A singular view on quantization

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[This talk summarizes many ideas and presents joint works (some, in progress) that would not have been possible without the deep insight on the essence of quantization, and more generally on the role of symmetries and of deformations in physics, of my friend and coworker for 35 years, Moshe Flato.]

Aarhus, December 2010

In Memoriam Moshe Flato (17 September 1937 – 27 November 1998)

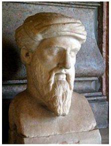


Abstract

In the Copenhagen interpretation of quantum theories, quantization manifests itself as a map (a functor) between a category of ‘classical’ observables (‘functions’ on some classical space) and a category of objects (’operators’) acting in some Hilbert space.

Many warned over time that “it ain’t necessarily so.” Deformation quantization gives a framework to get out of the dilemma, quantization being understood as a deformation of the classical (commutative) composition law of observables. It coincides with usual quantization when such a functor into the Procrustean bed of Hilbert space can be defined and permits generalizations which should play an important role, in particular when dealing with singular spaces. But these generalizations are often formal and not restrictive enough. It is therefore desirable to develop effective formalisms able to ‘focus’ the target of deformation quantization. We first present an overview of the main points of deformation quantization, its conceptual basis in the role of deformations in physics and its relations with usual quantization. We end by indicating some avenues susceptible to focus the ‘quantization’ part of deformation quantization, in particular in view of dealing with singular spaces.

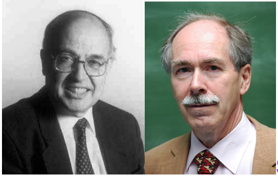
Some epistemological quotes

Pythagoras is the first to be recorded saying that *Mathematics is the way to understand the universe.* Many developed similar ideas, including Sir James Hopwood Jeans: “The Great Architect of the Universe now begins to appear as a pure mathematician.” “We may as well cut out the group theory. That is a subject that will never be of any use in physics.” [Discussing a syllabus in 1910.]

Einstein: “As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.” (“Geometry and Experience”, 27 January 1921) “Quantum mechanics is certainly imposing. But an inner voice tells me that it is not yet the real thing. The theory says a lot, but does not really bring us any closer to the secret of the ‘old one.’ I, at any rate, am convinced that He does not throw dice.” (Letter to Max Born, 4 December 1926).

A few epistemological comments



Sir Michael Atiyah (ICMP London 2000): "Mathematics and Physics are two communities separated by a common language." "Bismarck" quote: Laws, like sausages, cease to inspire respect in proportion as we know how they are made. John G. Saxe 1869.

Gerard 't Hooft (see web site): "I have deviating views on the physical interpretation of quantum theory, and its implications for Big Bang theories of the Universe."
"Most attempts at obtaining theories that unify quantum mechanics with general relativity require violation of locality and/or causality to some degree."

"How Does God Play Dice? (Pre-)Determinism at the Planck Scale."

Remarks. How can the human brain grasp a "Theory of Everything?"
"Curse" of experimental sciences. Mathematical logic: if A and $A \to B$, then B. In real life, imagine model or theory A. If $A \to B$ and "B is nice" (e.g. verified & more), then A! [Inspired by Kolmogorov quote.] (It ain't necessarily so.)

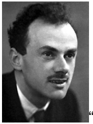
Three questions: Why, What, How? Physicists tell mathematicians what they are doing, not why.

Effectiveness: Physics and Mathematics

In 1960 Eugene P. Wigner wrote his famous provocative paper in Comm. Pure Applied Math. 13, 1-14, *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*, reproduced with many interesting essays in *Symmetries and Reflexions* (Indiana University Press 1967, MIT Press 1970).
Many elaborated on it, including the converse statement by Atiyah (*the unreasonable effectiveness of physics in mathematics*) on several occasions, lately with Dijkgraaf and Hitchin in Phil. Trans. R. Soc. A 2010 368, 913-926, *Geometry and physics*.

An aim of this talk is to indicate by examples dealing with deformation theory that these effectivenesses are quite reasonable but have their limitations.

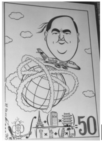
Dirac quote



"... One should examine closely even the elementary and the satisfactory features of our Quantum Mechanics and criticize them and try to modify them, because there may still be faults in them. The only way in which one can hope to proceed on those lines is by looking at the basic features of our present Quantum Theory from all possible points of view. Two points of view may be mathematically equivalent and you may think for that reason if you understand one of them you need not bother about the other and can neglect it. But it may be that one point of view may suggest a future development which another point does not suggest, and although in their present state the two points of view are equivalent they may lead to different possibilities for the future. Therefore, I think that we cannot afford to neglect any possible point of view for looking at Quantum Mechanics and in particular its relation to Classical Mechanics. Any point of view which gives us any interesting feature and any novel idea should be closely examined to see whether they suggest any modification or any way of developing the theory along new lines. A point of view which naturally suggests itself is to examine just how close we can make the connection between Classical and Quantum Mechanics. That is essentially a purely mathematical problem – how close can we make the connection between an algebra of non-commutative variables and the ordinary algebra of commutative variables?

In both cases we can do addition, multiplication, division..." **Dirac**, *The relation of Classical to Quantum Mechanics* (2$^{\text{nd}}$ Can. Math. Congress, Vancouver 1949). U.Toronto Press (1951) pp 10-31.

Flato's deformation philosophy

Physical theories have domain of applicability defined by the relevant distances, velocities, energies, etc. involved. The passage from one domain (of distances, etc.) to another doesn't happen in an uncontrolled way: experimental phenomena appear that cause a paradox and contradict [Fermi quote] accepted theories. Eventually a new fundamental constant enters, the formalism is modified: the attached structures (symmetries, observables, states, etc.) *deform* the initial structure to a new structure which in the limit, when the new parameter goes to zero, "contracts" to the previous formalism. The question is, in which category to seek for deformations? Physics is conservative: if start with e.g. category of associative or Lie algebras, tend to deform in same category. But there are important generalizations: e.g. quantum groups are deformations of (some commutative) Hopf algebras.

Quantization in physics

   Max Planck and black body radiation [ca. 1900]. Niels Bohr atom [1913]. Louis de Broglie [1924]: “wave mechanics” (waves and particles are two manifestations of the same physical reality).

  Traditional quantization (Heisenberg, Born, Hilbert, Schrödinger) of classical system $(\mathbb{R}^{2n}, \{\cdot,\cdot\}, H)$: Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n) \ni \psi$ where acts “quantized” Hamiltonian **H**, energy levels $\mathbf{H}\psi = \lambda\psi$, and von Neumann representation of CCR. [Goethe quote] Define $\hat{q}_\alpha(f)(q) = q_\alpha f(q)$ and $\hat{p}_\beta(f)(q) = -i\hbar \frac{\partial f(q)}{\partial q_\beta}$ for f differentiable in $\mathcal{H}$. Then (CCR) $[\hat{p}_\alpha, \hat{q}_\beta] = i\hbar\delta_{\alpha\beta} I \quad (\alpha, \beta = 1, \ldots, n)$.

Orderings, Weyl, Wigner, Dirac constraints



The couple $(\hat{q}, \hat{p})$ quantizes the coordinates (q, p). A polynomial classical Hamiltonian H is quantized once chosen an operator ordering, e.g. (Weyl) complete symmetrization of $\hat{p}$ and $\hat{q}$. In general the quantization on $\mathbb{R}^{2n}$ of a function $H(q, p)$ with inverse Fourier transform $\tilde{H}(\xi, \eta)$ can be given by (Hermann Weyl [1927] with weight $\varpi = 1$):

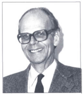
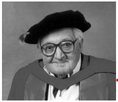
$$H \mapsto \mathbf{H} = \Omega_{\varpi}(H) = \int_{\mathbb{R}^{2n}} \tilde{H}(\xi, \eta) \exp(i(\hat{p}.\xi + \hat{q}.\eta)/\hbar)\varpi(\xi, \eta) d^n\xi d^n\eta.$$

E. Wigner [1932] inverse $H = (2\pi\hbar)^{-n}\mathrm{Tr}[\Omega_1(H)\exp((\xi.\hat{p} + \eta.\hat{q})/i\hbar)]$.

Ω_1 defines an isomorphism of Hilbert spaces between $L^2(\mathbb{R}^{2n})$ and Hilbert–Schmidt operators on $L^2(\mathbb{R}^n)$. Can extend e.g. to distributions.

Constrained systems (e.g. constraints $f_j(p, q) = 0$): Dirac formalism [1950].

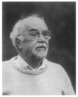
Classical $\leftrightarrow$ Quantum correspondence

The correspondence $H \mapsto \Omega(H)$ is not an algebra homomorphism, neither for ordinary product of functions nor for the Poisson bracket P ("Van Hove theorem"). Take two functions u_1 and u_2, then (Groenewold [1946], Moyal [1949]):
$\Omega_1^{-1}(\Omega_1(u_1)\Omega_1(u_2)) = u_1 u_2 + \frac{i\hbar}{2}\{u_1, u_2\} + O(\hbar^2)$, and similarly for bracket.
More precisely Ω_1 maps into product and bracket of operators (resp.):
$u_1 *_M u_2 = \exp(tP)(u_1, u_2) = u_1 u_2 + \sum_{r=1}^{\infty} \frac{t^r}{r!} P^r(u_1, u_2)$ (with $2t = i\hbar$),
$M(u_1, u_2) = t^{-1} \sinh(tP)(u_1, u_2) = P(u_1, u_2) + \sum_{r=1}^{\infty} \frac{t^{2r}}{(2r+1)!} P^{2r+1}(u_1, u_2)$
We recognize formulas for deformations of algebras.

Deformation quantization: forget the correspondence principle Ω and work in an *autonomous* manner (but with care) with "functions" on phase spaces.

Some other mathematicians' approaches

Geometric quantization (Kostant, Souriau).

[1970's. Mimic correspondence principle for general phase spaces M. Look for generalized Weyl map from functions on M:] Start with "prequantization" on $L^2(M)$ and tries to halve the number of degrees of freedom using (complex, in general) polarizations to get Lagrangian submanifold $\mathcal{L}$ of dimension half of that of M and quantized observables as operators in $L^2(\mathcal{L})$. Fine for representation theory (M coadjoint orbit, e.g. solvable group) but few observables can be quantized.

Berezin quantization. (ca.1975). Quantization is an algorithm by which a quantum system corresponds to a classical dynamical one, i.e. (roughly) is a functor between a category of algebras of classical observables (on phase space) and a category of algebras of operators (in Hilbert space).

Examples: Euclidean and Lobatchevsky planes, cylinder, torus and sphere, Kähler manifolds and duals of Lie algebras. [Only (M, π), no H here.]

Gukov – Witten


Branes and Quantization (Gukov–Witten, ATMP 2009)
Quantization via Mirror Symmetry (Gukov, Takagi Lecture 23 November 2010)
Quantization is an art (Gukov).
Very interesting theory – it makes no sense at all [Groucho Marx, about Quantum Mechanics.]
The idea is: (M, ω) symplectic $\sim$ [something] where [something] may be e.g.
Geometric quantization (prequantum line bundle and auxilliary choice of polarization)
Deformation of algebra, "with no auxilliary choices" (deformation quantization)
Quantization "via Categorification" (via Branes, via Mirror Symmetry):
(A-side) Symplectic geometry, and try to relate to
complex geometry of "mirror" (and concepts related to Hilbert space) (B-side).
Note: In deformation quantization, one does not have a priori "auxilliary" choices, but such choices come (and came) back in, when one wants to have a practical theory. An open question is to introduce these in a more geometric and systematic manner.

When we do not (can not) have the Procrustean bed of Hilbert space, it's really an art.

The framework of deformation quantization

Poisson manifold (M, π), deformations of product of functions.

Inspired by deformation philosophy, based on Gerstenhaber's deformation theory [Flato, Lichnerowicz, Sternheimer; and Vey; mid 70's] [Bayen, Flato, Fronsdal, Lichnerowicz, Sternheimer, LMP '77 & Ann. Phys. '78]

• $\mathcal{A}_t = C^\infty(M)[[t]]$, formal series in t with coefficients in $C^\infty(M) = A$. Elements: $f_0 + tf_1 + t^2 f_2 + \cdots$ (t formal parameter, not fixed scalar.)
• Star product $\star_t \colon \mathcal{A}_t \times \mathcal{A}_t \to \mathcal{A}_t$; $f \star_t g = fg + \sum_{r \geq 1} t^r C_r(f, g)$
- C_r are bidifferential operators null on constants: $(1 \star_t f = f \star_t 1 = f)$.
- $\star_t$ is associative and $C_1(f, g) - C_1(g, f) = 2\{f, g\}$, so that $[f, g]_t \equiv \frac{1}{2t}(f \star_t g - g \star_t f) = \{f, g\} + O(t)$ is Lie algebra deformation.

Basic paradigm. Moyal product on $\mathbb{R}^{2n}$ with the canonical Poisson bracket P:
$F \star_M G = \exp\left(\frac{i\hbar}{2} P\right)(F, G) \equiv FG + \sum_{k \geq 1} \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k P^k(F, G)$.

Applications and Equivalence

Equation of motion (time τ): $\frac{dF}{d\tau} = [H, F]_M \equiv \frac{1}{i\hbar}(H \star_M F - F \star_M H)$
Link with Weyl's rule of quantization: $\Omega_1(F \star_M G) = \Omega_1(F)\Omega_1(G)$

Equivalence of two star-products $\star_1$ and $\star_2$.

- Formal series of differential operators $T(f) = f + \sum_{r\geq 1} t^r T_r(f)$.
- $T(f \star_1 g) = T(f) \star_2 T(g)$.

For symplectic manifolds, equivalence classes of star-products are parametrized by the 2nd de Rham cohomology space $H^2_{dR}(M)$: $\{\star_t\}/\sim\, = H^2_{dR}(M)[[t]]$ (Nest-Tsygan [1995] and others). In particular, $H^2_{dR}(\mathbb{R}^{2n})$ is trivial, all deformations are equivalent.
Kontsevich: $\{$Equivalence classes of star-products$\} \equiv \{$equivalence classes of formal Poisson tensors $\pi_t = \pi + t\pi_1 + \cdots\}$.

Remarks: - The choice of a star-product fixes a quantization rule.
- Operator orderings can be implemented by good choices of T (or ϖ).
- On $\mathbb{R}^{2n}$, all star-products are equivalent to Moyal product (cf. von Neumann uniqueness theorem on projective UIR of CCR).

Existence and Classification

Let (M, π) be a Poisson manifold. $f \tilde{\star} g = fg + t\{f, g\}$ does not define an associative product. But $(f \tilde{\star} g) \tilde{\star} h - f \tilde{\star} (g \tilde{\star} h) = O(t^2)$.

Is it always possible to modify $\tilde{\star}$ in order to get an associative product?

Existence, symplectic case:
– DeWilde-Lecomte [1982]: Glue local Moyal products.
– Omori-Maeda-Yoshioka [1991]: Weyl bundle and glueing.
– Fedosov [1985,1994]: Construct a flat abelian connection on the Weyl bundle over the symplectic manifold.

General Poisson manifold M with Poisson bracket P:
Solved by Kontsevich [1997, LMP 2003]. "Explicit" local formula: $(f, g) \mapsto f \star g = \sum_{n \geq 0} t^n \sum_{\Gamma \in G_{n,2}} w(\Gamma) B_\Gamma(f, g)$, defines a differential star-product on $(\mathbb{R}^d, P)$; globalizable to M. Here $G_{n,2}$ is a set of graphs Γ, $w(\Gamma)$ some weight defined by Γ and $B_\Gamma(f, g)$ some bidifferential operators.

Particular case of Formality Theorem. Operadic approach

"Singular" deformation quantization

Algebraic varieties. Kontsevich LMP 2001 (semi-formal deformation quantization is either canonical – e.g. Poisson Lie Groups – or impossible – e.g. K3 surfaces –), Yekutieli 2005-10. Gerbes and algebroid stacks, Nest–Tsygan et al. etc.

"Manifolds with singularities." Frønsdal (cones), Frønsdal–Kontsevich (singular planar curves, LMP 2007), Frønsdal (minimal coadjoint orbit, LMP 2009): nontrivial Harrison cohomology.

Analytic manifolds, complex Poisson. Palamodov (LMP 2007) based on Grothendieck's "Éléments de Géométrie Analytique" in Séminaire Cartan 1960/61. Schapira et al.

Manifolds with corners. Melrose's b-calculus, ΨDO and index theorems (no star yet). "Resolution of singularities". Cf. also Boutet de Monvel calculus in Toeplitz operators context.

Nambu mechanics with n-linear brackets, e.g. evolution of F with 2 Hamiltonians G,H given by $\frac{dF}{dt} = \frac{\partial(F,G,H)}{\partial(x,y,z)} \equiv F, G, H$, Jacobian of map $\mathbb{R}^3 \ni (x, y, z) \to (F, G, H) \in \mathbb{R}^3$. "Zariski" second quantized (based on factorisation of real polynomials into irreducibles, morally $\hbar^2 = 0$) [DFST]. If Harrison cohomology nontrivial has simpler quantization with abelian nontrivial star product.

More general deformations ("parameter" acts on algebra right and/or left, Pinczon–Nadaud).

This is Quantization

A star-product provides an *autonomous* quantization of a manifold M. BFFLS '78: Quantization is a deformation of the composition law of observables of a classical system: $(A, \cdot) \to (A[[t]], \star_t)$, $A = C^\infty(M)$.

Star-product $\star$ ($t = \frac{i}{2}\hbar$) on Poisson manifold M and Hamiltonian H; introduce the star-exponential: $\mathrm{Exp}_\star(\frac{\tau H}{i\hbar}) = \sum_{r\geq 0} \frac{1}{r!}(\frac{\tau}{i\hbar})^r H^{\star r}$.

Corresponds to the unitary evolution operator, is a singular object i.e. belongs not to the quantized algebra $(A[[t]], \star)$ but to $(A[[t, t^{-1}]], \star)$. Singularity at origin of its trace, Harish Chandra character for UIR of semi-simple Lie groups.

Spectrum and states are given by a spectral (Fourier-Stieltjes in the time τ) decomposition of the star-exponential.

Paradigm: Harmonic oscillator $H = \frac{1}{2}(p^2 + q^2)$, Moyal product on $\mathbb{R}^{2\ell}$.

$\mathrm{Exp}_\star(\frac{\tau H}{i\hbar}) = \big(\cos(\frac{\tau}{2})\big)^{-1} \exp\big(\frac{2H}{i\hbar}\tan(\frac{\tau}{2})\big) = \sum_{n=0}^\infty \exp\big(-i(n + \frac{\ell}{2})\tau\big)\pi_n^\ell$.

Here ($\ell = 1$ but similar formulas for $\ell \geq 1$, L_n is Laguerre polynomial of degree n)

$\pi_n^1(q,p) = 2\exp\big(\frac{-2}{\hbar}H(q,p)\big)(-1)^n L_n\big(\frac{4}{\hbar}H(q,p)\big)$.

Conventional vs. deformation quantization

- It is a matter of practical feasibility of calculations, when there are Weyl and Wigner maps to intertwine between both formalisms, to choose to work with operators in Hilbert spaces or with functional analysis methods (distributions etc.) Dealing e.g. with spectroscopy (where it all started; cf. also Connes) and finite dimensional Hilbert spaces where operators are matrices, the operatorial formulation is easier.
- When there are no precise Weyl and Wigner maps (e.g. very general phase spaces, maybe infinite dimensional) one does not have much choice but to work (maybe "at the physical level of rigor") with functional analysis. Contrarily to what some (excellent physicists) assert, deformation quantization is quantization and not a mere reformulation: it permits (in concrete cases) to take for $\hbar$ its value, when there are Weyl and Wigner maps one can translate its results in Hilbert space, and e.g. for the 2-sphere there is a special behavior when the radius of the sphere has quantized values related to the Casimir values of $SO(3)$.

Composite electrodynamics

Photon (composite QED) and new infinite dimensional algebras. Flato, M.; Fronsdal, C. *Composite electrodynamics*. J. Geom. Phys. 5 (1988), no. 1, 37–61.
Singleton theory of light, based on a pure gauge coupling of scalar singleton field to electromagnetic current. Like quarks, singletons are essentially unobservable. The field operators are not local observables and therefore need not commute for spacelike separation, hence (like for quarks) generalized statistics. Then a pure gauge coupling generates real interactions – ordinary electrodynamics in AdS space. Singleton field operator $\phi(x) = \sum_j \phi^j(x) a_j$ +h.c. A concept of normal ordering in theory with unconventional statistics is worked out; there is a natural way of including both photon helicities.

Quantization (in this context) is a study in representation theory of certain infinite-dimensional, nilpotent Lie algebras (generated by the a_j), of which the Heisenberg algebra is the prototype (and included in it for the photon). Compatible with QED.

Singleton-based field theory in AdS

Dis and Racs and around (mostly M. Flato & C. Frønsdal)
Interacting singletons. Lett. Math. Phys. 44 (1998), no. 3, 249–259. (MF, CF)
Singleton fields, in the context of strings and membranes, have been regarded as topological gauge fields that can interact only at the boundary of anti-de Sitter space. At spatial infinity they may have a more physical manifestation as constituents of massless fields in spacetime. The composite character of massless fields is expressed by field-current identities that relate ordinary massless field operators to singleton currents and stress-energy tensors. Naive versions of such identities do not make sense, but when the singletons are described in terms of dipole structures, such constructions are at least formally possible. The new proposal includes and generalizes an early composite version of QED, and includes quantum gravity, super gravity and models of QCD. Unitarity of such theories is conjectural.

Singleton field theory and neutrino oscillations in AdS

Singletons, Physics in AdS Universe and Oscillations of Composite Neutrinos,
Lett. Math. Phys. 48 (1999), no. 1, 109–119. (MF, CF, DS)

The study starts with the kinematical aspects of singletons and massless particles. It extends to the beginning of a field theory of composite elementary particles and its relations with conformal field theory, including very recent developments and speculations about a possible interpretation of neutrino oscillations and CP violation in this context. This framework was developed since the 70's. Based on our deformation philosophy of physical theories, it deals with elementary particles composed of singletons in anti-de Sitter spacetime.

Composite neutrinos' oscillations

Developing a field theory of composite neutrinos (neutrinos composed of singleton pairs with, e.g., three flavors of singletons) it might be possible to correlate oscillations between the three kinds of neutrinos with the AdS_4 description of these 'massless' particles. Of course any reasonable estimate of the value of the cosmological constant rules out a direct connection to the value of experimental parameters like PC violation coupling constants or neutrino masses. PC violation is a feature of composite electrodynamics and any direct observation of singletons, even at infinity, will imply PC violation. If more than one singleton flavor is used, as is appropriate in the context of neutrinos, then PC invariance can be restored in the electromagnetic sector, but in that case, neutrino oscillations will imply PC violation. The structure of Anti de Sitter field theory, especially that of singleton field theory, may provide a natural framework for a description of neutrino oscillations.

Composite leptons and flavor symmetry

The electroweak model is based on “the weak group”, $S_W = SU(2) \times U(1)$, on the Glashow representation of this group, carried by the triplet $(\nu_e, e_L; e_R)$ and by each of the other generations of leptons.

Suppose that:
(a) There are three bosonic singletons $(R^N R^L; R^R) = (R^A)_{A=N,L,R}$ (three “Rac”s) that carry the Glashow representation of S_W;

(b) There are three spinorial singletons $(D_\varepsilon, D_\mu; D_\tau) = (D_\alpha)_{\alpha=\varepsilon,\mu,\tau}$ (three “Di”s). They are insensitive to S_W but transform as a Glashow triplet with respect to another group S_F (the “flavor group”), isomorphic to S_W;

(c) The vector mesons of the standard model are Rac-Rac composites, the leptons are Di-Rac composites, and there is a set of vector mesons that are Di-Di composites and that play exactly the same role for S_F as the weak vector bosons do for S_W: $W_A^B = \bar{R}^B R_A$, $L_\beta^A = R^A D_\beta$, $F_\beta^\alpha = \bar{D}_\beta D^\alpha$.

These are initially massless, massified by interaction with Higgs.

Composite leptons massified

Let us concentrate on the leptons ($A = N, L, R$; $\beta = \varepsilon, \mu, \tau$)

$$(L^A_\beta) = \begin{pmatrix} \nu_e & e_L & e_R \\ \nu_\mu & \mu_L & \mu_R \\ \nu_\tau & \tau_L & \tau_R \end{pmatrix} . \tag{1}$$

A factorization $L^A_\beta = R^A D_\beta$ is strongly urged upon us by the nature of the phenomenological summary in (1). Fields in the first two columns couple horizontally to make the standard electroweak current, those in the last two pair off to make Dirac mass-terms. Particles in the first two rows combine to make the (neutral) flavor current and couple to the flavor vector mesons. The Higgs fields have a Yukawa coupling to lepton currents, $\mathcal{L}_{\rm Yu} = -g_{\rm Yu}\bar{L}^\beta_A L^B_\alpha H^{\alpha A}_{\beta B}$. The electroweak model was constructed with a single generation in mind, hence it assumes a single Higgs doublet. We postulate additional Higgs fields, coupled to leptons in the following way, $\mathcal{L}'_{\rm Yu} = h_{\rm Yu} L^A_\alpha L^B_\beta K^{\alpha\beta}_{AB} + \text{h.c.}$. This model predicts 2 new mesons, parallel to the W and Z of the electroweak model (Frønsdal, LMP 2000). But too many free parameters.

Do the same for quarks (and gluons), adding color?

Questions and facts

Even if know “intimate structure” of particles (as composites of quarks etc. or singletons): How, when and where happened “baryogenesis”? [Creation of ‘our matter’, now 4% of universe mass, vs. 74% ‘dark energy’ and 22 % ‘dark matter’; and matter–antimatter asymetry, Sakharov 1967.] Everything at “big bang”?! [Shrapnel of ‘stem cells’ of initial singularity?]

Facts: $SO_q(3,2)$ at even root of unity has finite-dimensional UIRs (“compact”?).
Black holes à la ’t Hooft: can communicate with them, by interaction at surface.

Noncommutative (quantized) manifolds. E.g. quantum 3- and 4-spheres (Connes with Landi and Dubois-Violette); spectral triples $(\mathcal{A}, \mathcal{H}, D)$).

Connes’ Standard Model with neutrino mixing, minimally coupled to gravity. Space-time is Riemannian compact spin 4-manifold (Barrett has Lorentzian version) $\times$ finite (32) NCG. More economical than SUSYSM and predicts Higgs mass at upper limit (SUSYSM gives lower). [Recent with Marcolli and Chamseddine. (Aug. 2009) Marcolli’s early universe “Linde” models from NCG, with negative gravity & dark matter models with sterile neutrinos.]

Conjectures and a speculative answer

[Odessa Rabbi anecdote] Space-time could be, at very small distances, not only deformed (to AdS_4 with tiny negative curvature ρ, which does not exclude at cosmological distances to have a positive curvature or cosmological constant, e.g. due to matter) but also "quantized" to some $qAdS_4$. Such $qAdS_4$ could be considered, in a sense to make more precise (e.g. with some measure or trace) as having "finite" (possibly "small") volume (for q even root of unity). At the "border" of these one would have, for most practical purposes at "our" scale, the Minkowski space-time, obtained by $q\rho \to 0$. They could be considered as some "black holes" from which "q-singletons" would emerge, create massless particles that would be massified by interaction with dark matter or dark energy. That could (and should, otherwise there would be manifestations closer to us, that were not observed) occur mostly at or near the "edge" of our universe in accelerated expansion. These "qAdS black holes" ("inside" which one might find compactified extra dimensions) could be a kind of "shrapnel" resulting from the Big Bang (in addition to background radiation) and provide a clue to baryogenesis.

A NCG model for $qAdS_4$

To AdS_n, $n \geq 3$, we associate *naturally* a symplectic symmetric space (M, ω, s). The data of any invariant (formal or not) deformation quantization on (M, ω, s) yields canonically universal deformation formulae (procedures associating to a topological algebra $\mathbb{A}$ having a symmetry $\mathcal{G}$ a deformation $\mathbb{A}_\theta$ in same category) for the actions of a non-Abelian solvable Lie group $\mathcal{R}_0$ (one-dimensional extension of the Heisenberg group $\mathcal{H}_n$), given by an oscillatory integral kernel.

Using it we (P.Bieliavsky, LC, DS & YV) define a noncommutative Lorentzian spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ where $\mathcal{A}^\infty := (L^2_{\mathrm{right}}(\mathcal{R}_0))^\infty$ is a NC Fréchet algebra modelled on the space $\mathcal{H}^\infty$ of smooth vectors of the regular representation on the space $\mathcal{H}$ of square integrable functions on $\mathcal{R}_0$, and D a Dirac operator acting as a derivation of the noncommutative bi-module structure, and for all $a \in \mathcal{A}^\infty$, the commutator $[D, a]$ extends to $\mathcal{H}$ as a bounded operator. The underlying commutative limit is endowed with a causal black hole structure (for $n \geq 3$) encoded in the $\mathcal{R}_0$-group action.

Perspectives and cosmological speculations

1. Define within the present Lorentzian context the notion of causality at the operator algebraic level.
2. Representation theory for $\mathrm{SO}_q(2, n)$ (e.g. new reps. at root of unity, analogs of singletons, 'square root' of massless reps. of AdS or Poincaré, etc.) Also maybe quantized exceptional groups.
3. Define a kind of trace giving finite "q-volume" for qAdS at even root of unity (possibly in TVS context).
4. Find analogs of all the 'good' properties (e.g. compactness of the resolvent of D) of Connes' spectral triples in compact Riemannian case, possibly with quadruples $(\mathcal{A}, \mathcal{E}, D, \mathcal{G})$ where $\mathcal{A}$ is some topological algebra, $\mathcal{E}$ an appropriate TVS, D some (bounded on $\mathcal{E}$) "Dirac" operator and $\mathcal{G}$ some symmetry.
5. Limit $\rho q \to 0$ ($\rho < 0$ being AdS curvature)?
6. Unify (groupoid?) Poincaré in Minkowski space (possibly modified locally by the presence of matter) with these $\mathrm{SO}_q(2, 3)$ in the qAdS "black holes".
7. Field theory on such q-deformed spaces, etc.