An index theorem for Lorentzian manifolds with boundary

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Atiyah(-Patodi)-Singer index theorem

Lorentzian manifolds

Application to quantum field theory

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Atiyah-Singer index theorem

- M Riemannian manifold, compact, without boundary
- spin structure $\rightsquigarrow$ spinor bundle $SM \to M$
- $n = \dim(M)$ even $\rightsquigarrow$ splitting $SM = S_R M \oplus S_L M$
- Hermitian vector bundle $E \to M$ with connection $\rightsquigarrow$ twisted Dirac operator $D : C^\infty(M, V_R) \to C^\infty(M, V_L)$ where $V_{R/L} = S_{R/L} M \otimes E$

Atiyah-Singer index theorem

Theorem (M. Atiyah, I. Singer, 1968)

The operator D is Fredholm and

$$\mathrm{ind}(D) = \int_M \widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E)$$

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Moreover,

$$\begin{aligned}\mathrm{ind}(D) = &\dim\ker[D : C^\infty(M; V_R) \to C^\infty(M; V_L)] \\ &- \dim\ker[D : C^\infty(M; V_L) \to C^\infty(M; V_R)]\end{aligned}$$

Boundary conditions

Now let M have nonempty boundary.
Need boundary conditions:
Choose "Fermi coordinate function" $t : M \to \mathbb{R}$ and write

$$D = \gamma \left(\frac{\partial}{\partial t} + A_t \right)$$

A_0 is a selfadjoint Dirac-type operator on ∂M.
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APS-boundary conditions:

$$P_+(f|_{\partial M}) = 0$$

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Theorem (M. Atiyah, V. Patodi, I. Singer, 1975)

Under APS-boundary conditions D is Fredholm and



$$\mathrm{ind}(D_{\mathrm{APS}}) = \int_M \widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E) + \int_{\partial M} T(\widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E)) - \frac{h(A_0) + \eta(A_0)}{2}$$

Here

- $h(A) = \dim \ker(A)$
- $\eta(A) = \eta_A(0)$ where $\eta_A(s) = \sum_{\substack{\lambda \in \mathrm{spec}(A) \\ \lambda \neq 0}} \mathrm{sign}(\lambda) \cdot |\lambda|^{-s}$

Warning

APS-boundary conditions cannot be replaced by anti-Atiyah-Patodi-Singer boundary conditions,

$$P_-(f|_{\partial M}) = \chi_{(-\infty,0)}(A_0)(f|_{\partial M}) = 0$$

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Example

- M = unit disk $\subset \mathbb{C}$
- $D = \overline{\partial} = \frac{\partial}{\partial \overline{z}}$
- Fourier expansion: $u|_{\partial M} = \sum_{n\in\mathbb{Z}} \alpha_n e^{in\theta}$
- $A_0 = i\frac{d}{d\theta}$
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Replace “spaces” by “spacetimes”,
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Dirac operator no longer elliptic, but hyperbolic.
In particular, no elliptic regularity theory

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Problem 2: hyperbolic PDE theory does not work on such spacetimes
$\Rightarrow$ no Lorentzian analog to Atiyah-Singer index theorem

Globally hyperbolic spacetimes

A subset $\Sigma \subset M$ is called Cauchy hypersurface if each inextendible timelike curve in M meets Σ exactly once.

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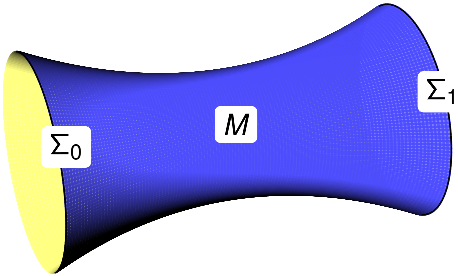
Examples:

- Minkowski spacetime (Special Relativity)
- Schwarzschild Model (Black Hole)
- Friedmann cosmos (Big Bang, cosmic expansion)
- deSitter spacetime
- $\dots$

The Lorentzian index theorem

Let M be a globally hyperbolic Lorentzian manifold
with boundary $\partial M = \Sigma_0 \sqcup \Sigma_1$
Σ_j compact smooth spacelike Cauchy hypersurfaces
D twisted Dirac operator



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Theorem (C. B., A. Strohmaier, 2015)

Under APS-boundary conditions D is a Fredholm operator. The kernel consists of smooth spinor fields and

$$\mathrm{ind}(D_{\mathrm{APS}}) = \int_M \widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E) + \int_{\partial M} T(\widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E)) - \frac{h(A_0) + h(A_1) + \eta(A_0) - \eta(A_1)}{2}$$

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Moreover,

$$\mathrm{ind}(D_{\mathrm{APS}}) = \dim\ker[D : C^\infty_{\mathrm{APS}}(M; V_R) \to C^\infty(M; V_L)] - \dim\ker[D : C^\infty_{\mathrm{aAPS}}(M; V_R) \to C^\infty(M; V_L)]$$

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aAPS conditions are as good as APS-boundary conditions

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- similarly, APS along $\Sigma_1 \Rightarrow$ $\mathrm{WF}(\Phi) \subset \{\text{past-directed lightlike covectors}\}$
- $\Rightarrow \mathrm{WF}(\Phi) = \emptyset$, i.e. Φ is smooth

The evolution picture

Wave propagator $U : C^\infty(\Sigma_0; V_R) \to C^\infty(\Sigma_1; V_R)$:
For $\varphi \in C^\infty(\Sigma_0; V_R)$ solve $D\Phi = 0$ with initial conditions $\Phi|_{\Sigma_0} = \varphi$. Then put $U\varphi = \Phi|_{\Sigma_1}$.

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Decompose

$$U = \begin{pmatrix} U_{++} & U_{+-} \\ U_{-+} & U_{--} \end{pmatrix}$$

w.r.t. decomposition

$$L^2(\Sigma_0; V_R) = L^2_{[0,\infty)}(\Sigma_0; V_R) \oplus L^2_{(-\infty,0)}(\Sigma_0; V_R),$$
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Then

$$\mathrm{ind}(D_{\mathrm{APS}}) = \dim\ker(U_{--}) - \dim\ker(U_{++})$$

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The chiral anomaly

No natural physical interpretation of APS boundary conditions in the Riemannian case.

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3) **But:** relative current does exist

$$J^{\Sigma_0,\Sigma_1} = J^{\Sigma_0} - J^{\Sigma_1}$$

Charge creation and index

Theorem (B.-Strohmaier 2015)

For any two Cauchy hypersurfaces with product structure near them, the relative current J^{Σ_0,Σ_1} is coclosed and its integral Q_R over any Cauchy hypersurface equals

$$\mathrm{ind}(U_{--}) = \mathrm{ind}(D_{\mathrm{APS}}).$$

Hence

$$Q_R = \int_M \widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E) - \frac{h(D_{\Sigma_0}) - h(D_{\Sigma_1}) + \eta(D_{\Sigma_0}) - \eta(D_{\Sigma_1})}{2}.$$

Similarly

$$Q_L = -\int_M \widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E) + \frac{h(D_{\Sigma_0}) - h(D_{\Sigma_1}) + \eta(D_{\Sigma_0}) - \eta(D_{\Sigma_1})}{2}.$$

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Total charge $Q = Q_R + Q_L$ is zero.
Chiral charge $Q_{\mathrm{chir}} = Q_R - Q_L$ is not!

Example

- Spacetime $M = \mathbb{R} \times S^{4k-1}$ with metric $-dt^2 + g_t$ where g_t are Berger metrics.
- Flat connection on trivial bundle E.
- Chiral anomaly:

$$Q_{\text{chir}}^{\Sigma_0,\Sigma_1} = (-1)^k 2\binom{2k}{k}$$

- See Gibbons 1979 (using results of Hitchin) for $k = 1$.

References

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