Max Planck Institute for Mathematics
California Institute of Technology

Resurgence in complex Chern-Simons theory and new invariants of 3-manifolds

based on: w/ P.Putrov, C.Vafa, arXiv:1602.05302

w/ M.Marino, P.Putrov, arXiv:1605.07615

"I am attracted by simply-stated questions about very concrete objects which nevertheless have a trail which leads back into some serious mathematics."

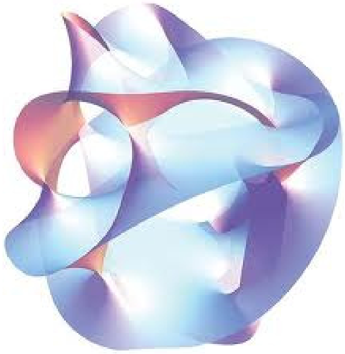
Nigel Hitchin

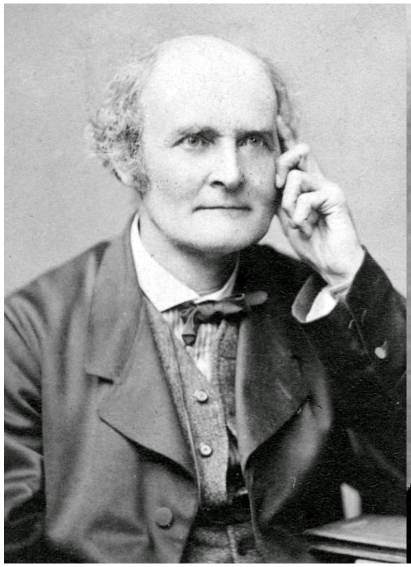
"Mathematics plays the role of experiment in String Theory."

Shing-Tung Yau

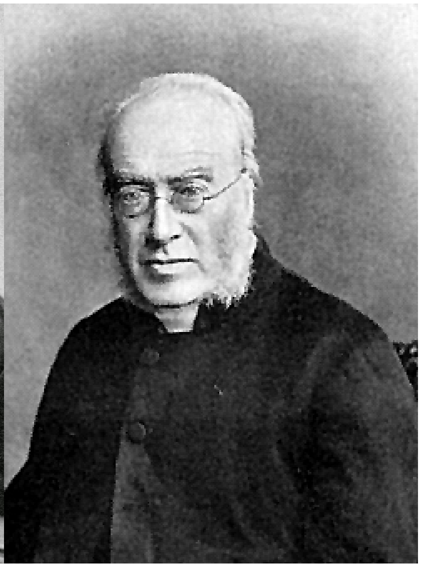
27 lines on a general cubic surface

Hermann Schubert
(1848-1911)





Arthur Cayley
(1821-1895)

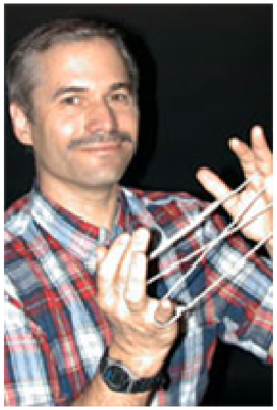


George Salmon
(1819-1904)

2,875 lines (degree-1) on a quintic 3-fold

609,250 conics (degree-2) on a quintic 3-fold

Sheldon Katz, 1986

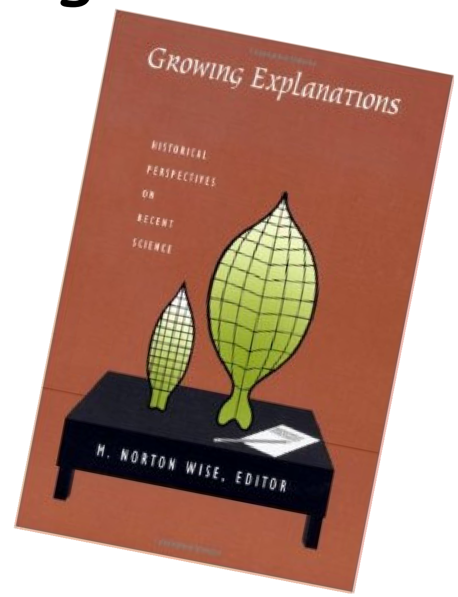


317,206,375 degree-3 curves on a quintic 3-fold

"Suddenly, in late 1990, Candelas and his collaborators Xenia de la Ossa, Paul Green, and Linda Parkes (COGP) saw a way to use mirror symmetry to barge into the geometers' garden.

...

Candelas, who liked calculating things, thought maybe it was in fact tractable using some algebra and a home computer."



2,682,549,425 degree-3 curves on a quintic 3-fold

Geir Ellingssrud and Stein Arild Stromme, May 1991

Chapter One
BHAG

Big Hairy Audacious Goal

I believe that this nation should commit itself to achieving the goal, before this decade is out, of landing a man on the moon and returning him safely to the earth.

John F. Kennedy, Address to Congress on Urgent National Needs, May 25, 1961

Before the decade is out, categorify this q-series:

$$\sum_{m=0}^{\infty}(-1)^m q^{-\frac{m(m+1)}{2}}(q^{m+1})_{m+1} =$$

$$= 1 - q - q^5 + q^{10} - q^{11} - q^{18} - q^{30} - q^{41} + \ldots$$

$$S^3_{-1}(\mathbf{4_1})$$

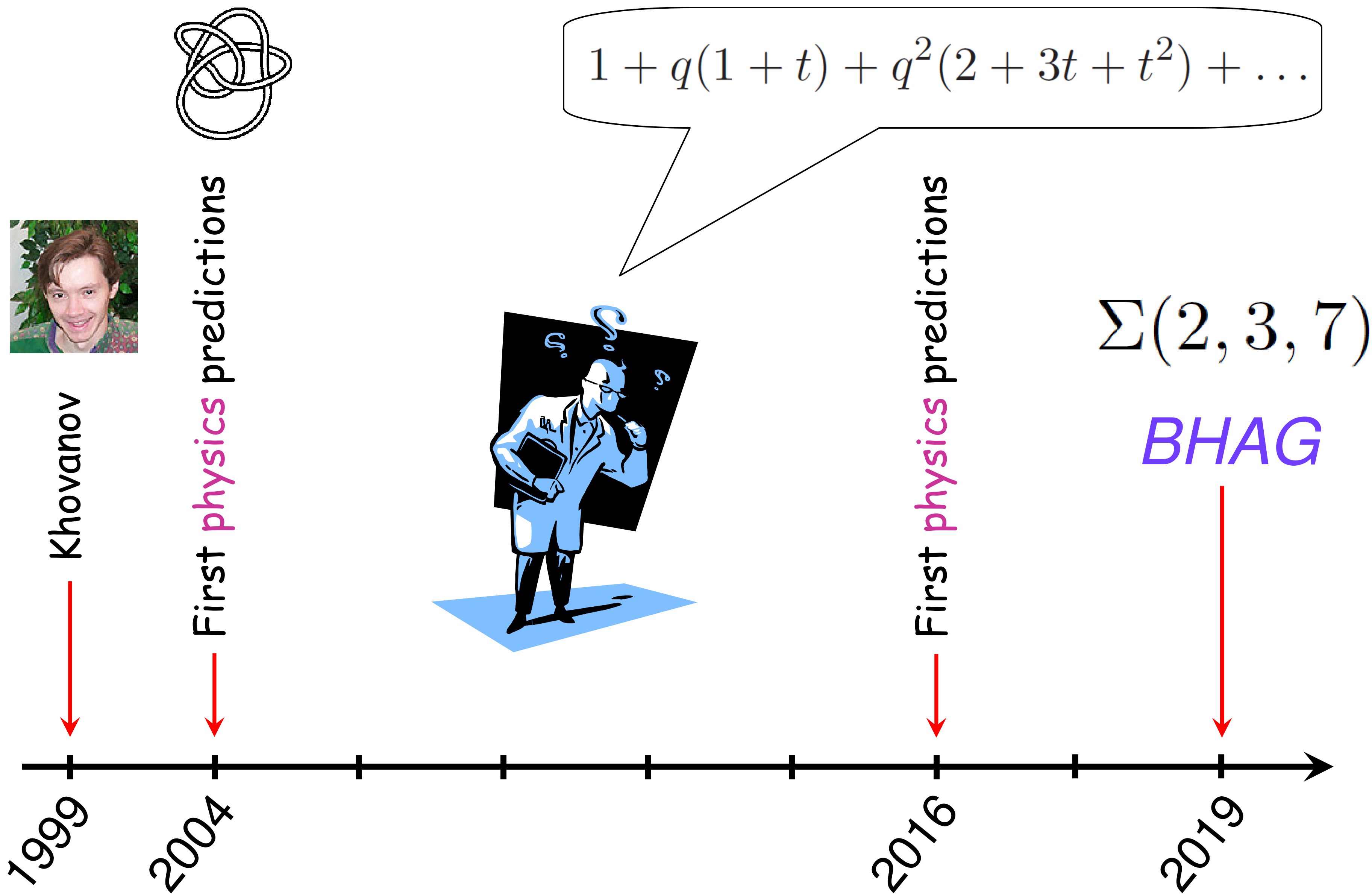
Back in 2003 …

Categorify sl(4) knot invariant:

$$1 + \frac{1}{q^4} - \frac{1}{q} - q + q^4$$

[S.G., A.Schwarz, C.Vafa, 2004]

$$1 + \frac{1}{q^4 t^2} + \frac{1}{qt} + qt + q^4 t^2$$



$1 + q(1+t) + q^2(2+3t+t^2) + \ldots$
Khovanov
First physics predictions
First physics predictions
$\Sigma(2,3,7)$
BHAG
1999
2004
2016
2019

Chapter Two
Lessons from Knots

Knot homology circa 2003

- Only a few homological knot invariants:
  - Khovanov homology (N=2, combinatorial)

Kh(T(7,6)).

In 1 day

Old techniques:

~1,000 years,

~1GGb RAM.

j \ r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
57																	1			1
55																	1	1		
53															1	2		1	1	
51													1	1		2	1			
49														3	1		1			
47												3	1		1					
45										2		1	2							
43								1		1	2									
41						1		1	2		1									
39						1	1		1											
37				1	1		1													
35					1															
33			1																	
31	1																			
29	1																			

the alternating knots zone

Knot homology circa 2003

- Only a few homological knot invariants:
  - Khovanov homology (N=2, combinatorial)
  - Knot Floer homology (N=0, symplectic)
- Categorification of HOMFLY not expected

sl(3) link homology

Mikhail Khovanov

Abstract We define a bigraded homology theory whose Euler characteristic is the quantum sl(3) link invariant.

AMS Classification 81R50, 57M27; 18G60

Keywords Knot, link, homology, quantum invariant, sl(3)

Figure 1: Quantum $\mathfrak{sl}(n)$ skein formula

When $\mathfrak{g} = \mathfrak{sl}(n)$ and each components of L is labelled either by the defining representation V or its dual, the invariant is determined by the skein relation in Figure 1. If we introduce a second variable $p = q^n$, the skein relation gives rise to the HOMFLY polynomial, a 2-variable polynomial invariant of oriented links [2]. We do not believe in a triply-graded homology theory categorifying the HOMFLY polynomial. Instead, for each $n \geq 0$ there should exist a bi-graded theory categorifying the (q, q^n) specialization of HOMFLY. For $n = 0$

Knot homology circa 2003

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  - Khovanov homology (N=2, combinatorial)
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- Categorification of HOMFLY not expected
- Higher rank not computable
- No SO/Sp groups
- No colors

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- Categorification of HOMFLY not expected
- Higher rank not computable
- No SO/Sp groups
- No colors
- Many unexplained patterns

Chapter Three
Connection to Physics

"Connecting the dots"

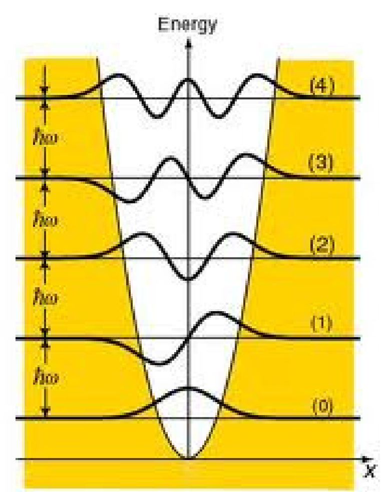
[S.G., A.Schwarz, C.Vafa, 2004]

Knot Homology
(Khovanov,...)

=

BPS spectrum
(Q-cohomology)

$$P(q) = \sum q^i (-1)^j \dim \mathcal{H}^{i,j}$$



$\mathcal{H}_{\text{BPS}}$

“Connecting the dots”

Physical Setup

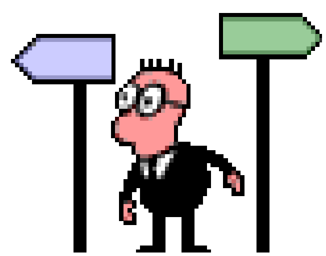
space-time:	$\mathbb{R} \times T^*M_3 \times TN_4$	duality	$\mathbb{R} \times X \times TN_4$
N M5-branes:	$\mathbb{R} \times M_3 \times \mathbb{R}^2_q$	⟷	$\mathbb{R} \times L_K \times \mathbb{R}^2_q$
M5′-branes:	$\mathbb{R} \times L_K \times \mathbb{R}^2_q$		(+ flux)

$\mathcal{H}_{\text{BPS}}$ doubly graded

fixed N

$\mathcal{H}_{\text{BPS}}$ triply graded

packages all ranks

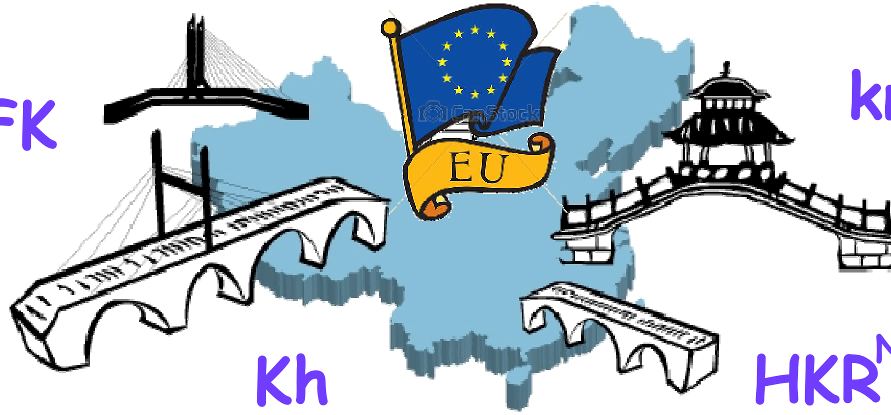


[H.Ooguri, C.Vafa, 1999]

What physics gives us?

- HOMFLY homology & new bridges:

HFK

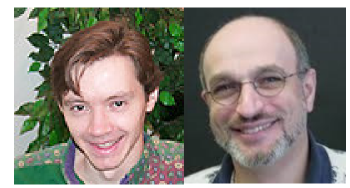


knot contact homology

A-polynomial

Kh

HKR^N



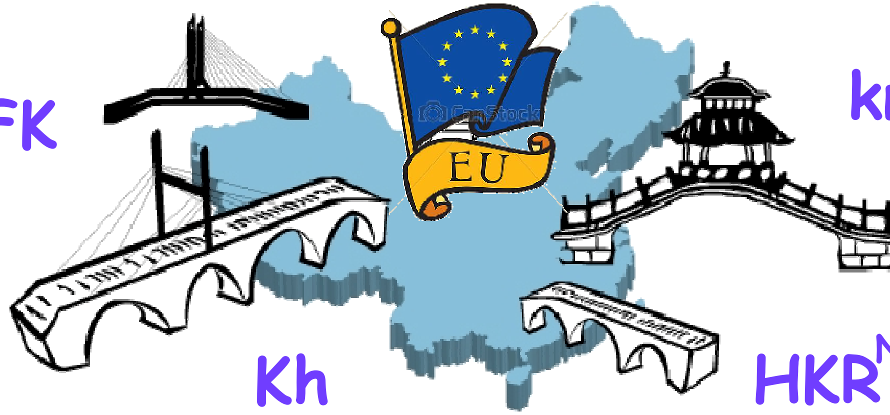
What physics gives us?

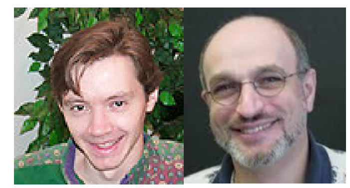
- HOMFLY homology & new bridges:

- New structural properties (differentials, recursion relations, …)

What physics gives us?

- HOMFLY homology & new bridges:



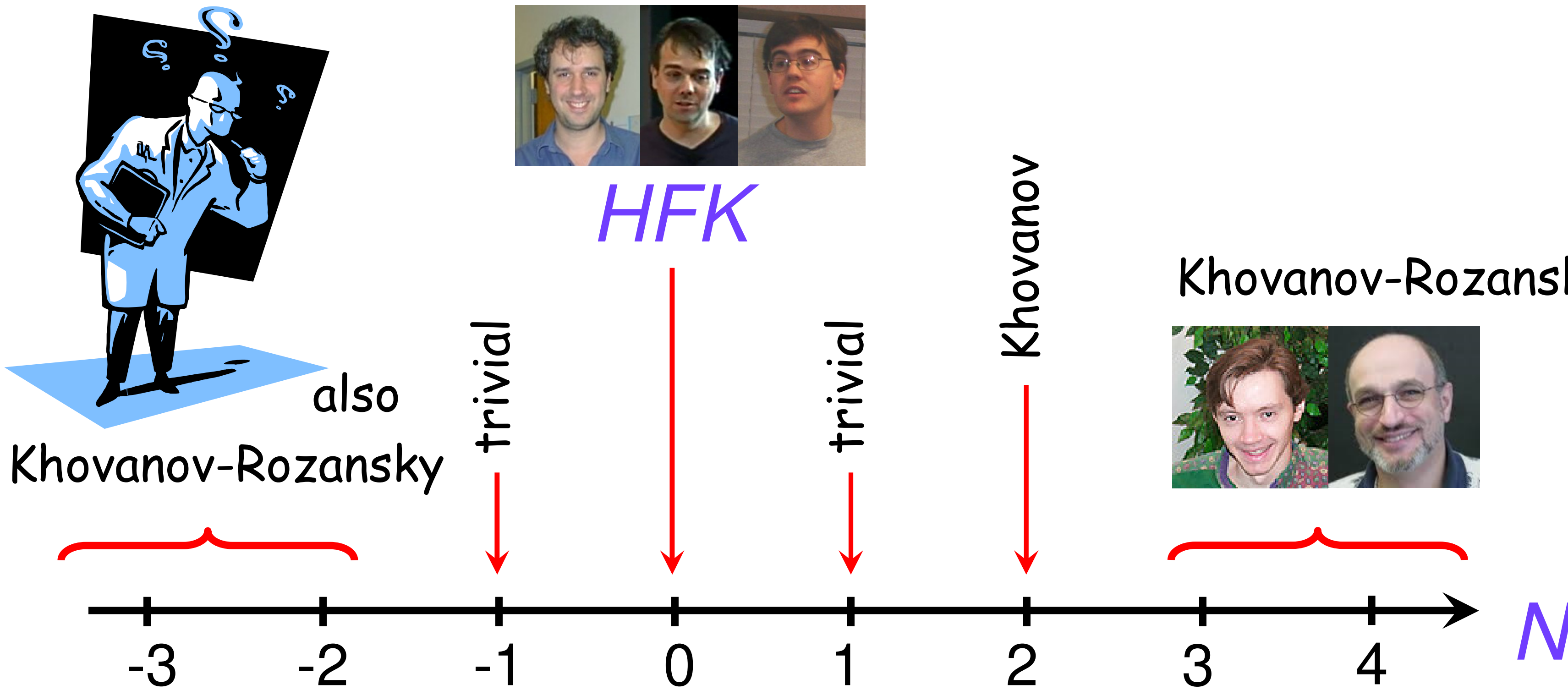


- New structural properties (differentials, recursion relations, …)
- New computational techniques

Example: trefoil

$$\mathscr{H}^{\mathfrak{e}_6,\mathbf{27}}(\mathbf{3}_1) = 1 + q^2t^2 + q^5t^2 + q^{10}tu + q^{13}tu + q^{10}t^4 + q^{15}t^3u + q^{18}t^3u + q^{23}t^2u^2$$

Unification of different theories



In categorification of quantum group invariants of knots, *HFK* is an oddball ... Will play an important role in categorification of 3-manifold invariants.

Chapter Four
3-manifolds

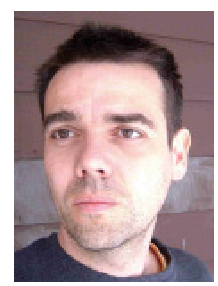
3-manifold homology in 2015

- Three 3-manifold homologies:

1. "monopole Floer homology"
 based on Seiberg-Witten equations

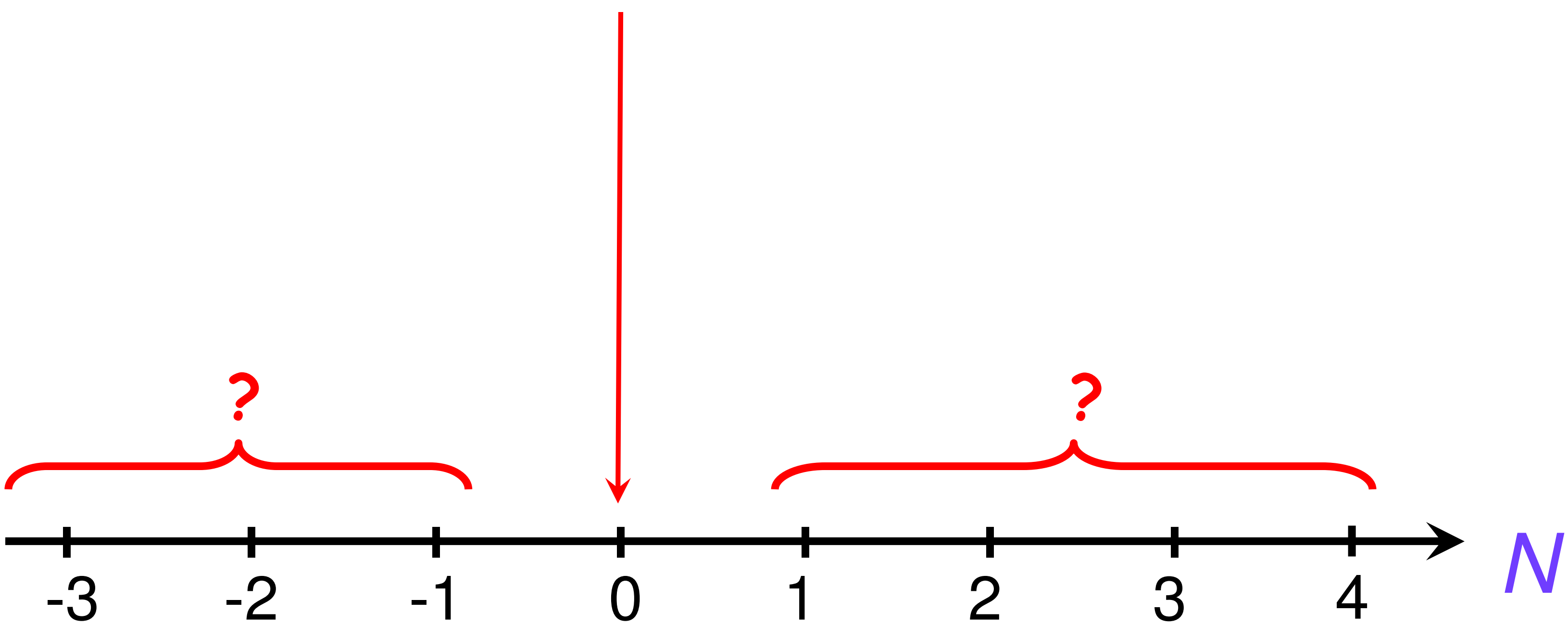

2. "embedded contact homology"
 homology version of *SW=Gr*


3. "Heegaard Floer homology"
 ("Atiyah-Floer conjecture")

1 + 2 + 3 = 1

$$HM(M_3) \cong HF(M_3) \cong ECH(M_3)$$

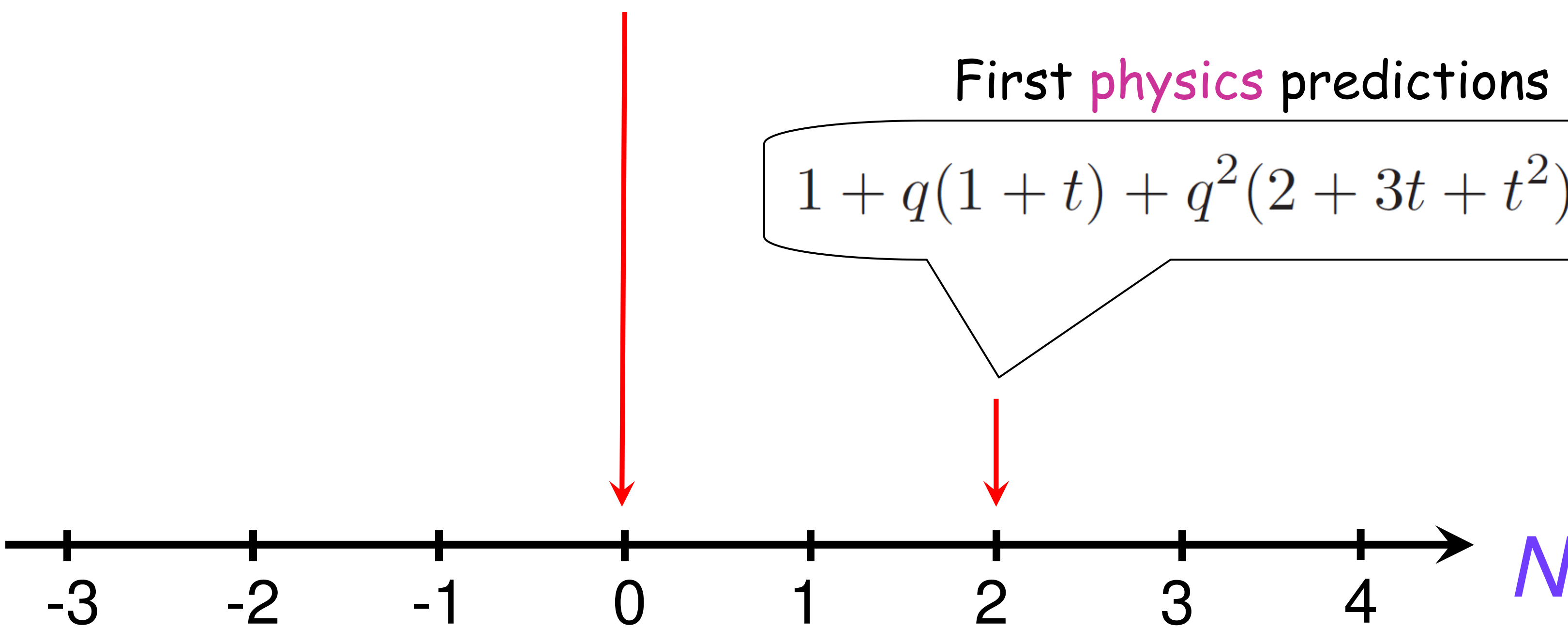


space-time:	$\mathbb{R} \times T^*M_3 \times TN_4$	duality $\longleftrightarrow$	$\mathbb{R} \times X \times TN_4$
N M5-branes:	$\mathbb{R} \times M_3 \times \mathbb{R}^2_q$		~~$\mathbb{R} \times L_K \times \mathbb{R}^2_q$~~
~~M5′-branes:~~	~~$\mathbb{R} \times L_K \times \mathbb{R}^2_q$~~		(+ flux)

$$HF(M_3)$$

First physics predictions

$$1 + q(1+t) + q^2(2 + 3t + t^2) + \ldots$$



Example: $M_3 = S^1 \times \Sigma$ $\qquad G = SU(2)$

$$\left(\frac{k}{2}\right)^{g-1} \sum_{j=1}^{k-1} \left(\sin \frac{\pi j}{k}\right)^{2-2g}$$

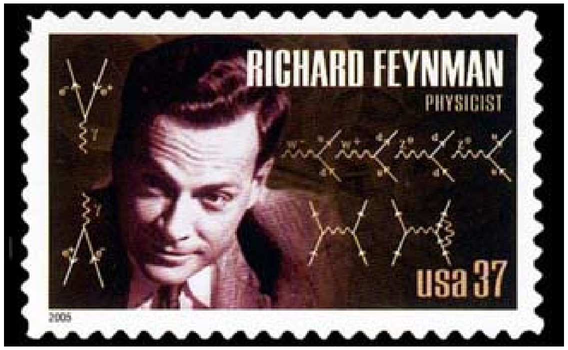
$$q = e^{\hbar} = e^{2\pi i/k}$$

Example: $M_3 = S^1 \times \Sigma$ $\qquad G = SU(2)$

$$\left(\frac{k}{2}\right)^{g-1} \sum_{j=1}^{k-1} \left(\sin \frac{\pi j}{k}\right)^{2-2g}$$

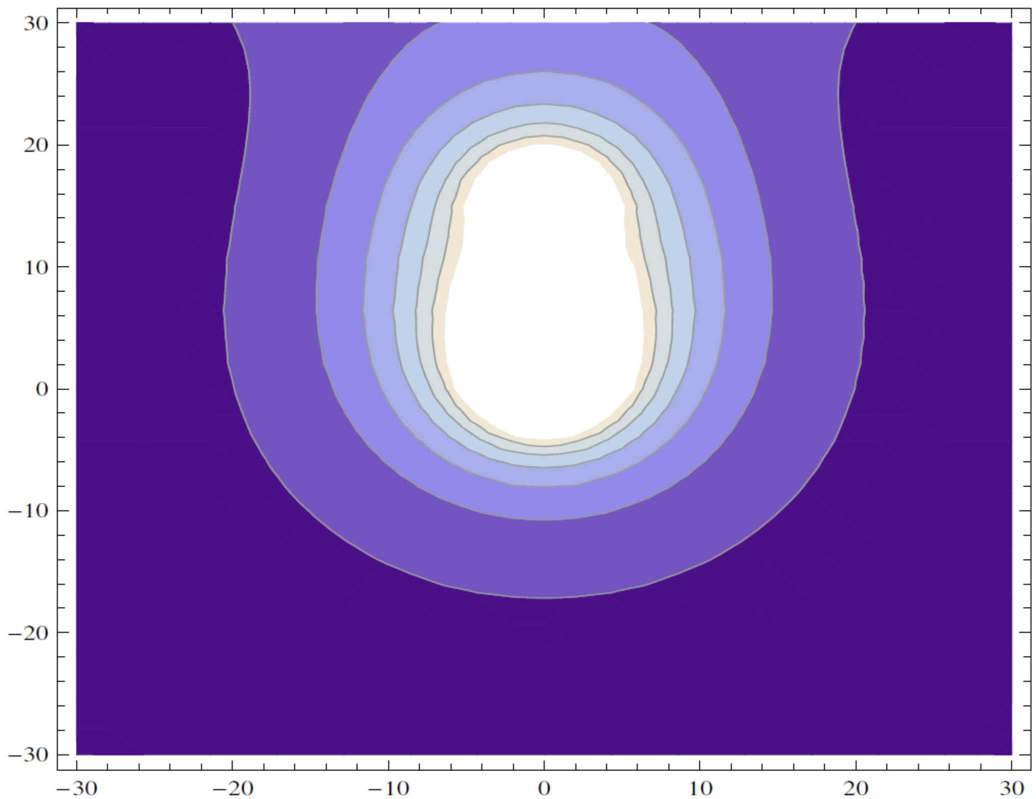
Example: $M_3 = L(5,1)$ $\qquad q = e^{\hbar} = e^{2\pi i/k}$

$$\frac{q^{1/2}}{\sqrt{2k}}\Big[\frac{1}{\sqrt{5}}(q^{1/5}-1) + \frac{e^{2\pi i k/5}}{2\sqrt{5}}((-1-\sqrt{5})q^{1/5}-4) + \\ + \frac{e^{-2\pi i k/5}}{2\sqrt{5}}((-1+\sqrt{5})q^{1/5}-4)\Big]$$



Perturbative (2-loop) invariant for the Abelian (trivial) flat connection = Non-Abelian SU(2) Casson invariant

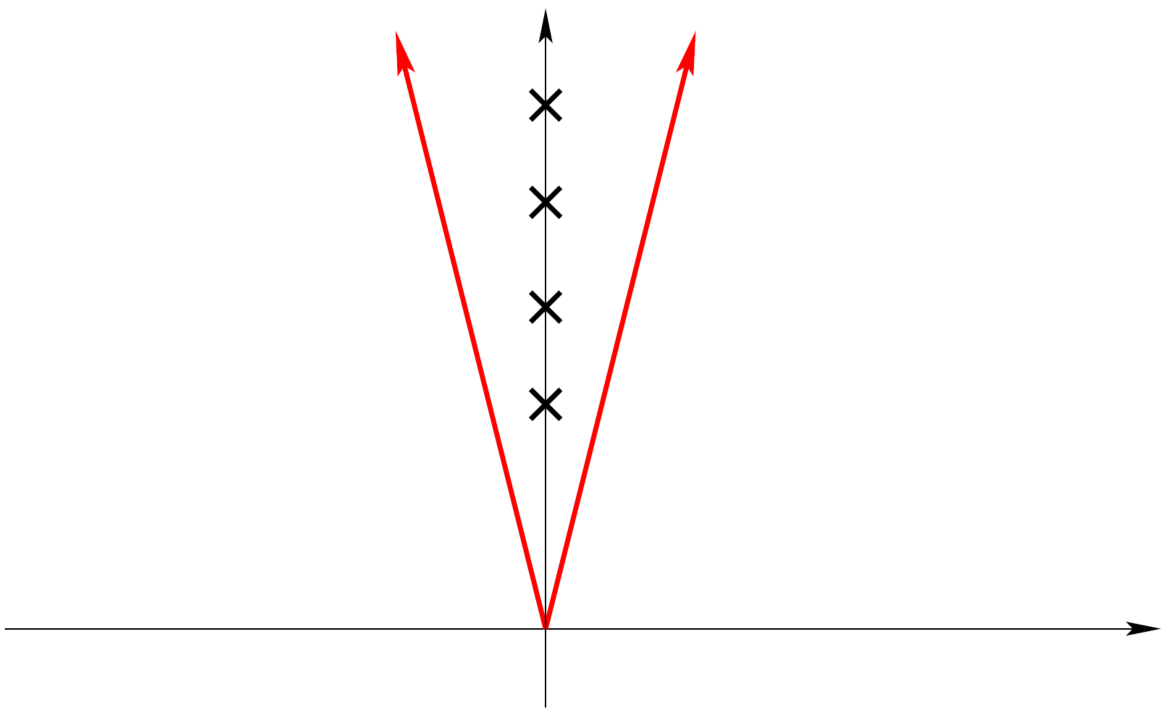
Borel plane



Emile Borel
(1871-1956)

$$Z(\hbar) = \sum_{k=0}^{\infty} p_k \hbar^k \quad \Rightarrow \quad BZ(t) = \sqrt{2\pi} \sum_{k=0}^{\infty} \frac{p_k}{k!} t^k$$

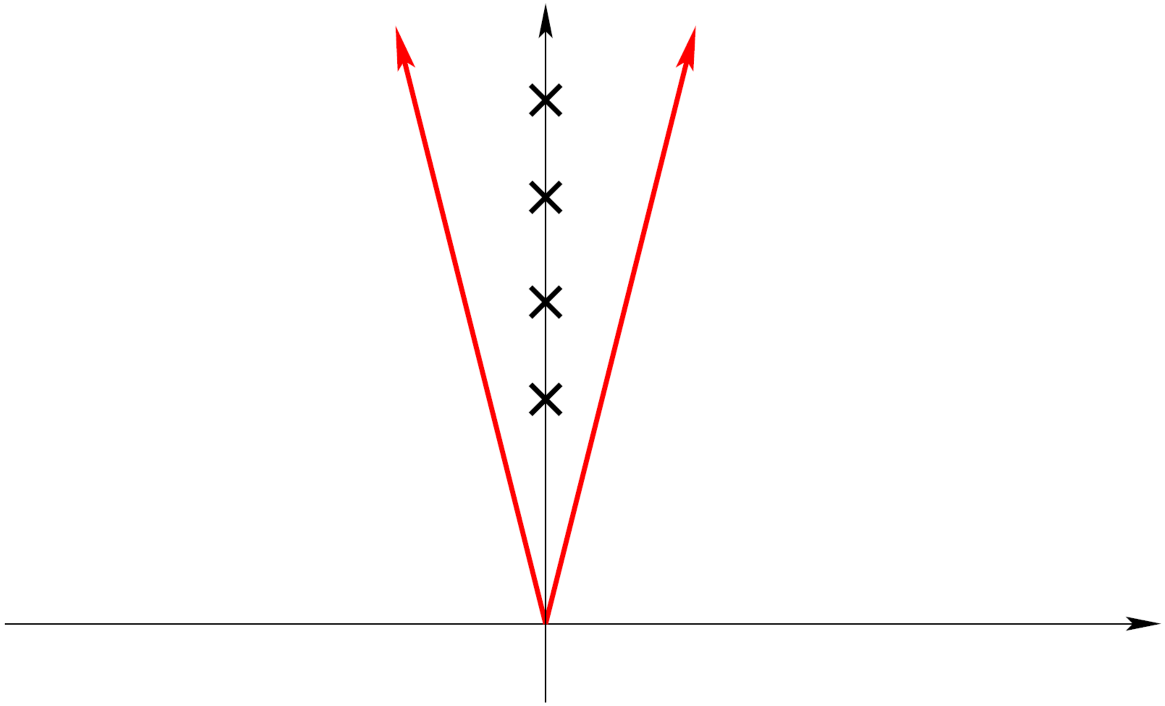
Borel plane



Emile Borel
(1871-1956)

$$\mathcal{S}_\theta Z(\hbar) = \frac{1}{\hbar}\int_0^{+\infty e^{i\theta}} dt e^{-t/\hbar} \widetilde{BZ}(t)$$

Borel plane



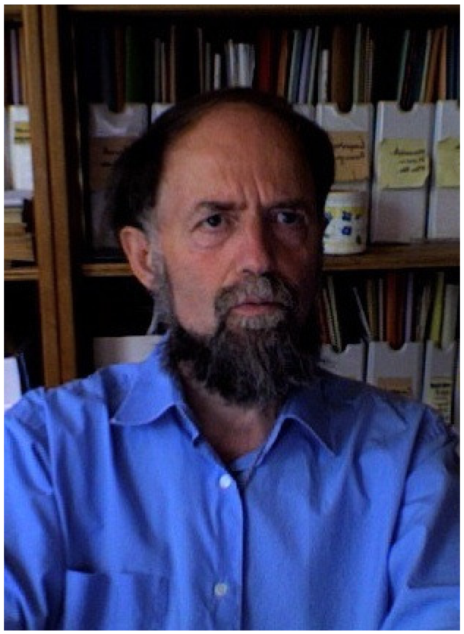
Emile Borel
(1871-1956)

1) position of singularities

2) “residues”

position of singularities: $\ell_{\alpha\beta} = S_\beta - S_\alpha$

$$\mathcal{S}_\theta Z_\alpha^{\mathrm{pert}}(\hbar) = Z_\alpha^{\mathrm{pert}}(\hbar) + \sum_\beta n_\beta^\alpha e^{-\frac{\ell_{\alpha\beta}}{\hbar}} Z_\beta^{\mathrm{pert}}(\hbar)$$


Jean Ecalle, 1980

"Resurgent functions display at each of their singular points a behavior closely related to their behavior at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were – at their singularities."

Theorem 1 [G-Marino-Putrov]:

$$n^{\alpha}_{\beta} = 0 \qquad \begin{array}{rcl} \alpha & = & \text{irreducible} \\ \beta & = & \text{abelian} \end{array}$$

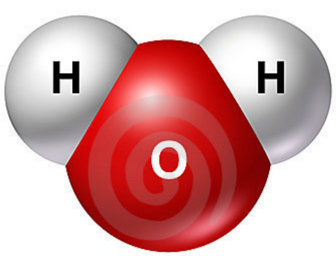
Corollary:

new invariants of 3-mflds!

$$Z_{\text{CS}}(M_3) = \sum_{a \in \text{abelian}} e^{2\pi i k S_a} \underbrace{Z_a(M_3)}$$

q-series convergent inside the unit disk

$|q|<1$



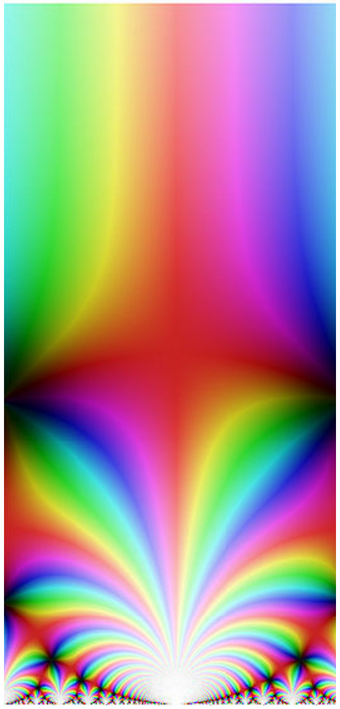
Theorem 2 [G-Putrov-Vafa]:

Modular group $\mathbf{SL(2,\mathbb{Z})}$ acts

on (connected components) of $\mathcal{M}_{\text{flat}}(M_3; G_{\mathbb{C}})$

e.g.

$$S = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 \\ 2 & \frac{1}{2}\left(-\sqrt{5}-1\right) & \frac{1}{2}\left(\sqrt{5}-1\right) \\ 2 & \frac{1}{2}\left(\sqrt{5}-1\right) & \frac{1}{2}\left(-\sqrt{5}-1\right) \end{pmatrix}$$



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* via instanton counting on 4-manifolds bounded by M_3

$$\sum_n q^n \chi(\mathcal{M}_n) = \text{modular form}$$

Formal power series

q-series, converges in a unit disk

q-series with integer powers and coefficients

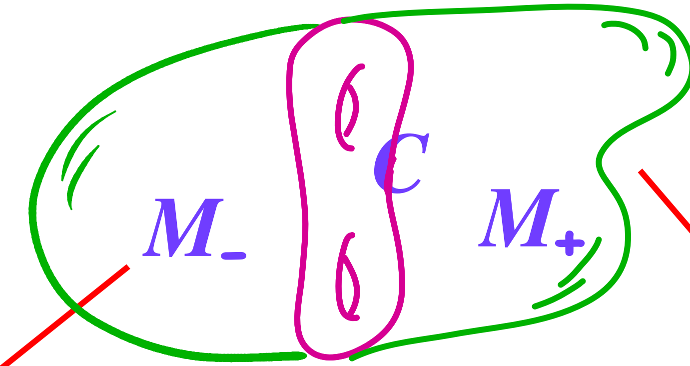
$$Z_{a}^{\text{pert}}(\hbar) \xrightarrow[\text{(Borel sum)}]{\text{resurgence}} Z_a(q = e^{\hbar}) \xrightarrow[\text{transform}]{\text{modular}} \widehat{Z}_a(q)$$

labeled by Abelian flat connection on M_3

$$\widehat{Z}_a(q) = \sum q^i (-1)^j \dim \mathcal{H}^{i,j}$$

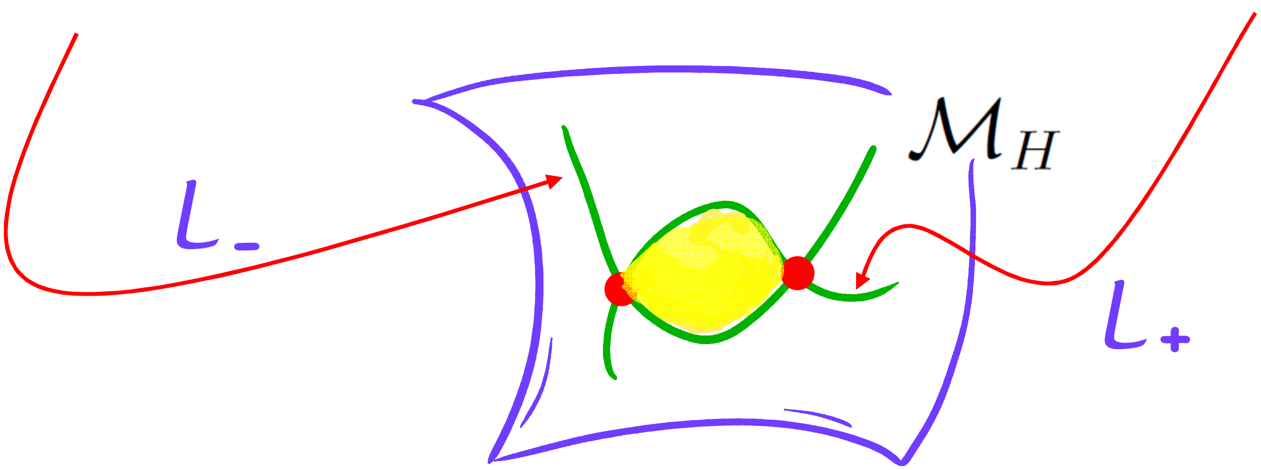
Srinivasa Ramanujan
(1887-1920)

S-duality as Mirror Symmetry



(A,B,A) brane (A,B,A) brane

$$\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, M_-) \subset \mathcal{M}_H(G, C) \supset \mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, M_+)$$



S-duality as Mirror Symmetry

Challenge: compute (for $H_1(M_3) = \mathbb{Z}_5$ and **G=SU(2)**)

$$S = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 \\ 2 & \frac{1}{2}(-\sqrt{5}-1) & \frac{1}{2}(\sqrt{5}-1) \\ 2 & \frac{1}{2}(\sqrt{5}-1) & \frac{1}{2}(-\sqrt{5}-1) \end{pmatrix}$$

(A,B,A) branes ⟷ (A,B,A) branes

$$\begin{array}{ccc} \mathcal{M}_H(G,C) & & \mathcal{M}_H({}^LG,C) \\ \pi \searrow & & \swarrow \widetilde{\pi} \\ & B & \end{array}$$

Happy Birthday, Nigel !

3d-3d interpretation

$$\begin{array}{rccc} \text{space-time:} & \mathbb{R} \times T^*M_3 \times TN_4 & \overset{\text{duality}}{\longleftrightarrow} & \mathbb{R} \times X \times TN_4 \\ N \ \text{M5-branes:} & \mathbb{R} \times \ M_3 \ \times \ \mathbb{R}^2_q & & \mathbb{R} \times L_K \times \mathbb{R}^2_q \\ \text{M5}'\text{-branes:} & \mathbb{R} \times \ L_K \ \times \ \mathbb{R}^2_q & & \end{array}$$

depends on topology and geometry of M_3