On Deformations of Singular Poisson Algebras

M. Bordemann, H.-C. Herbig, M. Pflaum arXiv:math-ph/0603078

H.-C. Herbig, S. Iyengar, M. Pflaum arXiv:0811.2152 [LMP 89, 2009, no. 2]

H.-C. Herbig, S. Iyengar (work in progress)

(M,ω) a symplectic manifold, $\{\,,\}$ Poisson bracket

K a compact, connected Lie group

$K \times M \to M$ a Hamiltonian K-action with equivariant moment map $J : M \to \mathfrak{k}^*$

Notation: $A_s := C^\infty(M)$ s = "smooth"

$Z := J^{-1}(0)$, $M_0 := Z/K$

$I_Z^s := \{ f \in A_s \mid f|_Z = 0 \}$, for $\xi \in \mathfrak{k}$: $J_\xi := \langle J, \xi \rangle \in A_s$

$I_J^s := \langle J_\xi \mid \xi \in \mathfrak{k} \rangle_{A_s}$

Thm^s [Bordemann, Herbig, Pflaum '06]

Assume that in addition

(gr)$_s$ $\quad I_Z^s = I_J^s$

(ci)$_s$ $\quad$ The Koszul complex $K(A_s, J)$ on J is acyclic, i.e. has $H_i = 0$ for $i \neq 0$.

Then there exist a continuous formal deformation of the Poisson algebra of smooth functions $C^\infty(M_0)$ on M_0.

remarks: a formal deformation of a Poisson algebra $(A, \cdot, \{,\}, 1)$ is

$$a * b = ab + \sum_{i \geq 1} \nu^i m_i(a,b) \text{ where}$$

↑ formal parameter

$m_i : A \otimes A \to A$ are such that $(A[[\nu]], *, 1)$ is associative and $m_1(a,b) - m_1(b,a) = \{a,b\}$

$$\nu \mathrel{\hat{=}} \frac{\hbar}{i}$$

$$C^\infty(M_0) = A_s^K / (I_Z^s)^K$$ [Arms/Cushman/Gotay '91]

VARIATION: $r =$ "real"

$\mathbb{C}^n$ with standard Kähler structure

$\{z_i, \bar z_j\} = \frac{2}{\sqrt{-1}} \delta_{ij}$

K compact, connected $K \xrightarrow{\rho} U_n = U(\mathbb{C}^n)$

For $\xi \in \mathfrak{k}$, $\rho_\xi \in \mathfrak{u}(n)$. We set $A_\xi := \frac{1}{\sqrt{-1}} \rho_\xi$.

$\Rightarrow A_\xi = (A_\xi)_{ij}$ is a hermitean matrix

moment map: $J_\xi(z, \bar z) := \frac{1}{2} \sum_{i,j} (A_\xi)_{ij} z_i \bar z_j$

Notation: $A_r := \mathbb{C}[z, \bar z]^{-}$ $\quad {}^{-}$ = inv. under ${}^{-}$

Thm^r Assume that in addition

(gh)$_r$ $\langle J_\xi \mid \xi \in \mathfrak{k} \rangle_{A_r} \subset A_r$ is real,

(ci)$_r$ The Koszul complex $K(A_r, J)$ is acyclic,

Then there is a countable set $X \subset \mathbb{R}^\times$ and a family of associative products $(*_\hbar)_{\hbar \in \mathbb{R}^\times \setminus X}$ which deform the Poisson algebra $\mathbb{R}[M_0] \otimes_{\mathbb{R}} \mathbb{C}$. We can achieve that $*_\hbar$ are hermitean, i.e. $\overline{f *_\hbar g} = \bar g *_\hbar \bar f$.

remarks:

- real Nullstellensatz: $I_Z^r = \sqrt[\mathbb{R}]{\langle J_\xi \mid \xi \in \mathfrak{k} \rangle_{A_r}}$

(gh)$_r$ $:\Leftrightarrow$ $\langle J_\xi \rangle_{A_r} = I_Z^r$

I is real $:\Longleftrightarrow I = \sqrt[\mathbb{R}]{I}$ where

$$\sqrt[\mathbb{R}]{I} = \{ a \in I \mid \exists i \in \mathbb{N},\ b_1,\dots,b_r \in A : a^{2i} + \sum_{j=1}^{r} b_j^2 \in I \}$$

- $\mathbb{R}[M_0] = A_r^K / (I_Z^r)^K$

VARIATION $c =$ "complex"

G a connected reductive complex algebraic group

V a finite dimensional representation of G, $n := \dim_{\mathbb{C}} V$

$V \times V^*$ the cotangent lifted representation of G

choose a basis $v_1,\dots,v_n$ of V with dual basis $v^1,\dots,v^n$

a generic point $V \times V^*$ is written as $\sum_{i=1}^{n} q^i v_i + p_i v^i$

$$A_c = \mathbb{C}[V \times V^*] = \mathbb{C}[q^1,\dots,q^n,p_1,\dots,p_n]$$
$$= \mathbb{C}[q,p]$$

For $\xi \in \mathfrak{g}$: $(\varrho_\xi)_i^j = \langle v^j, \xi . v_i \rangle$.

moment map $\phi_\xi = \sum (\varrho_\xi)_i^j q^i p_j \in A_c$

Thm^c Assume that in addition

(gh)$_c$ the ideal $\langle \phi_\xi \mid \xi \in \mathfrak{g} \rangle_{A_c}$ is radical

(ci)$_c$ the Koszul complex $K_\cdot(A_c, \Phi)$ is acyclic

Then there exists a countable set $\mathcal{Y} \subset \mathbb{C}^\times$ and a family of associative deformations $(*_\nu)_{\nu \in \mathbb{C}^\times \setminus \mathcal{Y}}$ of the Poisson algebra of regular functions on the invariant theory symplectic quotient.

remarks

- Hilbert's Nullstellensatz
$$I_Z^c = \sqrt{\langle \phi_\xi \mid \xi \in \mathfrak{g} \rangle}$$
Hence (gh)$_c$ means that $I_Z^c = I_J^c$.
- reduced algebra:
$$(A_c / I_Z^c)^G = A_c^G / (I_Z^c)^G$$
The invariant theory symplectic quotient is Spec of this algebra.
(it might be regular)

In all three cases: (we use (gh))

$$A^K/I_Z^K \cong (A/I_Z)^K \cong N(I_Z)/I_Z$$

as Poisson algebras (for case c = "complex" replace K by G!)

Here $N(I_Z) = \{a \in A \mid \{a, I_Z\} \subset I_Z\}$
= Lie normalizer of the
Lie subalgebra I_Z.

In the s = "smooth" case the $\cong$'s are also isomorphisms of Fréchet algebras.

So, what is the Koszul complex?

(Let us work in the s = "smooth" ~~or~~ r = "real" setup.)

Choose a basis $e_1, \dots, e_\ell$ for $\mathfrak{k}$, and write

$$J_a := J_{e_a} \quad \text{for } a = 1, \dots, \ell.$$

$\xi_a := e_a[-1]$

$K_\bullet(A, J) := A[\xi_1, \dots, \xi_\ell]$ (Koszul sign rule!)

Koszul differential: $\partial = \sum_{a=1}^{\ell} J_a \frac{\partial}{\partial \xi_a}$

$K(A_c, \Phi)$ is defined analogously

remarks

- $K_\cdot(A,J)$ is a supercommutative dg algebra; there is a canonical algebra morphism

$$B := A/I_Z \xleftarrow[res]{} K_\cdot(A,J)$$ (here we use (gh))

- if in addition (ci) holds res is a quasiisomorphism of complexes
- in all three cases we can make out of this a contraction (in the sense of (HPT)):

$$\begin{tikzcd}(B,0) \arrow[r, shift right, "prol"'] & (K_\cdot(A,J),\partial), h \arrow[l, shift right, "res"'] \end{tikzcd}$$

[Huebschmann et al]

i.e. res, prol are morphisms of complexes s.th.

$$res \circ prol = id_B, \quad id_K - prol \circ res = [\partial, h]$$

- in the cases r = "real" and c = "complex" prol and h are constructed by hand
- in the case s = "smooth" we use a two results of [Bierstone/Schwarz 1983] to construct h and prol

Duke Math. J. 50 (1983) no. 1

- note that ∂ is equivariant
- we can choose h to be equivariant; we can also achieve $h^2 = 0$

How are the three setups related?

- "r → s" by interpreting $\mathbb{C}^n$ as a smooth mfd:
  $(gh)_r \Leftrightarrow (gh)_s$ [Arms / Gotay / Jennings 1990]
  $(ci)_r \Rightarrow (ci)_s$ flatness tricks → Malgrange
  $(\overset{?}{\Leftarrow})$
- "s → r" linearize around $z \in Z$ [AG] 90]
  $(gh)_s \Leftrightarrow \forall z \in Z: (gh)_r$ [AG] 90]
  $(ci)_s \overset{?}{\Leftarrow} \forall z \in Z: (ci)_r$ still open
- "r → c" complexify the representation
  $(gh)_r \Leftrightarrow (gh)_c$ & $\dim_{\mathbb{R}} W_i \cap \mathbb{R}^n = \dim_{\mathbb{C}} W_i$
  $(ci)_r \Leftrightarrow (ci)_c$ for all $i = 1, \dots, s$, where
  $Z^c = W_1 \cap \dots \cap W_s$ irr. dec.

Classical BFV-construction:

Introduce odd variables: $\mathcal{P}_a := e_a[1]$, $\eta^a = e^a[-1]$

$$\mathcal{A}_{BFV} := A[\mathcal{P}_1, \dots, \mathcal{P}_\ell, \eta^1, \dots, \eta^\ell] = A[\mathcal{P}, \eta]$$

$\{\mathcal{P}_a, \eta^b\} = \delta_a^b$ extends to a super-Poisson-bracket

Charge: $\theta = \sum_a \eta^a J_a - \frac{1}{2} \sum_{a,b,c} \underbrace{f_{ab}^c}_{\langle e^c, [e_a, e_b] \rangle} \eta^a \eta^b \mathcal{P}_c$

$$\{\theta, \theta\} = 0$$

BFV-differential: $D = \{\theta, -\}$
$= \partial + \delta$

$\partial = \sum_{a=1}^{\ell} J_a \frac{\partial}{\partial \mathcal{P}_a}$ (Koszul), $\delta = \dots$ codiff. of Lie algebra cohom. of $\mathfrak{g}$-module $A[\mathcal{P}]$

Proposition There is a contraction of cochain complexes

$$\begin{tikzcd}(C^{\cdot}(\mathfrak{k},B), d) \arrow[r, shift right, "\phi"'] & (A_{BFV}, D) \arrow[l, shift right, "res"'] \end{tikzcd}, \mathcal{H}$$

$\uparrow$

Lie alg. cocomplex of $\mathfrak{k}$-module B

| if h is equiv.: $\phi = prol$, $\mathcal{H} = h$

Transfer of Poisson structure: $[a],[b] \in H^{\cdot}(\mathfrak{k},B)$

$$\{[a],[b]\} = \left[res\{\phi(a), \phi(b)\}\right]$$

Induced bracket in $H^0(\mathfrak{k},B) = B^K$ is the aforementioned one

Examples

(1) unitary representations of tori: $K = (S^1)^\ell$

$$J_a = \frac{1}{2}\sum_{i=1}^{n} A_{ai}|z_i|^2, \quad A = (A_{ai}) \in \mathbb{Z}^{\ell \times n}$$ matrix of weights

Thm [Herbig, Iyengar, Pflaum '08]

$(gh)_r \Leftrightarrow 0 \in \operatorname{relint}(\operatorname{conv}(A))$

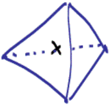
$\Leftrightarrow \operatorname{im}(J)$ is a vector subspace of $\mathfrak{k}^* = \mathbb{R}^\ell$

Moreover, for the shifted moment map $J_\mu := J - \mu$

for $\xi \in \mathfrak{k}^*$: $(gh)_r \Leftrightarrow \mu \in \operatorname{relint}(\operatorname{im}(J))$

J fullfills $(ci)_r \Leftrightarrow \operatorname{rank}(A) = \ell$

$(\Rightarrow J_\mu$ fullfills $(ci)_r)$



Conj.: all unireps of tori can be BFV-quantized.

(2) unireps of SU_2

- SU_2 acts on $\mathbb{C}^2$, hence on $\mathbb{C}[x,y] = \bigoplus \underset{\uparrow}{V_d}$

  binary forms of degree d
- $(V_d)_{d \geq 0}$ give all irreps

Thm [Herbig, Iyengar]

(i) all unireps, except $V_0, V_1, V_1 \oplus V_1$, fulfill $(gh)_r$.

(ii) all unireps, exept V_0, V_1, V_2, fulfill $(ci)_r$.

remark

approach to reduced algebra via invariant theory breaks down at $d \geq 9$

for $d = 7$ see [Bedratyuk'07]

(3) angular momentum for dimension $N \geq 3$:

2×2 minors of $\begin{bmatrix} q^1 & q^2 & \cdots & q^N \\ p_1 & p_2 & \cdots & p_N \end{bmatrix}$

$(gh)_r$ is ok [Bos/Gotay '86, AGJ 90]

$\cancel{(ci)_r}$

observation (Herbig 09): BFV-Quantization can still be performed!

corollary: V_2 and $\underbrace{V_1 \oplus V_1}_{\text{enlarge the group to } SO(4)}$ can be quantized

[AGJ'90]

"kill homology by adding cycles" [Tate '58, Batalin et.al '8x]

[ε_n = number of $\mathcal{P}$'s in homological degree n
"deviations"

$$M_n(t) := \frac{1}{n}\sum_{d|n} \mu\left(\frac{n}{d}\right) t^d$$

"Colonel Moreau's necklace counting fct."

$$(-1)^n M_n(-t) = \begin{cases} M_n(t) + M_{\frac{n}{2}}(t) & \text{for } n \equiv 2 \bmod 4 \\ M_n(t) & \text{else} \end{cases}$$

[Stanley]

$$\boxed{\varepsilon_n = (-1)^{n+1} M_{n+1}(-(N-1))}$$

[Herbig '10]

Quantum BFV-construction

(for r = "real")

- $A = A_r \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[\mathbf{z}, \bar{\mathbf{z}}]$ with hermitean star product $*$, e.g.

$$*_\lambda := m \circ e^{\hbar((\lambda+1)Q + (\lambda-1)\bar{Q})} \quad (\lambda \in \mathbb{R}),$$

where $Q = \sum_{i=1}^{n} \frac{\partial}{\partial z_i} \otimes \frac{\partial}{\partial \bar{z}_i}$, $\bar{Q} = \sum_{i=1}^{n} \frac{\partial}{\partial \bar{z}_i} \otimes \frac{\partial}{\partial z_i}$

$*_{\lambda=0}$ = Moyal-Weyl

$*_{\lambda=1}$ = Wick

$$\overline{a *_\lambda b} = \bar{b} *_\lambda \bar{a}$$

quantum moment map:

$$\mathbf{J}_\xi = J_\xi + \gamma(\hbar)\underbrace{\mathrm{trace}(\mathrm{ad}(\xi))}_{\text{central}} \qquad \xi \in k \qquad \gamma \in \hbar\mathbb{R}[\hbar]$$

important property: $\frac{\hbar}{i}[\mathbf{J}_\xi, -] = \{J_\xi, -\}$

$$\mathbf{J}_\xi = \overline{\mathbf{J}_\xi}$$

---

associative product on $\mathbb{C}[P,\eta]$:

$$a \cdot_\kappa b = m \circ e^{-i\hbar(\kappa P_1 + (1-\kappa)P_2)}$$

$$P_1 = \sum_a \frac{\partial}{\partial P_a} \otimes \frac{\partial}{\partial \eta^a}$$

$$P_2 = \sum_a \frac{\partial}{\partial \eta^a} \otimes \frac{\partial}{\partial P_a}$$

Special cases: $\cdot_{\kappa=\frac12} = \cdot_W$

$\cdot_{\kappa=1} = \cdot_N$

The Weyl product $\cdot_W$ is hermitean:

$$\overline{a \cdot_W b} = (-1)^{|a||b|}\, \bar{b} \cdot_W \bar{a}$$

Weylordered BFV-construction:

- Introduce $*_W$ on $\mathcal{A}_{BFV} = A[P,\eta]$ by tensoring one of the $*_\lambda$ with $\cdot_W$.
- Quantum BFV-charge:

$$\hat{\Theta} = \sum_a \mathbf{J}_a \eta^a - \frac{1}{2}\sum_{a,b,c} f_{ab}^c\, \eta^a \eta^b P_c$$

Important property: $\Theta *_W \Theta = 0$, $\overline{\Theta} = \Theta$

Introduce Weylordered BFV-differential:

$$\mathcal{W} := \frac{\hbar}{i}[\Theta, -]$$

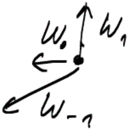
Prop.

$\mathcal{W} = W_1 + W_0 + W_{-1}$

$W_1 = \delta$

$W_0 = \sum_a \mathrm{sym}(J_a)\frac{\partial}{\partial P_a}$

$W_{-1} = \frac{\hbar}{8}\sum_{a,b,c} f_{ab}^{c}\frac{\partial^3}{\partial P_a\,\partial P_b\,\partial \eta^c}$



$\mathrm{sym}(f): A \to A$

$g \mapsto \frac{1}{2}(f * g + g * f)$

⚠ W_0 is a differential $\Leftrightarrow W_{-1} = 0$.

This is essentially the case iff K is a torus!

Solution: Change to normal order:

$$N(a *_n b) = N(a) *_W N(b)$$

$$N = e^{-\frac{i\hbar}{2}\sum \frac{\partial}{\partial P_a}\frac{\partial}{\partial \eta_a}}$$

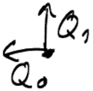
Normal ordered BFV diff.

$$Q := N \circ W \circ N^{-1} = \frac{\hbar}{i}\,\mathrm{ad}_{*_n}(N^{-1}(\Theta)),$$

$$N^{-1}(\Theta) = \Theta + \frac{i\hbar}{4}\sum_{a,b} f_{ab}^{b}\eta^a$$

change quantum moment map

$$J'_a := J_a + \frac{i\hbar}{4}\sum f_{ab}^{b}.$$



Prop. $Q = Q_0 + Q_1$

$Q_1 = \delta$

$$\partial_\hbar := Q_0 = \sum_a \mathrm{right}(J'_a)\frac{\partial}{\partial p_a} + i\hbar \sum_{a,b,c} f_{ab}^{c}\, p_c \frac{\partial^2}{\partial p_a \partial p_b}$$

Introduce involution: $a \mapsto a^* = N\left(\overline{N^{-1}(a)}\right)$
$= N^2(\bar{a})$

$(\mathcal{A}_{BFV}, *_\hbar, Q, {}^*)$ is a $*$-dg algebra

Prop. There exists a countable set $X \subset \mathbb{R}^\times$ and families of maps $(\mathrm{res}_\hbar)_{\hbar \in \mathbb{R}\setminus X}$, $\mathrm{res}_\hbar : A \to B$ and $(h_\hbar)_{\hbar \in \mathbb{R}^\times \setminus X}$, $h_\hbar : K_\bullet \to K_{\bullet+1}$ such that

$$\begin{tikzcd}(B, 0) \arrow[r, shift right, "\mathrm{prol}"'] & (K_\bullet(A, J), \partial_\hbar) \arrow[l, shift right, "\mathrm{res}_\hbar"'] \end{tikzcd}, h_\hbar$$

is a contraction of complexes.

Idea of proof: $h_\hbar = h \circ \left(\mathrm{id}_K + [\partial_\hbar - \partial, h]\right)^{-1}$

- $h_\hbar$ and $\mathrm{res}_\hbar$ can be chosen to be equivariant
- we can also achieve $h_\hbar^2 = 0$

⚠ $\mathrm{res}_\hbar$ is not compatible with complex conjugation.

Thm There are families of linear maps

$(\phi_\hbar)_{\hbar\in\mathbb{R}^\times\setminus X}$, $\phi_\hbar : C^\cdot(\mathfrak{g},B) \to \mathcal{A}^\cdot_{BFV}$, and

$(\mathcal{K}_\hbar)_{\hbar\in\mathbb{R}^\times\setminus X}$, $\mathcal{K}_\hbar : \mathcal{A}^\cdot_{BFV} \to \mathcal{A}^{\cdot-1}_{BFV}$ such that

$$\begin{tikzcd} C^\cdot(\mathfrak{g},B) \arrow[r, shift right, "\phi_\hbar"'] & (\mathcal{A}_{BFV}, Q) \arrow[l, shift right, "res_\hbar"'] \end{tikzcd}$$

, $\mathcal{K}_\hbar$ is a contraction.

one can achieve: $\phi_\hbar = prol$
$\mathcal{K}_\hbar = h_\hbar$

Corollary We can transfer ass. alg. structure

$$[a] * [b] := \left[res_\hbar\left(\phi_\hbar(a) *_\hbar \phi_\hbar(b) \right) \right]$$