcal{O}_X & \to & \mathcal{O}_X \end{array}$$

Deformation

- Let $(X, \mathcal{O}_X)$ be a c.a.s. and $(S, \mathcal{O}_S)$ be an Artin germ. An *associative deformation* of X with the base S is called any $\mathcal{O}_S$-sheaf $\mathcal{A}$ on X endowed with an associative $\mathcal{O}_S$-bilinear operation $\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$
- and a isomorphism $\alpha : \mathcal{A} \otimes \mathbb{C} \cong \mathcal{O}_X$ of $\mathbb{C}$-sheaves on X such that
- (i) $\mathcal{O}_S$-sheaf $\mathcal{A}$ is flat,
- (ii) the diagram is commutative:

$$\begin{array}{rccc} \mu \otimes \mathbb{C} : & \mathcal{A} \otimes_{\mathrm{O}_S} \mathbb{C} \times \mathcal{A} \otimes_{\mathrm{O}_S} \mathbb{C} & \rightarrow & \mathcal{A} \otimes_{\mathrm{O}_S} \mathbb{C} \\ & \alpha \times \alpha \downarrow & & \alpha \downarrow \\ & \mathcal{O}_X \times \mathcal{O}_X & \rightarrow & \mathcal{O}_X \end{array}$$

- (iii) For any $x \in X$ there exist a nbd Y, a proper embedding $\phi : Y \to U \subset \mathbb{C}^n$, and a $\mathcal{O}_S$-surjection

$$\pi_S : \mathcal{O}_{U \times S} \to \phi_* (\mathcal{A} | Y)$$

such that $\operatorname{Ker} \pi_S$ is a coherent $\mathcal{O}_{U \times S}$-subsheaf.

Deformation (cont.)

- Because of α each fiber $\mathcal{A}_x, x \in X$ is a local $\mathbb{C}$-algebra with the maximal two-side ideal $M_x = \rho^{-1}\left(\mathfrak{m}\left(\mathcal{O}_{X,x}\right)\right)$

$$\begin{array}{ccc} M_x & \rightarrow & \mathfrak{m}\left(\mathcal{O}_{X,x}\right) \\ \downarrow & & \downarrow \\ \mathcal{A} & \stackrel{\alpha_\bullet}{\rightarrow} & \mathcal{O}_{X,x} \end{array}$$

where $\alpha_\bullet$ is the surjection generated by α.

Deformation (cont.)

- Because of α each fiber $\mathcal{A}_x, x \in X$ is a local $\mathbb{C}$-algebra with the maximal two-side ideal $M_x = \rho^{-1}\left(\mathfrak{m}\left(\mathcal{O}_{X,x}\right)\right)$

$$\begin{array}{ccc} M_x & \rightarrow & \mathfrak{m}\left(\mathcal{O}_{X,x}\right) \\ \downarrow & & \downarrow \\ \mathcal{A} & \stackrel{\alpha_\bullet}{\rightarrow} & \mathcal{O}_{X,x} \end{array}$$

  where $\alpha_\bullet$ is the surjection generated by α.
- (iv) (continuity) that for any compact set $K \subset X$ and a number k there exists a number l such that for any $x \in K$ we have $\mu(a,b) \in M_x^k$ if $ab \in M_x^l$.

Generalities

- Deformations $(X, S, \mathcal{A}, \mu_{\mathcal{A}})$ and $(X, S, \mathcal{B}, \mu_{\mathcal{B}})$ of X with the same base S are *isomorph*, if there exist a isomorphism of $\mathcal{O}_S$-sheaves of algebras $\varepsilon : \mathcal{B} \to \mathcal{A}$ on X such that

$$\mu_{\mathcal{A}}(\varepsilon(b_1), \varepsilon(b_2)) = \varepsilon\mu_{\mathcal{B}}(b_1, b_2)$$

for arbitrary $b_1, b_2 \in \mathcal{B}$.

Generalities

- Deformations $(X, S, \mathcal{A}, \mu_{\mathcal{A}})$ and $(X, S, \mathcal{B}, \mu_{\mathcal{B}})$ of X with the same base S are *isomorph*, if there exist a isomorphism of $\mathcal{O}_S$-sheaves of algebras $\varepsilon : \mathcal{B} \to \mathcal{A}$ on X such that

$$\mu_{\mathcal{A}}\left(\varepsilon\left(b_1\right), \varepsilon\left(b_2\right)\right) = \varepsilon\mu_{\mathcal{B}}\left(b_1, b_2\right)$$

  for arbitrary $b_1, b_2 \in \mathcal{B}$.
- *Pull back*. Let $F = (X, S, \mathcal{A}, \mu)$ be a deformation of X and : $\phi : (R, \mathcal{O}_R) \to (S, \mathcal{O}_S)$ be a morphism of germs. The product $\mathcal{B} = \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_R$ is a $\mathcal{O}_R$-sheaf. The bilinear operation $\mu_R = \mu \otimes \mathcal{O}_R$ is well defined in $\mathcal{B}$ and fulfils $\mu_R \otimes \mathbb{C} = \mu \otimes \mathbb{C}$. The quadruple $\phi^*(F) \doteq (X, R, \mathcal{B}, \mu_R)$ is a deformation of X.

Generalities

- Deformations $(X, S, \mathcal{A}, \mu_{\mathcal{A}})$ and $(X, S, \mathcal{B}, \mu_{\mathcal{B}})$ of X with the same base S are *isomorph*, if there exist a isomorphism of $\mathcal{O}_S$-sheaves of algebras $\varepsilon : \mathcal{B} \to \mathcal{A}$ on X such that

$$\mu_{\mathcal{A}}\left(\varepsilon\left(b_1\right), \varepsilon\left(b_2\right)\right) = \varepsilon\mu_{\mathcal{B}}\left(b_1, b_2\right)$$

  for arbitrary $b_1, b_2 \in \mathcal{B}$.
- *Pull back*. Let $F = (X, S, \mathcal{A}, \mu)$ be a deformation of X and : $\phi : (R, \mathcal{O}_R) \to (S, \mathcal{O}_S)$ be a morphism of germs. The product $\mathcal{B} = \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{O}_R$ is a $\mathcal{O}_R$-sheaf. The bilinear operation $\mu_R = \mu \otimes \mathcal{O}_R$ is well defined in $\mathcal{B}$ and fulfils $\mu_R \otimes \mathbb{C} = \mu \otimes \mathbb{C}$. The quadruple $\phi^*(F) \doteq (X, R, \mathcal{B}, \mu_R)$ is a deformation of X.
- A *formal* germ $\tilde{S}$ is the direct limit a sequence of imbeddings of Artin germs:

$$\phi_k : S_k \to S_{k+1},\ k = 0, 1, 2, \ldots$$

  up to a natural equivalence relation.

Versal deformations

- A *formal deformation* of a space X is the direct limit $\tilde{F} \to \tilde{S}$ of the diagram

$$\begin{array}{ccccccc} \ldots \to & F_k & \to & F_{k+1} & \to \ldots \\ & \downarrow & & \downarrow & \\ \ldots \to & S_k & \stackrel{\phi_k}{\to} & S_{k+1} & \to \ldots \end{array}$$

Each square is cartesian, that is $\phi_k^* \left(F_{k+1} \right) \cong F_k$ for any k.

Versal deformations

- A *formal deformation* of a space X is the direct limit $\tilde{F} \to \tilde{S}$ of the diagram
$$\begin{array}{ccccccc} \ldots \to & F_k & \to & F_{k+1} & \to \ldots \\ & \downarrow & & \downarrow & \\ \ldots \to & S_k & \stackrel{\phi_k}{\to} & S_{k+1} & \to \ldots \end{array}$$
Each square is cartesian, that is $\phi_k^* (F_{k+1}) \cong F_k$ for any k.
- A morphism of formal germs $\psi : \tilde{S} \to \tilde{T}$ is the limit of commutative diagram of morphisms of Artin germs
$$\psi_k : S_k \to T_k, k = 0, 1, \ldots$$

Versal deformations

- A *formal deformation* of a space X is the direct limit $\tilde{F} \to \tilde{S}$ of the diagram

$$\begin{array}{ccccccc} \ldots \to & F_k & \to & F_{k+1} & \to \ldots \\ & \downarrow & & \downarrow & \\ \ldots \to & S_k & \xrightarrow{\phi_k} & S_{k+1} & \to \ldots \end{array}$$

  Each square is cartesian, that is $\phi_k^* \left(F_{k+1}\right) \cong F_k$ for any k.
- A morphism of formal germs $\psi : \tilde{S} \to \tilde{T}$ is the limit of commutative diagram of morphisms of Artin germs

$$\psi_k : S_k \to T_k, k = 0, 1, \ldots$$

- A formal deformation $\tilde{F} = \left(X, \tilde{S}, \mathcal{A}, \mu\right)$ is called *versal*, if for any deformation $\tilde{G}$ of X with a formal base $\tilde{R}$ there exist a morphism of formal germs :$\psi : \tilde{R} \to \tilde{S}$ and a isomorphism $\tilde{G} \cong \psi^* \left(\tilde{F}\right)$ over $\tilde{R}$. It is called *minimal* if the dimension of the base is minimal.

Versal deformations

- A *formal deformation* of a space X is the direct limit $\tilde{F} \rightarrow \tilde{S}$ of the diagram

$$\begin{array}{ccccccc} \ldots \rightarrow & F_k & \rightarrow & F_{k+1} & \rightarrow \ldots \\ & \downarrow & & \downarrow & \\ \ldots \rightarrow & S_k & \stackrel{\phi_k}{\rightarrow} & S_{k+1} & \rightarrow \ldots \end{array}$$

Each square is cartesian, that is $\phi_k^* \left(F_{k+1}\right) \cong F_k$ for any k.

- A morphism of formal germs $\psi : \tilde{S} \rightarrow \tilde{T}$ is the limit of commutative diagram of morphisms of Artin germs

$$\psi_k : S_k \rightarrow T_k, k = 0, 1, \ldots$$

- A formal deformation $\tilde{F} = \left(X, \tilde{S}, \mathcal{A}, \mu\right)$ is called *versal*, if for any deformation $\tilde{G}$ of X with a formal base $\tilde{R}$ there exist a morphism of formal germs :$\psi : \tilde{R} \rightarrow \tilde{S}$ and a isomorphism $\tilde{G} \cong \psi^* \left(\tilde{F}\right)$ over $\tilde{R}$. It is called *minimal* if the dimension of the base is minimal.
- The versal deformation is called *universal* if the morphism ψ is uniquely defined.

Linearization

- The set $T(S)$ of all $\mathbb{C}$-derivations $t : \mathcal{O}_S \to \mathbb{C}$ is called the tangent space to the germ S.

Linearization

- The set $T(S)$ of all $\mathbb{C}$-derivations $t : \mathcal{O}_S \to \mathbb{C}$ is called the tangent space to the germ S.
- Let Δ be the Artin germ $(\bullet, \mathcal{O}_\Delta)$ with the algebra $\mathcal{O}_\Delta = \mathbb{C}\{z\}/\left(z^2\right)$. The tangent space to Δ is spanned by the vector $d/dz : a + bz \mapsto b$. Any morphism of analytic germs $\delta_t : \Delta \to S$ defines a derivation $t : \mathcal{O}_S \to \mathbb{C}$ which is the composition of the homomorphism $\mathcal{O}_S \to \mathcal{O}_\Delta$ and of the vector d/dz (and vice versa).

Linearization

- The set $T(S)$ of all $\mathbb{C}$-derivations $t : \mathcal{O}_S \to \mathbb{C}$ is called the tangent space to the germ S.
- Let Δ be the Artin germ $(\bullet, \mathcal{O}_\Delta)$ with the algebra $\mathcal{O}_\Delta = \mathbb{C}\{z\}/\left(z^2\right)$. The tangent space to Δ is spanned by the vector $d/dz : a + bz \mapsto b$. Any morphism of analytic germs $\delta_t : \Delta \to S$ defines a derivation $t : \mathcal{O}_S \to \mathbb{C}$ which is the composition of the homomorphism $\mathcal{O}_S \to \mathcal{O}_\Delta$ and of the vector d/dz (and vice versa).
- Let $\mathrm{D}(X)$ be the set of isomorphism classes of deformations of a space X with the base Δ. For an arbitrary deformation F of X with a base S and a tangent vector t to S the pull back $\delta_t^*(F)$ a deformation with the base Δ. The map

$$\mathrm{D}F : T(S) \to \mathrm{D}(X), \ t \mapsto \mathrm{cl}\left(\delta_t^*(F)\right)$$

is linear for a natural vector space structure in $\mathrm{D}(X)$ (generalized Kodaira-Spencer map).

Resolution of analytic algebra

- A free graded differential analytic sheaf-algebra is a graded sheaf $\mathcal{R}^* = \oplus_{k \leq 0} \mathcal{R}^k$ with a grading $|\cdot|$ and a differential d of grading 1 such that:

Resolution of analytic algebra

- A free graded differential analytic sheaf-algebra is a graded sheaf $\mathcal{R}^* = \oplus_{k \leq 0} \mathcal{R}^k$ with a grading $|\cdot|$ and a differential d of grading 1 such that:
- (i) $\mathcal{R}$ has a structure of graded commutative algebra;

Resolution of analytic algebra

- A free graded differential analytic sheaf-algebra is a graded sheaf $\mathcal{R}^* = \oplus_{k \leq 0} \mathcal{R}^k$ with a grading $|\cdot|$ and a differential d of grading 1 such that:
- (i) $\mathcal{R}$ has a structure of graded commutative algebra;
- (ii) $\mathcal{R}^0 = \mathcal{O}\left(D^n\right)$ for some n, where D^n is the unit polydisc in $\mathbb{C}^n$;

Resolution of analytic algebra

- A free graded differential analytic sheaf-algebra is a graded sheaf $\mathcal{R}^* = \oplus_{k \leq 0} \mathcal{R}^k$ with a grading $|\cdot|$ and a differential d of grading 1 such that:
- (i) $\mathcal{R}$ has a structure of graded commutative algebra;
- (ii) $\mathcal{R}^0 = \mathcal{O}\left(D^n\right)$ for some n, where D^n is the unit polydisc in $\mathbb{C}^n$;
- (iii) a system of homogeneous sections of $\mathcal{R}$ is distinguished; it consists of the set $z_1, ..., z_n$ of coordinate functions in $\mathbb{C}^n$ and of the set E of free generators of $\mathcal{R}$ over $\mathcal{R}^0$, that is $\mathcal{R} = \mathcal{O}\left(D^n\right)[E]$;

Resolution of analytic algebra

- A free graded differential analytic sheaf-algebra is a graded sheaf $\mathcal{R}^* = \oplus_{k \leq 0} \mathcal{R}^k$ with a grading $|\cdot|$ and a differential d of grading 1 such that:
- (i) $\mathcal{R}$ has a structure of graded commutative algebra;
- (ii) $\mathcal{R}^0 = \mathcal{O}(D^n)$ for some n, where D^n is the unit polydisc in $\mathbb{C}^n$;
- (iii) a system of homogeneous sections of $\mathcal{R}$ is distinguished; it consists of the set $z_1, ..., z_n$ of coordinate functions in $\mathbb{C}^n$ and of the set E of free generators of $\mathcal{R}$ over $\mathcal{R}^0$, that is $\mathcal{R} = \mathcal{O}(D^n)[E]$;
- $|z_j| = 0, j = 1, ..., ; |e| < 0$ for $e \in E$; the number of generators is finite in each grading;

Resolution of analytic algebra

- A free graded differential analytic sheaf-algebra is a graded sheaf $\mathcal{R}^* = \oplus_{k \leq 0} \mathcal{R}^k$ with a grading $|\cdot|$ and a differential d of grading 1 such that:
- (i) $\mathcal{R}$ has a structure of graded commutative algebra;
- (ii) $\mathcal{R}^0 = \mathcal{O}(D^n)$ for some n, where D^n is the unit polydisc in $\mathbb{C}^n$;
- (iii) a system of homogeneous sections of $\mathcal{R}$ is distinguished; it consists of the set $z_1, ..., z_n$ of coordinate functions in $\mathbb{C}^n$ and of the set E of free generators of $\mathcal{R}$ over $\mathcal{R}^0$, that is $\mathcal{R} = \mathcal{O}(D^n)[E]$;
- $|z_j| = 0, j = 1, ..., ; |e| < 0$ for $e \in E$; the number of generators is finite in each grading;
- (iv) d is a derivative that is $d(ab) = d(a)\,b + (-)^{|a|}\,ad(b)$ for any homogeneous a and any $b \in \mathcal{R}$.

Resolution of analytic algebra

- A free graded differential analytic sheaf-algebra is a graded sheaf $\mathcal{R}^* = \oplus_{k \leq 0} \mathcal{R}^k$ with a grading $|\cdot|$ and a differential d of grading 1 such that:
- (i) $\mathcal{R}$ has a structure of graded commutative algebra;
- (ii) $\mathcal{R}^0 = \mathcal{O}(D^n)$ for some n, where D^n is the unit polydisc in $\mathbb{C}^n$;
- (iii) a system of homogeneous sections of $\mathcal{R}$ is distinguished; it consists of the set $z_1, ..., z_n$ of coordinate functions in $\mathbb{C}^n$ and of the set E of free generators of $\mathcal{R}$ over $\mathcal{R}^0$, that is $\mathcal{R} = \mathcal{O}(D^n)[E]$;
- $|z_j| = 0, j = 1, ..., ; |e| < 0$ for $e \in E$; the number of generators is finite in each grading;
- (iv) d is a derivative that is $d(ab) = d(a)b + (-)^{|a|} ad(b)$ for any homogeneous a and any $b \in \mathcal{R}$.
- Observation: d vanishes on $\mathcal{R}^0$ and is a $\mathcal{R}^0$-morphism.

Resolution of algebra (cont.)

- FGDA be the category of free graded differential analytic sheaf-algebras. A morphism of the category FGDA is an injective homomorphism of graded differential algebras $\varphi : p^* (\mathcal{O}(D^n)[E], d) \to (\mathcal{O}(D^m)[F], d')$ where $p : \mathbb{C}^m \to \mathbb{C}^n$ is a coordinate projection such that $\varphi(z_j) = z_j$, $j = 1, ..., n$ and $\varphi(E) \subset F$.

Resolution of algebra (cont.)

- FGDA be the category of free graded differential analytic sheaf-algebras. A morphism of the category FGDA is an injective homomorphism of graded differential algebras $\varphi : p^* (\mathcal{O}(D^n)[E], d) \rightarrow (\mathcal{O}(D^m)[F], d')$ where $p : \mathbb{C}^m \rightarrow \mathbb{C}^n$ is a coordinate projection such that $\varphi(z_j) = z_j$, $j = 1, ..., n$ and $\varphi(E) \subset F$.
- Let $\mathcal{O}$ be a coherent sheaf of analytic algebras defined in a neighborhood of the closure of the polydisc $D^n \subset \mathbb{C}^n$. *Tate-Tyurina* resolution of $\mathcal{O}$ is an object $(\mathcal{R}, d)$ of the category FGDA together with a $\mathbb{C}$-sheaf-algebra homomorphism $(\mathcal{R}, d) \rightarrow \mathcal{O}$ that induces

$$H^k(\mathcal{R}) = 0,\ \ k < 0;\ \ H^0(\mathcal{R}) \cong \mathcal{O}.$$

Polyhedral covering

- A *polyhedron* in a c.a.s. X is an open subset $P \subset X$ together with a proper holomorphic imbedding $\phi_P : Y \to U$ where $P \Subset Y \subset \mathbb{C}^N$ such that $P = \phi_P^{-1}\left(D^N\right)$ where $D^N \Subset U$.

Polyhedral covering

- A *polyhedron* in a c.a.s. X is an open subset $P \subset X$ together with a proper holomorphic imbedding $\phi_P : Y \to U$ where $P \Subset Y \subset \mathbb{C}^N$ such that $P = \phi_P^{-1}\left(D^N\right)$ where $D^N \Subset U$.
- ϕ_P is called the *barrier* mapping of P.

Polyhedral covering

- A *polyhedron* in a c.a.s. X is an open subset $P \subset X$ together with a proper holomorphic imbedding $\phi_P : Y \to U$ where $P \Subset Y \subset \mathbb{C}^N$ such that $P = \phi_P^{-1}\left(D^N\right)$ where $D^N \Subset U$.
- ϕ_P is called the *barrier* mapping of P.
- If P, Q are polyhedrons in X; the intersection $P \cap Q$ is also a polyhedron with the barrier map $\phi_P \times \phi_Q$. A polyhedral *covering* of the space X is a locally finite covering $P = \{P_\alpha, \alpha \in \mathbb{N}\}$ by polyhedrons.

Polyhedral covering

- A *polyhedron* in a c.a.s. X is an open subset $P \subset X$ together with a proper holomorphic imbedding $\phi_P : Y \to U$ where $P \Subset Y \subset \mathbb{C}^N$ such that $P = \phi_P^{-1}\left(D^N\right)$ where $D^N \Subset U$.
- ϕ_P is called the *barrier* mapping of P.
- If P, Q are polyhedrons in X; the intersection $P \cap Q$ is also a polyhedron with the barrier map $\phi_P \times \phi_Q$. A polyhedral *covering* of the space X is a locally finite covering $P = \{P_\alpha, \alpha \in \mathbb{N}\}$ by polyhedrons.
- A *simplex* of the covering P is an ordered subset $A = \{\alpha_0, ..., \alpha_d\}$ such that the set $P_{\alpha_0} \cap ... \cap P_{\alpha_d}$ is not empty.

Polyhedral covering

- A *polyhedron* in a c.a.s. X is an open subset $P \subset X$ together with a proper holomorphic imbedding $\phi_P : Y \to U$ where $P \Subset Y \subset \mathbb{C}^N$ such that $P = \phi_P^{-1}\left(D^N\right)$ where $D^N \Subset U$.
- ϕ_P is called the *barrier* mapping of P.
- If P, Q are polyhedrons in X; the intersection $P \cap Q$ is also a polyhedron with the barrier map $\phi_P \times \phi_Q$. A polyhedral *covering* of the space X is a locally finite covering $P = \{P_\alpha, \alpha \in \mathbb{N}\}$ by polyhedrons.
- A *simplex* of the covering P is an ordered subset $A = \{\alpha_0, ..., \alpha_d\}$ such that the set $P_{\alpha_0} \cap ... \cap P_{\alpha_d}$ is not empty.
- The nerve N of the covering P is the category of all simplices of the covering and of all inclusions $B \subset A$ of simplices.

Resolution of a complex space

- Let P be a polyhedral covering of a c.a.s. $(X, \mathcal{O})$ with the nerve N. A *resolution* of X on P is a covariant functor

$$\mathcal{R} : \mathrm{N} \rightsquigarrow \mathrm{FGDA}$$

together with a functor transform $A \mapsto \mathcal{R}(A) \Rightarrow \mathcal{O}|P_A$ such that for any simplex A the sheaf $\mathcal{R}(A)$ is a resolution of the sheaf $\phi_A^* (\mathcal{O}|P_A)$.

Resolution of a complex space

- Let P be a polyhedral covering of a c.a.s. $(X, \mathcal{O})$ with the nerve N. A *resolution* of X on P is a covariant functor

$$\mathcal{R} : \mathrm{N} \rightsquigarrow \mathrm{FGDA}$$

  together with a functor transform $A \mapsto \mathcal{R}(A) \Rightarrow \mathcal{O}|P_A$ such that for any simplex A the sheaf $\mathcal{R}(A)$ is a resolution of the sheaf $\phi_A^* (\mathcal{O}|P_A)$.
- Functoriality of $\mathcal{R}$ means (in particular) that $d_A \varphi_{AB} = \varphi_{AB} d_B$ for any simplices $B \subset A$, where $\varphi_{AB} : p^* (\mathrm{R}(B)) \rightarrow \mathrm{R}(A)$ is the corresponding morphism of category FGDA, $d_A.d_B$ are differentials.

Resolution of a complex space

- Let P be a polyhedral covering of a c.a.s. $(X, \mathcal{O})$ with the nerve N. A *resolution* of X on P is a covariant functor

$$\mathcal{R} : \mathrm{N} \rightsquigarrow \mathrm{FGDA}$$

together with a functor transform $A \mapsto \mathcal{R}(A) \Rightarrow \mathcal{O}|P_A$ such that for any simplex A the sheaf $\mathcal{R}(A)$ is a resolution of the sheaf $\phi_A^* (\mathcal{O}|P_A)$.

- Functoriality of $\mathcal{R}$ means (in particular) that $d_A \varphi_{AB} = \varphi_{AB} d_B$ for any simplices $B \subset A$, where $\varphi_{AB} : p^* (\mathrm{R}(B)) \to \mathrm{R}(A)$ is the corresponding morphism of category FGDA, $d_A . d_B$ are differentials.

- **Theorem.** For an arbitrary c.a.s. X and any polyhedral covering P of X there exists a resolution $\mathcal{R}$ of X on P.

Basic bicomplex

- Let $\mathrm{P} = \{P_\alpha\}$ be a polyhedral covering of X with a nerve N, $\mathcal{R}$ be a resolution of X on this covering. The sequence of functors $\mathrm{R}^{\odot n} : \mathrm{N} \rightsquigarrow \mathrm{FGDA}$ $n = 1, 2, \ldots$ is defined, where $\mathrm{R}^{\odot n}(A) = \mathrm{R}(A)^{\odot n}$ for any simplex $A \in \mathrm{N}$.

Basic bicomplex

- Let $\mathrm{P} = \{P_\alpha\}$ be a polyhedral covering of X with a nerve N, $\mathcal{R}$ be a resolution of X on this covering. The sequence of functors $\mathrm{R}^{\odot n} : \mathrm{N} \rightsquigarrow \mathrm{FGDA}$ $n = 1, 2, ...$ is defined, where $\mathrm{R}^{\odot n}(A) = \mathrm{R}(A)^{\odot n}$ for any simplex $A \in \mathrm{N}$.
- Given a linear N-transform $\phi : \mathrm{R}^{\otimes n} \to \mathrm{R}$, we set $\deg \phi = n - 1$ (shifted by -1 Hochschild degree) and the *grading* $|\cdot|$ by the rule: $|\phi| = |\phi(\alpha)| - |\alpha|$ for any homogeneous $\alpha = [a_1|...|a_n] \in \mathrm{R}^{\otimes n}$, where $|\alpha| = |a_1| + ... + |a_n|$.

Basic bicomplex

- Let $\mathrm{P} = \{P_\alpha\}$ be a polyhedral covering of X with a nerve N, $\mathcal{R}$ be a resolution of X on this covering. The sequence of functors $\mathrm{R}^{\odot n} : \mathrm{N} \rightsquigarrow \mathrm{FGDA}$ $n = 1, 2, ...$ is defined, where $\mathrm{R}^{\odot n}(A) = \mathrm{R}(A)^{\odot n}$ for any simplex $A \in \mathrm{N}$.
- Given a linear N-transform $\phi : \mathrm{R}^{\otimes n} \to \mathrm{R}$, we set $\deg \phi = n - 1$ (shifted by -1 Hochschild degree) and the *grading* $|\cdot|$ by the rule: $|\phi| = |\phi(\alpha)| - |\alpha|$ for any homogeneous $\alpha = [a_1|...|a_n] \in \mathrm{R}^{\otimes n}$, where $|\alpha| = |a_1| + ... + |a_n|$.
- *Extension.* For any $k > 0$ the transform ϕ can be extended to a functor transform $\Phi : \mathrm{R}^{\otimes n+k} \to \mathrm{R}^{\otimes k+1}$ by

$$\Phi([a_1|...|a_{n+k}]) = \sum_{i=0}^{k} (-)^{\sigma} [a_1|...|a_i|\phi([a_{i+1}|...|a_{i+n}])|a_{i+n+2}|...|a_{k+n}]$$

where $\sigma = i(n-1) + (|a_1| + ... + |a_i|)|\phi|$. Set $\Phi([a_1|...|a_m]) = 0$ for $m < n$.

Basic bicomplex

- Let $\mathrm{P} = \{P_\alpha\}$ be a polyhedral covering of X with a nerve N, $\mathcal{R}$ be a resolution of X on this covering. The sequence of functors $\mathrm{R}^{\odot n} : \mathrm{N} \rightsquigarrow \mathrm{FGDA}$ $n = 1, 2, ...$ is defined, where $\mathrm{R}^{\odot n}(A) = \mathrm{R}(A)^{\odot n}$ for any simplex $A \in \mathrm{N}$.
- Given a linear N-transform $\phi : \mathrm{R}^{\otimes n} \to \mathrm{R}$, we set $\deg \phi = n - 1$ (shifted by -1 Hochschild degree) and the *grading* $|\cdot|$ by the rule: $|\phi| = |\phi(\alpha)| - |\alpha|$ for any homogeneous $\alpha = [a_1|...|a_n] \in \mathrm{R}^{\otimes n}$, where $|\alpha| = |a_1| + ... + |a_n|$.
- *Extension.* For any $k > 0$ the transform ϕ can be extended to a functor transform $\Phi : \mathrm{R}^{\otimes n+k} \to \mathrm{R}^{\otimes k+1}$ by

$$\Phi([a_1|...|a_{n+k}]) = \sum_{i=0}^{k} (-)^{\sigma} [a_1|...|a_i|\phi([a_{i+1}|...|a_{i+n}])|a_{i+n+2}|...|a_{k+n}]$$

where $\sigma = i(n-1) + (|a_1| + ... + |a_i|)|\phi|$. Set $\Phi([a_1|...|a_m]) = 0$ for $m < n$.
- The transform Φ has bigrading (degree and grading) $(-n+1, |\phi|)$.

Differentials

- The extension of the differential d in $\mathrm{R}(A)$ is a differential in $\mathrm{R}^{\otimes n}(A)$ of bidegree $(0,1)$ acting as follows

$$d\left[a_1|\ldots|a_m\right] = \sum (-)^{|a_1|+\ldots+|a_i|}\left[a_1|\ldots|a_i|da_{i+1}|a_{i+2}|\ldots|a_m\right].$$

This construction is functorial in the argument A that is a transform of functors $\mathrm{N} \rightsquigarrow \mathrm{Hom}(\mathrm{R}^{\otimes n} \to \mathrm{R}^{\otimes n})$.

Differentials

- The extension of the differential d in $\mathrm{R}(A)$ is a differential in $\mathrm{R}^{\otimes n}(A)$ of bidegree $(0,1)$ acting as follows

$$d[a_1|...|a_m] = \sum (-)^{|a_1|+...+|a_i|} [a_1|...|a_i|da_{i+1}|a_{i+2}|...|a_m].$$

  This construction is functorial in the argument A that is a transform of functors $\mathrm{N} \rightsquigarrow \mathrm{Hom}(\mathrm{R}^{\otimes n} \to \mathrm{R}^{\otimes n})$.
- The extension of the multiplication operator $\partial : \mathrm{R}^{\otimes 2}(A) \to \mathrm{R}(A), \partial(a|b) = ab$ is the standard chain differential of bidegree $(1,0)$

$$\partial [a_1|...|a_n] = \sum_1^{n-1} (-)^i [a_1|a_2|...|a_{i-1}|a_i a_{i+1}|a_{i+2}|, ..., |a_n]$$

Bicomplex (cont.)

- The space $\mathrm{Hom}\,(\mathrm{R}^{\otimes *}, \mathrm{R})$ of linear N-transforms $f : \mathrm{R}^{\otimes *} \to \mathrm{R}$ is bigraded. The total degree $\mathrm{Deg}\, f = \deg f + |f|$.

Bicomplex (cont.)

- The space $\mathrm{Hom}\left(\mathrm{R}^{\otimes *}, \mathrm{R}\right)$ of linear N-transforms $f : \mathrm{R}^{\otimes *} \to \mathrm{R}$ is bigraded. The total degree $\mathrm{Deg}\, f = \deg f + |f|$.
- Composition of cochains: $(f \circ g)(a) = f(G(a))$ for $a \in \mathrm{R}^{\otimes *}$, where G is the extension of g. We have $\deg\,(f \circ g) = \deg f + \deg g$ and $|f \circ g| = |f| + |g|$. The commutator

$$[f, g] = f \circ g - (-)^{\mathrm{Deg}\, f\, \mathrm{Deg}\, g}\, g \circ f$$

is a graded Lie bracket operation with respect to Deg, since it fulfils the Jacobi identity

$$[f, [g, h]] = [[f, g], h] + (-)^{\mathrm{Deg}\, f\, \mathrm{Deg}\, g}\, [g, [f, h]]$$

In particular, $(-)^{\deg f}\, [\partial, \cdot] = \partial^*$.

Bicomplex (cont.)

- The space $\mathrm{Hom}\left(\mathrm{R}^{\otimes *}, \mathrm{R}\right)$ of linear N-transforms $f : \mathrm{R}^{\otimes *} \to \mathrm{R}$ is bigraded. The total degree $\mathrm{Deg}\, f = \deg f + |f|$.
- Composition of cochains: $(f \circ g)(a) = f(G(a))$ for $a \in \mathrm{R}^{\otimes *}$, where G is the extension of g. We have $\deg\,(f \circ g) = \deg f + \deg g$ and $|f \circ g| = |f| + |g|$. The commutator

  $$[f, g] = f \circ g - (-)^{\mathrm{Deg}\, f\, \mathrm{Deg}\, g}\, g \circ f$$

  is a graded Lie bracket operation with respect to Deg, since it fulfils the Jacobi identity

  $$[f, [g, h]] = [[f, g], h] + (-)^{\mathrm{Deg}\, f\, \mathrm{Deg}\, g}\, [g, [f, h]]$$

  In particular, $(-)^{\deg f}\, [\partial, \cdot] = \partial^*$.
- By the Jacobi identity implies the equation

  $$\partial^* [f, g] = [\partial^* f, g] + (-)^{\mathrm{Deg}\, f}\, [f, \partial^* g]$$

Polydifferential operators

- $\mathrm{Diff}\,(\mathrm{R}^{\otimes *}, \mathrm{R})$ be the subcomplex of $\mathrm{Hom}\,(\mathrm{R}^{\otimes *}, \mathrm{R})$ of polydifferential operators. It is stable under ∂ and d.

Polydifferential operators

- $\mathrm{Diff}\left(\mathrm{R}^{\otimes *}, \mathrm{R}\right)$ be the subcomplex of $\mathrm{Hom}\left(\mathrm{R}^{\otimes *}, \mathrm{R}\right)$ of polydifferential operators. It is stable under ∂ and d.
- *Deformation complex* is the total complex $\mathrm{Tot} \doteq \mathrm{Tot}^{*}\left(\mathcal{R}\right)$ of the bicomplex $\mathrm{Diff}\left(\mathrm{R}^{\otimes *}, \mathrm{R}\right)$ with the total degree Deg and the differential

$$[D, f] = (-)^{\deg f}[d, f] + [\partial, f] = (-)^{\deg f}\left([d, f] + \partial^{*} f\right)$$

where $D = (-)^{\deg f} d + \partial$.

Polydifferential operators

- $\mathrm{Diff}\left(\mathrm{R}^{\otimes *}, \mathrm{R}\right)$ be the subcomplex of $\mathrm{Hom}\left(\mathrm{R}^{\otimes *}, \mathrm{R}\right)$ of polydifferential operators. It is stable under ∂ and d.
- *Deformation complex* is the total complex $\mathrm{Tot} \doteq \mathrm{Tot}^{*}\left(\mathcal{R}\right)$ of the bicomplex $\mathrm{Diff}\left(\mathrm{R}^{\otimes *}, \mathrm{R}\right)$ with the total degree Deg and the differential

$$[D, f] = (-)^{\deg f}[d, f] + [\partial, f] = (-)^{\deg f}\left([d, f] + \partial^{*} f\right)$$

  where $D = (-)^{\deg f} d + \partial$.
- *Lie bracket.* The bracket operator applied to polydifferential operators is again a polydifferential operator, which implies that Tot is a graded Lie algebra.

Associative cohomology

- The graded space $\mathrm{A}^*(\mathcal{R}) \doteq H^*(\mathrm{Tot}^*(\mathcal{R}), D)$ is called the *associative* cohomology of $\mathcal{R}$. The bracket is inherited in the deformation cohomology and satisfies the Jacobi identity; $\mathrm{A}^*(\mathcal{R})$ has structure of graded Lie algebra.

Associative cohomology

- The graded space $\mathrm{A}^*(\mathcal{R}) \doteq H^*(\mathrm{Tot}^*(\mathcal{R}), D)$ is called the *associative* cohomology of $\mathcal{R}$. The bracket is inherited in the deformation cohomology and satisfies the Jacobi identity; $\mathrm{A}^*(\mathcal{R})$ has structure of graded Lie algebra.

- **Proposition.** For any c.a.s. X and any polyhedral coverings P and $\mathrm{P}'$and respective resolutions $\mathcal{R}$ and $\mathcal{R}'$ of a X there exists a homotopy of graded differential Lie algebras $\mathrm{Tot}^*(\mathcal{R}) \sim \mathrm{Tot}^*(\mathcal{R}')$. The graded Lie algebra $\mathrm{A}^*(\mathcal{R})$ does not depend on a choice of the covering and of the resolution of X.

Associative cohomology

- The graded space $\mathrm{A}^*(\mathcal{R}) \doteq H^*(\mathrm{Tot}^*(\mathcal{R}), D)$ is called the *associative* cohomology of $\mathcal{R}$. The bracket is inherited in the deformation cohomology and satisfies the Jacobi identity; $\mathrm{A}^*(\mathcal{R})$ has structure of graded Lie algebra.

- **Proposition.** For any c.a.s. X and any polyhedral coverings P and $\mathrm{P}'$and respective resolutions $\mathcal{R}$ and $\mathcal{R}'$ of a X there exists a homotopy of graded differential Lie algebras $\mathrm{Tot}^*(\mathcal{R}) \sim \mathrm{Tot}^*(\mathcal{R}')$. The graded Lie algebra $\mathrm{A}^*(\mathcal{R})$ does not depend on a choice of the covering and of the resolution of X.

- This algebra is denoted $\mathrm{A}^*(X)$ and called *associative* cohomology of X.

Analytic Hochschild cohomology

- The analytic cochain space $\mathrm{Ca}^k(A)$ is the module of all continuous linear maps $h : A^{\odot k} \rightarrow A$ with respect to the topology in these analytic algebras. The cohomology of $(\mathrm{Ca}^*(A), \partial^*)$ is called *analytic* Hochschild cohomology; notation: $\mathrm{Hoch}^*_{an}(A, A)$.

Analytic Hochschild cohomology

- The analytic cochain space $\mathrm{Ca}^{k}(A)$ is the module of all continuous linear maps $h: A^{\odot k} \to A$ with respect to the topology in these analytic algebras. The cohomology of $(\mathrm{Ca}^{*}(A), \partial^{*})$ is called *analytic* Hochschild cohomology; notation: $\mathrm{Hoch}^{*}_{an}(A, A)$.
- For $f, g \in \mathrm{Ca}^{*}(A)$ any composition $f \circ g$ is again a continuous map. Therefore the *Gerstenhaber* bracket is well defined in the analytic Hochschild cohomology.

Analytic Hochschild cohomology

- The analytic cochain space $\mathrm{Ca}^k(A)$ is the module of all continuous linear maps $h : A^{\odot k} \to A$ with respect to the topology in these analytic algebras. The cohomology of $(\mathrm{Ca}^*(A), \partial^*)$ is called *analytic* Hochschild cohomology; notation: $\mathrm{Hoch}^*_{an}(A, A)$.
- For $f, g \in \mathrm{Ca}^*(A)$ any composition $f \circ g$ is again a continuous map. Therefore the *Gerstenhaber* bracket is well defined in the analytic Hochschild cohomology.
- Denote by $\mathrm{Harr}^*_{an}(A, A)$ for the analytic cohomology of the subcomplex in $(\mathrm{Ca}^*(A), \partial^*)$ consisting of maps h that vanish on shuffle products.

Analytic Hochschild cohomology

- The analytic cochain space $\mathrm{Ca}^k(A)$ is the module of all continuous linear maps $h : A^{\odot k} \longrightarrow A$ with respect to the topology in these analytic algebras. The cohomology of $(\mathrm{Ca}^*(A), \partial^*)$ is called *analytic* Hochschild cohomology; notation: $\mathrm{Hoch}^*_{an}(A, A)$.
- For $f, g \in \mathrm{Ca}^*(A)$ any composition $f \circ g$ is again a continuous map. Therefore the *Gerstenhaber* bracket is well defined in the analytic Hochschild cohomology.
- Denote by $\mathrm{Harr}^*_{an}(A, A)$ for the analytic cohomology of the subcomplex in $(\mathrm{Ca}^*(A), \partial^*)$ consisting of maps h that vanish on shuffle products.
- Analytic cohomology is finitely generated for each degree.

Analytic Hochschild cohomology

- The analytic cochain space $\mathrm{Ca}^k(A)$ is the module of all continuous linear maps $h : A^{\odot k} \to A$ with respect to the topology in these analytic algebras. The cohomology of $(\mathrm{Ca}^*(A), \partial^*)$ is called *analytic* Hochschild cohomology; notation: $\mathrm{Hoch}^*_{an}(A, A)$.
- For $f, g \in \mathrm{Ca}^*(A)$ any composition $f \circ g$ is again a continuous map. Therefore the *Gerstenhaber* bracket is well defined in the analytic Hochschild cohomology.
- Denote by $\mathrm{Harr}^*_{an}(A, A)$ for the analytic cohomology of the subcomplex in $(\mathrm{Ca}^*(A), \partial^*)$ consisting of maps h that vanish on shuffle products.
- Analytic cohomology is finitely generated for each degree.
- In particular $\mathrm{Hoch}^2_{an}(A) \cong T^1(A) \oplus \mathcal{P}(A)$ for an arbitrary analytic algebra A,

Analytic Hochschild cohomology

- The analytic cochain space $\mathrm{Ca}^k(A)$ is the module of all continuous linear maps $h : A^{\odot k} \to A$ with respect to the topology in these analytic algebras. The cohomology of $(\mathrm{Ca}^*(A), \partial^*)$ is called *analytic* Hochschild cohomology; notation: $\mathrm{Hoch}^*_{an}(A, A)$.
- For $f, g \in \mathrm{Ca}^*(A)$ any composition $f \circ g$ is again a continuous map. Therefore the *Gerstenhaber* bracket is well defined in the analytic Hochschild cohomology.
- Denote by $\mathrm{Harr}^*_{an}(A, A)$ for the analytic cohomology of the subcomplex in $(\mathrm{Ca}^*(A), \partial^*)$ consisting of maps h that vanish on shuffle products.
- Analytic cohomology is finitely generated for each degree.
- In particular $\mathrm{Hoch}^2_{an}(A) \cong T^1(A) \oplus \mathcal{P}(A)$ for an arbitrary analytic algebra A,
- where $T^1(A) \cong \mathrm{Harr}^2_{an}(A, A)$ is the space of infinitesimal commutative deformations of A in the categrory of analytic algebras, and $\mathcal{P}(A)$ is the set of tangent bivectors on A.

Global cohomology

- **Proposition.** For an arbitrary coherent sheaf of analytic algebras $\mathcal{A}$ in X the sheaf $\mathcal{H}^*(\mathcal{A},\mathcal{A})$ whose fibers are cohomolgy groups $\mathrm{Hoch}^*_{an}(\mathcal{A}_x,\mathcal{A}_x)$ is coherent in each degree.

Global cohomology

- **Proposition.** For an arbitrary coherent sheaf of analytic algebras $\mathcal{A}$ in X the sheaf $\mathcal{H}^*(\mathcal{A},\mathcal{A})$ whose fibers are cohomolgy groups $\mathrm{Hoch}^*_{an}(\mathcal{A}_x,\mathcal{A}_x)$ is coherent in each degree.
- **Theorem**. There exists a spectral sequence E with the second term

$$E_2^{pq} = H^p\left(X, \mathcal{H}^{q+1}(\mathcal{O})\right) \Rightarrow \mathrm{A}^*(X)$$

that converges to the deformation cohomology.

Global cohomology

- **Proposition.** For an arbitrary coherent sheaf of analytic algebras $\mathcal{A}$ in X the sheaf $\mathcal{H}^* (\mathcal{A}, \mathcal{A})$ whose fibers are cohomolgy groups $\mathrm{Hoch}^*_{an} (\mathcal{A}_x, \mathcal{A}_x)$ is coherent in each degree.
- **Theorem**. There exists a spectral sequence E with the second term

$$E_2^{pq} = H^p \left(X, \mathcal{H}^{q+1} (\mathcal{O})\right) \Rightarrow \mathrm{A}^* (X)$$

that converges to the deformation cohomology.

- The spectral sequence is induced by the filtration $\left\{\mathrm{Tot}^{(p)}\right\}$, where $\mathrm{Tot}^{(p)}$ means the subspace of N-operators that vanishes on generators of $\mathrm{R}^{\otimes *} (A)$ for dim $A < p$, $p = 0, 1, 2, \ldots$

Degree 0 and 1

- The sheaf $\mathcal{T}^0 \cong \mathcal{H}^1$ coincides with the sheaf of germs of holomorphic tangent fields on X and the global cohomology coincide: $\mathrm{A}^0(X) = H^0(X, \mathcal{T}^0)$.

Degree 0 and 1

- The sheaf $\mathcal{T}^0 \cong \mathcal{H}^1$ coincides with the sheaf of germs of holomorphic tangent fields on X and the global cohomology coincide: $\mathrm{A}^0(X) = H^0(X, \mathcal{T}^0)$.
- For degree 1

$$\begin{aligned} \mathrm{gr}_0 \mathrm{A}^1(X) &= \tilde{H}^0(X, \mathcal{T}^1) \oplus H^0(X, \mathcal{P}), \\ \mathrm{gr}_1 \mathrm{A}^1(X) &= H^1(X, \mathcal{T}^0), \end{aligned}$$

where gr means the grading in the cohomology induced by the filtration $\{\mathrm{Tot}^{(p)}\}$ in the basic complex,

Degree 0 and 1

- The sheaf $\mathcal{T}^0 \cong \mathcal{H}^1$ coincides with the sheaf of germs of holomorphic tangent fields on X and the global cohomology coincide: $\mathrm{A}^0(X) = H^0(X, \mathcal{T}^0)$.
- For degree 1

$$\begin{aligned} \mathrm{gr}_0 \mathrm{A}^1(X) &= \tilde{H}^0(X, \mathcal{T}^1) \oplus H^0(X, \mathcal{P}), \\ \mathrm{gr}_1 \mathrm{A}^1(X) &= H^1(X, \mathcal{T}^0), \end{aligned}$$

  where gr means the grading in the cohomology induced by the filtration $\left\{\mathrm{Tot}^{(p)}\right\}$ in the basic complex,
- $\mathcal{T}^1$ is the symmetric and $\mathcal{P}$ is the skew-symmetric direct components of $\mathcal{H}^2$,

Degree 0 and 1

- The sheaf $\mathcal{T}^0 \cong \mathcal{H}^1$ coincides with the sheaf of germs of holomorphic tangent fields on X and the global cohomology coincide: $\mathrm{A}^0(X) = H^0(X, \mathcal{T}^0)$.
- For degree 1

$$\begin{aligned} \mathrm{gr}_0 \mathrm{A}^1(X) &= \tilde{H}^0(X, \mathcal{T}^1) \oplus H^0(X, \mathcal{P}), \\ \mathrm{gr}_1 \mathrm{A}^1(X) &= H^1(X, \mathcal{T}^0), \end{aligned}$$

where gr means the grading in the cohomology induced by the filtration $\left\{ \mathrm{Tot}^{(p)} \right\}$ in the basic complex,

- $\mathcal{T}^1$ is the symmetric and $\mathcal{P}$ is the skew-symmetric direct components of $\mathcal{H}^2$,
- $\tilde{H}^0(X, \mathcal{T}^1) = \mathrm{Ker}\, d_2 : H^0(X, \mathcal{T}^1) \to H^2(X, \mathcal{T}^0)$.

Degree 2

- $$\begin{aligned}
\mathrm{gr}_0 \mathrm{A}^2 &= \mathrm{Ker}\, d_3 : \mathrm{Ker}\left\{ d_2 : \Gamma\left(X, \mathcal{H}^3\right) \to H^2\left(X, \mathcal{H}^2\right)\right\} \to H^3\left(X, \mathcal{T}^0\right), \\
\mathrm{gr}_1 \mathrm{A}^2 &= H^1(X, \mathcal{P}) \oplus \mathrm{Ker}\, d_2 : H^1\left(X, \mathcal{T}^1\right) \to H^3\left(X, \mathcal{T}^0\right), \\
\mathrm{gr}_2 \mathrm{A}^2 &= H^2\left(X, \mathcal{T}^0\right),
\end{aligned}$$

Degree 2

- $$\begin{aligned} \mathrm{gr}_0 \mathrm{A}^2 &= \mathrm{Ker}\, d_3 : \mathrm{Ker}\left\{ d_2 : \Gamma\left(X, \mathcal{H}^3\right) \to H^2\left(X, \mathcal{H}^2\right)\right\} \to H^3\left(X, \mathcal{T}^0\right), \\ \mathrm{gr}_1 \mathrm{A}^2 &= H^1(X, \mathcal{P}) \oplus \mathrm{Ker}\, d_2 : H^1\left(X, \mathcal{T}^1\right) \to H^3\left(X, \mathcal{T}^0\right), \\ \mathrm{gr}_2 \mathrm{A}^2 &= H^2\left(X, \mathcal{T}^0\right), \end{aligned}$$
- Let $\mathrm{T}^*(X)$ be the tangent cohomology of X responsible for deformations of X in the category of c.a.s.

Degree 2

- $$\mathrm{gr}_0\mathrm{A}^2 = \mathrm{Ker}\, d_3 : \mathrm{Ker}\left\{ d_2 : \Gamma\left(X, \mathcal{H}^3\right) \to H^2\left(X, \mathcal{H}^2\right)\right\} \to H^3\left(X, \mathcal{T}^0\right),$$
$$\mathrm{gr}_1\mathrm{A}^2 = H^1(X, \mathcal{P}) \oplus \mathrm{Ker}\, d_2 : H^1\left(X, \mathcal{T}^1\right) \to H^3\left(X, \mathcal{T}^0\right),$$
$$\mathrm{gr}_2\mathrm{A}^2 = H^2\left(X, \mathcal{T}^0\right),$$
- Let $\mathrm{T}^*(X)$ be the tangent cohomology of X responsible for deformations of X in the category of c.a.s.
- **Theorem.** There is a canonical morphism of graded Lie algebras $t^* : \mathrm{T}^*(X) \to \mathrm{A}^*(X)$.

Degree 2

- $$\begin{aligned} \mathrm{gr}_0 \mathrm{A}^2 &= \mathrm{Ker}\, d_3 : \mathrm{Ker}\left\{ d_2 : \Gamma\left(X, \mathcal{H}^3\right) \to H^2\left(X, \mathcal{H}^2\right)\right\} \to H^3\left(X, \mathcal{T}^0\right), \\ \mathrm{gr}_1 \mathrm{A}^2 &= H^1(X, \mathcal{P}) \oplus \mathrm{Ker}\, d_2 : H^1\left(X, \mathcal{T}^1\right) \to H^3\left(X, \mathcal{T}^0\right), \\ \mathrm{gr}_2 \mathrm{A}^2 &= H^2\left(X, \mathcal{T}^0\right), \end{aligned}$$
- Let $\mathrm{T}^*(X)$ be the tangent cohomology of X responsible for deformations of X in the category of c.a.s.
- **Theorem.** There is a canonical morphism of graded Lie algebras $t^* : \mathrm{T}^*(X) \to \mathrm{A}^*(X)$.
- It is injective and admits a canonical linear splitting.

Degree 2

- $$\begin{aligned} \mathrm{gr}_0 \mathrm{A}^2 &= \mathrm{Ker}\, d_3 : \mathrm{Ker}\left\{ d_2 : \Gamma\left(X, \mathcal{H}^3\right) \to H^2\left(X, \mathcal{H}^2\right)\right\} \to H^3\left(X, \mathcal{T}^0\right), \\ \mathrm{gr}_1 \mathrm{A}^2 &= H^1(X, \mathcal{P}) \oplus \mathrm{Ker}\, d_2 : H^1\left(X, \mathcal{T}^1\right) \to H^3\left(X, \mathcal{T}^0\right), \\ \mathrm{gr}_2 \mathrm{A}^2 &= H^2\left(X, \mathcal{T}^0\right), \end{aligned}$$

- Let $\mathrm{T}^*(X)$ be the tangent cohomology of X responsible for deformations of X in the category of c.a.s.

- **Theorem.** There is a canonical morphism of graded Lie algebras $t^* : \mathrm{T}^*(X) \to \mathrm{A}^*(X)$.
- It is injective and admits a canonical linear splitting.

- In particular, $\mathrm{A}^0(X) = \mathrm{T}^0(X)$;

Degree 2

- $$\mathrm{gr}_0\mathrm{A}^2 = \mathrm{Ker}\, d_3 : \mathrm{Ker}\left\{ d_2 : \Gamma\left(X, \mathcal{H}^3\right) \to H^2\left(X, \mathcal{H}^2\right)\right\} \to H^3\left(X, \mathcal{T}^0\right),$$
  $$\mathrm{gr}_1\mathrm{A}^2 = H^1(X, \mathcal{P}) \oplus \mathrm{Ker}\, d_2 : H^1\left(X, \mathcal{T}^1\right) \to H^3\left(X, \mathcal{T}^0\right),$$
  $$\mathrm{gr}_2\mathrm{A}^2 = H^2\left(X, \mathcal{T}^0\right),$$
- Let $\mathrm{T}^*(X)$ be the tangent cohomology of X responsible for deformations of X in the category of c.a.s.
- **Theorem.** There is a canonical morphism of graded Lie algebras $t^* : \mathrm{T}^*(X) \to \mathrm{A}^*(X)$.
- It is injective and admits a canonical linear splitting.
- In particular, $\mathrm{A}^0(X) = \mathrm{T}^0(X)$;
- $\mathrm{A}^1(X) \cong \mathrm{T}^1(X) \oplus \Gamma(X, \mathcal{P}),\ \mathrm{T}^1(X) \cong H^1\left(X, \mathcal{T}^0\right) \oplus \tilde{H}^0\left(X, \mathcal{T}^1\right).$

Cohomology and deformation

- $\mathrm{A}^0(X) = H^0\left(X, \mathcal{T}^0\right)$ is the Lie algebra of infinitesimal automorphisms of X.

Cohomology and deformation

- $\mathrm{A}^0(X) = H^0(X, \mathcal{T}^0)$ is the Lie algebra of infinitesimal automorphisms of X.
- $\mathrm{A}^1(X)$ is canonically isomorphic to $\mathrm{D}(X)$.

Cohomology and deformation

- $\mathrm{A}^0(X) = H^0(X, \mathcal{T}^0)$ is the Lie algebra of infinitesimal automorphisms of X.
- $\mathrm{A}^1(X)$ is canonically isomorphic to $\mathrm{D}(X)$.
- $\mathrm{A}^2(X)$ is the space where all obstructions are evaluated.

Obstructions

- **Theorem** Let $\mathrm{F} : \mathrm{X} \to S$ be a deformation of a c.a.s. X with an Artin base S. For any small extension $\phi : S \to V$ with the ideal J there is defined an element $\mathrm{Ob}\,(\mathrm{F}, V) \in \mathrm{A}^2\,(X) \otimes J$ such that

Obstructions

- **Theorem** Let $\mathrm{F} : \mathrm{X} \to S$ be a deformation of a c.a.s. X with an Artin base S. For any small extension $\phi : S \to V$ with the ideal J there is defined an element $\mathrm{Ob}\,(\mathrm{F}, V) \in \mathrm{A}^2\,(X) \otimes J$ such that
- (i) vanishing of $\mathrm{Ob}\,(\mathrm{F}, V)$ is necessary and sufficient for existence of a deformation $\mathrm{G} : \mathrm{Y} \to V$ such that $\phi^*\,(\mathrm{G}) \simeq \mathrm{F}$;

Obstructions

- **Theorem** Let $\mathrm{F} : \mathrm{X} \to S$ be a deformation of a c.a.s. X with an Artin base S. For any small extension $\phi : S \to V$ with the ideal J there is defined an element $\mathrm{Ob}\,(\mathrm{F}, V) \in \mathrm{A}^2\,(X) \otimes J$ such that

- (i) vanishing of $\mathrm{Ob}\,(\mathrm{F}, V)$ is necessary and sufficient for existence of a deformation $\mathrm{G} : \mathrm{Y} \to V$ such that $\phi^*\,(\mathrm{G}) \simeq \mathrm{F}$;
- (ii) the element $\mathrm{Ob}\,(\mathrm{F}, V)$ is functorial in the argument V;

Obstructions

- **Theorem** Let $\mathrm{F}: \mathrm{X} \to S$ be a deformation of a c.a.s. X with an Artin base S. For any small extension $\phi : S \to V$ with the ideal J there is defined an element $\mathrm{Ob}\,(\mathrm{F}, V) \in \mathrm{A}^2\,(X) \otimes J$ such that

- (i) vanishing of $\mathrm{Ob}\,(\mathrm{F}, V)$ is necessary and sufficient for existence of a deformation $\mathrm{G} : \mathrm{Y} \to V$ such that $\phi^*\,(\mathrm{G}) \simeq \mathrm{F}$;
- (ii) the element $\mathrm{Ob}\,(\mathrm{F}, V)$ is functorial in the argument V;
- (iii) $\mathrm{Ob}\,(\mathrm{F}, V) = \mathrm{Ob}\,(\mathrm{F}', V)$ for any equivalent deformation $\mathrm{F}' \simeq \mathrm{F}$ over S.

  The map $\mathrm{Ob}\,(\mathrm{F}, V)$ is called *obstruction* to extension of the deformation F on V.

Master equation

- The basic equation of a c.a.s. X is $D^2 = 0$, which is equivalent to the system $\partial^2 = 0$, (associativity), $d^2 = 0$ (by the construction) and $[d, \partial] = 0$, which means that d is a derivation.

Master equation

- The basic equation of a c.a.s. X is $D^2 = 0$, which is equivalent to the system $\partial^2 = 0$, (associativity), $d^2 = 0$ (by the construction) and $[d, \partial] = 0$, which means that d is a derivation.
- Let $(S, \mathcal{O}_S)$ be an analytic germ and $\mathfrak{m}(S)$ be the maximal ideal in $\mathcal{O}_S$. Consider a sum $\mathrm{q} = \mathrm{q}_0 + \mathrm{q}_1 + \mathrm{q}_2 + \ldots \in \mathrm{Tot}^1(\mathcal{R}) \otimes_{\mathbb{C}} \mathcal{O}_S$, where $\mathrm{q}_k : \mathcal{R}^{\otimes k+1} \to \mathcal{R} \otimes \mathfrak{m}(S)$ is a N-functor of bidegree $(k, 1-k)$, $k \geq 0$, whose values are polydifferential operators. We extend q to an endomorphism of $\mathcal{R}^{\otimes *}$ and write

$$(D + \mathrm{q})^2 = 0.$$

the equation of the space of deformation $\mathcal{X}$

Master equation

- The basic equation of a c.a.s. X is $D^2 = 0$, which is equivalent to the system $\partial^2 = 0$, (associativity), $d^2 = 0$ (by the construction) and $[d, \partial] = 0$, which means that d is a derivation.
- Let $(S, \mathcal{O}_S)$ be an analytic germ and $\mathfrak{m}(S)$ be the maximal ideal in $\mathcal{O}_S$. Consider a sum $\mathrm{q} = \mathrm{q}_0 + \mathrm{q}_1 + \mathrm{q}_2 + \ldots \in \mathrm{Tot}^1(\mathcal{R}) \otimes_{\mathbb{C}} \mathcal{O}_S$, where $\mathrm{q}_k : \mathcal{R}^{\otimes k+1} \to \mathcal{R} \otimes \mathfrak{m}(S)$ is a N-functor of bidegree $(k, 1-k)$, $k \geq 0$, whose values are polydifferential operators. We extend q to an endomorphism of $\mathcal{R}^{\otimes *}$ and write

$$(D + \mathrm{q})^2 = 0.$$

  the equation of the space of deformation $\mathcal{X}$
- It is equivalent to the system equations

$$\begin{aligned} (d + \mathrm{q}_0)^2 &= 0, \\ [\partial + \mathrm{q}_1, d + \mathrm{q}_0] &= 0, \\ (\partial + \mathrm{q}_1)^2 + [\mathrm{q}_2, d + \mathrm{q}_0] &= 0, \\ &\ldots \end{aligned}$$

Existence

- **Theorem.** For an arbitrary c.a.s. X such that $\dim \mathrm{A}^1(X) < \infty$ there exists a formal minimal versal deformation $\mathrm{F} : \mathcal{X} \to \tilde{S}$. The base is $\tilde{S}$ is equal to $\mathrm{B}^{-1}(0)$ for a formal power map $\mathrm{B} : \mathrm{A}^1(X) \to \mathrm{A}^2(X)$ such that

Existence

- **Theorem.** For an arbitrary c.a.s. X such that $\dim \mathrm{A}^1(X) < \infty$ there exists a formal minimal versal deformation $\mathrm{F} : \mathcal{X} \to \tilde{S}$. The base is $\tilde{S}$ is equal to $\mathrm{B}^{-1}(0)$ for a formal power map $\mathrm{B} : \mathrm{A}^1(X) \to \mathrm{A}^2(X)$ such that
- $\mathrm{B}(0) = 0$, $\mathrm{d}^1\mathrm{B}(0) = 0$, $\mathrm{d}^2\mathrm{B}(0) = [\mathrm{A}^1(X), \mathrm{A}^1(X)]$.

Existence

- **Theorem.** For an arbitrary c.a.s. X such that $\dim \mathrm{A}^1(X) < \infty$ there exists a formal minimal versal deformation $\mathrm{F} : \mathcal{X} \to \tilde{S}$. The base is $\tilde{S}$ is equal to $\mathrm{B}^{-1}(0)$ for a formal power map $\mathrm{B} : \mathrm{A}^1(X) \to \mathrm{A}^2(X)$ such that
- $\mathrm{B}(0) = 0$, $\mathrm{d}^1\mathrm{B}(0) = 0$, $\mathrm{d}^2\mathrm{B}(0) = [\mathrm{A}^1(X), \mathrm{A}^1(X)]$.
- Write $\tilde{S} = \lim\,\mathrm{ind}\{S_k\}$, where S_k is a k-th small extension of the point $s = 0$ for $k = 1, 2, \ldots$

Existence

- **Theorem.** For an arbitrary c.a.s. X such that $\dim \mathrm{A}^1(X) < \infty$ there exists a formal minimal versal deformation $\mathrm{F} : \mathcal{X} \to \tilde{S}$. The base is $\tilde{S}$ is equal to $\mathrm{B}^{-1}(0)$ for a formal power map $\mathrm{B} : \mathrm{A}^1(X) \to \mathrm{A}^2(X)$ such that
- $\mathrm{B}(0) = 0$, $\mathrm{d}^1\mathrm{B}(0) = 0$, $\mathrm{d}^2\mathrm{B}(0) = [\mathrm{A}^1(X), \mathrm{A}^1(X)]$.
- Write $\tilde{S} = \lim \operatorname{ind}\{S_k\}$, where S_k is a k-th small extension of the point $s = 0$ for $k = 1, 2, \ldots$
- For $k \geq 3$ the equation holds $\mathrm{d}^k\mathrm{B}(0)\big|\, V = \mathrm{Ob}\left(\mathrm{F}|\, S_{k-1}, V\right)$ where V is an arbitrary small extension of S_{k-1}. Then $S_k = V$ for the maximal small extension V such that $\mathrm{Ob}\left(\mathrm{F}|\, S_{k-1}, V\right) = 0$.

Existence

- **Theorem.** For an arbitrary c.a.s. X such that $\dim \mathrm{A}^1(X) < \infty$ there exists a formal minimal versal deformation $\mathrm{F} : \mathcal{X} \to \tilde{S}$. The base is $\tilde{S}$ is equal to $\mathrm{B}^{-1}(0)$ for a formal power map $\mathrm{B} : \mathrm{A}^1(X) \to \mathrm{A}^2(X)$ such that
- $\mathrm{B}(0) = 0$, $\mathrm{d}^1\mathrm{B}(0) = 0$, $\mathrm{d}^2\mathrm{B}(0) = [\mathrm{A}^1(X), \mathrm{A}^1(X)]$.
- Write $\tilde{S} = \lim\operatorname{ind}\{S_k\}$, where S_k is a k-th small extension of the point $s = 0$ for $k = 1, 2, ...$
- For $k \geq 3$ the equation holds $\mathrm{d}^k\mathrm{B}(0)\big|\, V = \mathrm{Ob}\left(\mathrm{F}|\, S_{k-1}, V\right)$ where V is an arbitrary small extension of S_{k-1}. Then $S_k = V$ for the maximal small extension V such that $\mathrm{Ob}\left(\mathrm{F}|\, S_{k-1}, V\right) = 0$.
- Observation: the differential $\mathrm{d}^k\mathrm{B}$ is not uniquely determined for $k \geq 3$,

Existence

- **Theorem.** For an arbitrary c.a.s. X such that $\dim \mathrm{A}^1(X) < \infty$ there exists a formal minimal versal deformation $\mathrm{F} : \mathcal{X} \to \tilde{S}$. The base is $\tilde{S}$ is equal to $\mathrm{B}^{-1}(0)$ for a formal power map $\mathrm{B} : \mathrm{A}^1(X) \to \mathrm{A}^2(X)$ such that
- $\mathrm{B}(0) = 0$, $\mathrm{d}^1\mathrm{B}(0) = 0$, $\mathrm{d}^2\mathrm{B}(0) = [\mathrm{A}^1(X), \mathrm{A}^1(X)]$.
- Write $\tilde{S} = \lim\operatorname{ind}\{S_k\}$, where S_k is a k-th small extension of the point $s = 0$ for $k = 1, 2, \ldots$
- For $k \geq 3$ the equation holds $\mathrm{d}^k\mathrm{B}(0)\big|\, V = \mathrm{Ob}\left(\mathrm{F}|\, S_{k-1}, V\right)$ where V is an arbitrary small extension of S_{k-1}. Then $S_k = V$ for the maximal small extension V such that $\mathrm{Ob}\left(\mathrm{F}|\, S_{k-1}, V\right) = 0$.
- Observation: the differential $\mathrm{d}^k\mathrm{B}$ is not uniquely determined for $k \geq 3$,
- therefore the minimal versal deformation is may not canonically defined.

Existence

- **Theorem.** For an arbitrary c.a.s. X such that $\dim \mathrm{A}^1(X) < \infty$ there exists a formal minimal versal deformation $\mathrm{F} : \mathcal{X} \to \tilde{S}$. The base is $\tilde{S}$ is equal to $\mathrm{B}^{-1}(0)$ for a formal power map $\mathrm{B} : \mathrm{A}^1(X) \to \mathrm{A}^2(X)$ such that
- $\mathrm{B}(0) = 0$, $\mathrm{d}^1\mathrm{B}(0) = 0$, $\mathrm{d}^2\mathrm{B}(0) = [\mathrm{A}^1(X), \mathrm{A}^1(X)]$.
- Write $\tilde{S} = \lim\mathrm{ind}\{S_k\}$, where S_k is a k-th small extension of the point $s = 0$ for $k = 1, 2, \ldots$
- For $k \geq 3$ the equation holds $\mathrm{d}^k\mathrm{B}(0)\big|\, V = \mathrm{Ob}\left(\mathrm{F}|\, S_{k-1}, V\right)$ where V is an arbitrary small extension of S_{k-1}. Then $S_k = V$ for the maximal small extension V such that $\mathrm{Ob}\left(\mathrm{F}|\, S_{k-1}, V\right) = 0$.
- Observation: the differential $\mathrm{d}^k\mathrm{B}$ is not uniquely determined for $k \geq 3$,
- therefore the minimal versal deformation is may not canonically defined.
- However, if the bracket $[\mathrm{a}, \mathrm{b}]$ vanishes for all $\mathrm{a} \in \mathrm{A}^0(X), \mathrm{b} \in \mathrm{A}^1(X)$, any minimal versal deformation is universal.

Examples: K3-surfaces

- All complex analytic K3-surfaces (simply connected compact surfaces with trivial canonical class) form a irreducible family $\mathrm{f} : Y \to R$ of dimension $\dim R = 20$. This family is a universal commutative deformation for each fiber $X_s = \mathrm{f}^{-1}(s)$. The associative cohomology of each normal K3-surface is

$$H^0\left(X, \mathcal{T}^0\right) = 0,\ H^1\left(X, \mathcal{T}^0\right) = \mathbb{C}^{20},\ H^0(X, \mathcal{P}) = \mathbb{C},$$
$$H^2\left(X, \mathcal{T}^0\right) = 0,\ H^1(X, \mathcal{P}) = 0,$$

where $\mathcal{T}^0$ is the tangent sheaf on X and $\mathcal{P} = \wedge^2 \mathcal{T}^0 \cong \mathcal{O}_X$. We have $\mathrm{A}^1(X) \cong \mathbb{C}^{21}$ and $\mathrm{A}^2(X) = 0$. There exists a formal minimal versal deformation $\mathrm{F} : \mathcal{X} \to \tilde{S}$ of X, where $\tilde{S}$ is the formal completion of the germ $\left(\mathbb{C}^{21}, 0\right)$. The subgerm $\mathrm{T}^1(X) \cong \mathbb{C}^{20}$ is the base of a versal commutative deformation.

Examples: K3-surfaces

- All complex analytic K3-surfaces (simply connected compact surfaces with trivial canonical class) form a irreducible family $\mathrm{f} : Y \to R$ of dimension $\dim R = 20$. This family is a universal commutative deformation for each fiber $X_s = \mathrm{f}^{-1}(s)$. The associative cohomology of each normal K3-surface is

$$H^0\left(X, \mathcal{T}^0\right) = 0,\ H^1\left(X, \mathcal{T}^0\right) = \mathbb{C}^{20},\ H^0(X, \mathcal{P}) = \mathbb{C},$$
$$H^2\left(X, \mathcal{T}^0\right) = 0,\ H^1(X, \mathcal{P}) = 0,$$

  where $\mathcal{T}^0$ is the tangent sheaf on X and $\mathcal{P} = \wedge^2 \mathcal{T}^0 \cong \mathcal{O}_X$. We have $\mathrm{A}^1(X) \cong \mathbb{C}^{21}$ and $\mathrm{A}^2(X) = 0$. There exists a formal minimal versal deformation $\mathrm{F} : \mathcal{X} \to \tilde{S}$ of X, where $\tilde{S}$ is the formal completion of the germ $\left(\mathbb{C}^{21}, 0\right)$. The subgerm $\mathrm{T}^1(X) \cong \mathbb{C}^{20}$ is the base of a versal commutative deformation.
- The value $\mathrm{DF}(q)$ for the complementary direction q is given by a basic element of the space $H^0(X, \mathcal{P})$ and the deformation F in this direct is a "pure" quantization.

K3-surfaces (cont.)

- Consider the surface $X_4 = \{f_4 = 0\}$ in $\mathbb{CP}^3$ where f_4 is homogeneous polynomial of degree 4 of homogeneous coordinates z_0, z_1, z_2, z_3. A bivector field can be defined in the affine chart $\{z_l \neq 0\}$ by

$$p_{ijk}(a,b) = \varepsilon_{ijkl} z_l^{-1} \det \begin{vmatrix} \partial_i a & \partial_j a & \partial_k a \\ \partial_i b & \partial_j b & \partial_k b \\ \partial_i f & \partial_j f & \partial_k f \end{vmatrix},$$

where $\partial_i = \partial/\partial z_i$, (i,j,k,l) is a permutation of $(0,1,2,3)$ and ε_{ijkl} is the sign of the permutation. This is globally defined Poisson bracket in X which generates a quantization.

K3-surfaces (cont.)

- Consider the surface $X_4 = \{f_4 = 0\}$ in $\mathbb{CP}^3$ where f_4 is homogeneous polynomial of degree 4 of homogeneous coordinates z_0, z_1, z_2, z_3. A bivector field can be defined in the affine chart $\{z_l \neq 0\}$ by

$$p_{ijk}(a, b) = \varepsilon_{ijkl} z_l^{-1} \det \begin{vmatrix} \partial_i a & \partial_j a & \partial_k a \\ \partial_i b & \partial_j b & \partial_k b \\ \partial_i f & \partial_j f & \partial_k f \end{vmatrix},$$

where $\partial_i = \partial / \partial z_i$, (i, j, k, l) is a permutation of $(0, 1, 2, 3)$ and ε_{ijkl} is the sign of the permutation. This is globally defined Poisson bracket in X which generates a quantization.

- There are just two more families of projective complete intersections that contain K3-surfaces:

$$X_{32} = \{f_3 = g_2 = 0\} \subset \mathbb{CP}^4,\ X_{222} = \{f_2 = g_2 = h_2 = 0\} \subset \mathbb{CP}^5$$

Here $f_3, f_2, g_2, h_2 \neq 0$ are arbitrary homogeneous polynomials of respective degrees such that X_{32} and X_{222} are 2-dimensional complex analytic manifolds, occasionally with singularities. For each surface of

K3-surfaces (cont.)

- The surfaces X form a flat deformation in the category of compact complex spaces for each of three types $X = X_4, X_{32}, X_{222}$. The deformation parameters are just coefficients of the polynomials f, g, h, which run in the corresponding projective spaces. For the family of surfaces of types X_{32} and X_{222} the coefficients must avoid some Zariski closed subspaces, where dimension of X jumps above 2 so that the base of deformation is an affine manifold V for each of the three types of surfaces X. The infinitesimal quantization applied to each fiber X of the deformation yields a global associative deformation $\mathcal{X} \to V \times \Delta$, which is a fusion of a commutative deformation of complex spaces and of quantization deformation.

Complex tori

- Let Λ be a discrete lattice in $\mathbb{C}^n$, $n > 1$; the quotient $X = \mathbb{C}^n/\Lambda$ is a complex n-torus. We have $\mathcal{T}^0 \cong \mathcal{O}_X^n$, $\mathcal{P} = \wedge^2\mathcal{T}^0$ for any Λ. Let $\lambda_1, ..., \lambda_{2n} \in \mathbb{C}^n$ be some vectors that generate Λ. The versal commutative deformation of X is given by variation of the first n vectors $\lambda_1, ..., \lambda_n$ as the last n vectors are frozen. The space $\Gamma(X, \mathcal{P})$ consists of tangent bivector fields $q(s) = \sum s_{ij}\partial/\partial z_i \wedge \partial/\partial z_j$ with constant coefficients s_{ij}, Each field q can be extended to a star-product in $\mathcal{O}_X$ by means of the Groenewold-Moyal formula, which is translation invariant and can be lifted on Λ. We obtain a star product depending on a point s. This extension is a formal deformation-quantization of X with the base $\Gamma(X, \mathcal{P})$.

Complex tori (cont.)

- Finally we can combine both: the commutative deformation with the base $\mathrm{T}^1 = H^1\left(X, \mathcal{T}^0\right)$ and the star-product depending on $q \in \Gamma(X, \mathcal{P})$. This is a formal minimal versal deformation of X with the base A^1, which has dimension $n^2\left(n+1\right)/2$.

Complex tori (cont.)

- Finally we can combine both: the commutative deformation with the base $\mathrm{T}^1 = H^1\left(X, \mathcal{T}^0\right)$ and the star-product depending on $q \in \Gamma(X, \mathcal{P})$. This is a formal minimal versal deformation of X with the base A^1, which has dimension $n^2\left(n+1\right)/2$.
- Versal deformation is unobstructed despite the massive spaces $\mathrm{A}^2 \cong H^2\left(X, \mathcal{T}^0\right)$ and $H^2\left(X, \mathcal{O}\right)$. This deformation is universal, whereas the Lie group $\mathrm{A}^0 \cong \mathbb{C}^n$ is not trivial. The reason is that the bracket $\left[\mathrm{A}^0, \mathrm{A}^1\right]$ vanishes.