The quaternionic Feix–Kaledin construction

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1. Background and motivation
2. The quaternionic Feix–Kaledin construction
3. Examples and applications

Joint work with Alexandra Borowka

1. Hyperkähler metrics on cotangent bundles

The cotangent bundle T^*S of a complex manifold S is a holomorphic symplectic manifold, and is often hyperkähler

Examples: S is $\mathbb{C}P^n$ or a coadjoint orbit (Calabi, Kronheimer, Biquard).

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General picture (Feix 2001, Kaledin 1999)

- S a real-analytic Kähler manifold
- Then $\exists$ a germ-unique $U(1)$-invariant hyperkähler metric on a tubular nbhd of the zero section in T^*S.

1. Feix construction (via HKLR twistor theory)

- Complexify, work in holomorphic category, add real structures.
- Given a holomorphic Kähler $2n$-manifold S^c, construct a holomorphic $2n+1$ manifold Z from $\hat{Z} := S^c \times \mathbb{C}P^1$ by blowing down the zero section along the $(1,0)$-foliation and the infinity section along the $(0,1)$-foliation.

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- A $\mathbb{C}P^1$ fibre of $\hat{Z} \to S^c$ projects to a twistor line $\mathcal{C}$ in Z, i.e., a rational curve with normal bundle $\mathcal{O}(1) \otimes \mathbb{C}^{2n}$.
- The moduli space of deformations of $\mathcal{C}$ is a holomorphic $4n$-manifold with a holomorphic hyperkähler metric, since Z has a fibration over $\mathbb{C}P^1$ with symplectic leaves.

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Hypercomplex version (Feix and Kaledin)

When S is complex affine with type (1,1) curvature, can construct a hypercomplex structure on nbhd of zero section in TS.

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Complexification: an analogy
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Underlying complex manifold $({M^c}_{\mathbb{R}}, J)$ has M as a *totally real submanifold*, i.e., $TM \cap J(TM) = 0$, so $J(TM) \cong TM$ is the normal bundle to M in ${M^c}_{\mathbb{R}}$.

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Szoke and Bielawski: embed ${M^c}_{\mathbb{R}}$ as nbhd of zero section in TM.

Theorem. *A real-analytic projective manifold M has a complexification ${M^c}_{\mathbb{R}} \subseteq TM$ s.t. for any geodesic $\gamma \subseteq M$, ${M^c}_{\mathbb{R}} \cap T\gamma$ is a complex submanifold.*

1. Projective structures

Idea: seek common framework in projective geometry.

Let M be a (real) n-manifold. A (real) *affine connection* is a connection D on TM (e.g., $D = \nabla^g$ for a riemannian metric g).

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Affine connections D and $\tilde{D}$ on TM are *projectively equivalent* iff $\exists\, \gamma \in \Omega^1(M)$, a 1-form, with

$$\tilde{D}_X - D_X = [\![X, \gamma]\!]^r \in C^\infty(M, \mathfrak{gl}(TM)),$$

where $$[\![X, \gamma]\!]^r(Y) := \tfrac{1}{2}\big(\gamma(X)Y + \gamma(Y)X\big).$$

A *projective structure* on M^n ($n > 1$) is a projective class $\Pi^r = [D]$ of affine connections. Connections in Π^r have same unparametrized geodesics. (Also have same torsion, usually assumed zero.)

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If M is complex, $\exists$ a holomorphic version of this notion, often called a complex projective structure. However, the Levi-Civita connection of a Kähler metric is not holomorphic.

1. Complex projective structures

Let (M, J) be an almost complex manifold of real dimension $n = 2m$. A *complex affine connection* is a connection D on TM with $DJ = 0$ (e.g., $D = \nabla^g$ for a hermitian metric g).

Complex affine connections D and $\tilde{D}$ are *c-projectively equivalent* iff $\exists\, \gamma \in \Omega^1(M)$, a 1-form, with

$$\tilde{D}_X - D_X = [\![X, \gamma]\!]^c \in C^\infty(M, \mathfrak{gl}(TM, J)),$$
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A *c-projective structure* on M^{2m} $(m > 1)$ is an c-projective class $\Pi^c = [D]$ of complex affine connections. Connections in Π^c have the same torsion, often assumed type $(0, 2)$—and then given by the Nijenhuis tensor of J.

1. Quaternionic structures

Let (M, Q) be a quaternionic manifold of real dimension $n = 4\ell$ (thus $Q \subset \mathfrak{gl}(TM)$, with fibres isomorphic to $\mathfrak{sp}(1)$, spanned by imaginary quaternions J_1, J_2, J_3).

A *quaternionic affine connection* is a connection on TM preserving Q (e.g., $D = \nabla^g$ for a quaternion Kähler metric g on M).

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Fact. For any two quaternionic connections D and $\tilde{D}$ with the same torsion, $\exists\, \gamma \in \Omega^1(M)$ with

$$\tilde{D}_X - D_X = [\![X, \gamma]\!]^q \in C^\infty(M, \mathfrak{gl}(TM, Q)),$$

$$[\![X, \gamma]\!]^q(Y) := \tfrac{1}{2}\Big(\gamma(X)Y + \gamma(Y)X - \sum_i \big(\gamma(J_iX)J_iY + \gamma(J_iY)J_iX\big)\Big).$$

An equivalence class of quaternionic connections may be denoted $\Pi^q = [D]$. Thus any quaternionic manifold has a distinguished class of torsion-free quaternionic connections.

1. Submanifolds

Observation 1. Let (M, J, Π^c) be a c-projective $2m$-manifold, and let N be a maximal totally real submanifold, i.e., $J(TN) \cap TN = 0$ and $\dim N = m$ so that $TM|_N \cong TN \oplus J(TN)$.

By projecting c-projective connections onto TN, N inherits a projective structure: for $X, Y \in TN$, the projection of $[\![X, \gamma]\!]^c(Y)$ is $[\![X, i^*\gamma]\!]^r(Y)$, where $i\colon M \to N$ is the inclusion.

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Observation 2. Let $(M^{4\ell}, Q)$ be a quaternionic manifold. A submanifold N is *totally complex* if TN is invariant under some $J \in Q$ (along N), but $I(TN) \cap TN = 0$ for any $I \in Q$ anticommuting with J. If N is a maximal ($\dim N = 2\ell$), then $TM|_N \cong TN \oplus TN^\perp$ where $TN^\perp = I(TN)$ for any such I.

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Question: conversely, can complexify a real projective manifold, so can we quaternionify a c-projective manifold?

2. Model example

$S = \mathbb{C}P^m = \mathrm{P}(\mathbb{C}^{m+1})$ has complexification $S^c = \mathbb{C}P^m \times \mathbb{C}P^m$.
Let $\hat{Z} = \mathrm{P}(\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1))$, a $\mathbb{C}P^1$-bundle over S^c.
The map $\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1) \to \mathbb{C}^{m+1} \times \mathbb{C}^{m+1}$

$$(\ell_1, v_1, \ell_2, v_2) \mapsto (v_1, v_2),$$

where $v_i \in \ell_i \leq \mathbb{C}^{m+1}$, induces a map $\hat{Z} \to Z = \mathbb{C}P^{2m+1}$.
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The $\mathbb{C}P^1$ fibres of $\hat{Z} \to S^c$ map to projective lines in $\mathbb{C}P^{2m+1}$, which are twistor lines. The moduli space of such lines is $\mathrm{Gr}_2(\mathbb{C}^{2m+2})$, which is a complexification of $\mathbb{H}P^m$.

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Theorem (HKLR, LeBrun, Pedersen–Poon). Let Z be a holomorphic $(2m+1)$-manifold equipped with an antiholomorphic involution $\tau \colon Z \to Z$ containing a τ-invariant twistor line on which τ has no fixed points. Then the space of such twistor lines is a $4m$-dimensional quaternionic manifold (M, Q).

2. The main difficulty

To generalize the model, need to partially blow down 0 and ∞ sections of a projective line bundle over a complexification S^c of S.

Problem: S^c is not compact, nor even a complete variety.

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Essence of problem: the blow-up of $\mathbb{C}^{m+1}$ at the origin is the total space of $\pi\colon \mathcal{O}(-1) \to \mathbb{C}P^m$. Now let U be open in $\mathbb{C}P^m$.

- How to construct the blow-down knowing only $\pi^{-1}(U) \to U$?
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- How to construct the blow-down knowing only $\pi^{-1}(U) \to U$?
- The image of $\pi^{-1}(U)$ under the blow-down is singular at 0.

Resolution: U has a flat complex projective structure, hence a second order linear operator on sections of $\mathcal{O}(1)|_U$ whose solutions are "affine". The vector space $\mathcal{A}(\mathcal{O}(1)|_U)$ of affine sections has evaluation maps $\mathcal{A}(\mathcal{O}(1)|_U) \to \mathcal{O}(1)_u$ for all $u \in U$.

- If $V = \mathcal{A}(\mathcal{O}(1)|_U)^*$ then for each $u \in U$, the image of the transpose $\mathcal{O}(-1)_u \to V$ is a 1-dimensional subspace, and this defines a developing map (local biholomorphism) $U \to \mathrm{P}(V)$.
- Take the union of the image of $\pi^{-1}(U) = \mathcal{O}(-1)|_U$ in V with an open nbhd of the origin.

2. Main results

A. (M^{4m}, Q) quaternionic, with a quaternionic $U(1)$ action.

- Fixed point set has a component S of dimension $2m$ with no triholomorphic points (where stabilizer commutes with Q).

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- Then S is totally complex, with induced c-projective structure of type $(1,1)$, i.e., c-projective curvature has type $(1,1)$, i.e., complexification S^c has flat complex projective structures on $(1,0)$ and $(0,1)$ foliations.

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B. To any c-projective S of type $(1,1)$ and any line bundle $\mathcal{L}$ with connection of type $(1,1)$, have a complexification S^c and a projective line bundle $\hat{Z} = \mathrm{P}(\mathcal{L}^*_{0,1} \oplus \mathcal{L}^*_{1,0}) \to S^c$.

- $\hat{Z}$ has a blow down which is a twistor space of a $U(1)$-invariant quaternionic structure Q on a nbhd M of zero section in $TS \otimes (\mathcal{L}^*_{0,1} \otimes \mathcal{L}_{1,0})|_S$.

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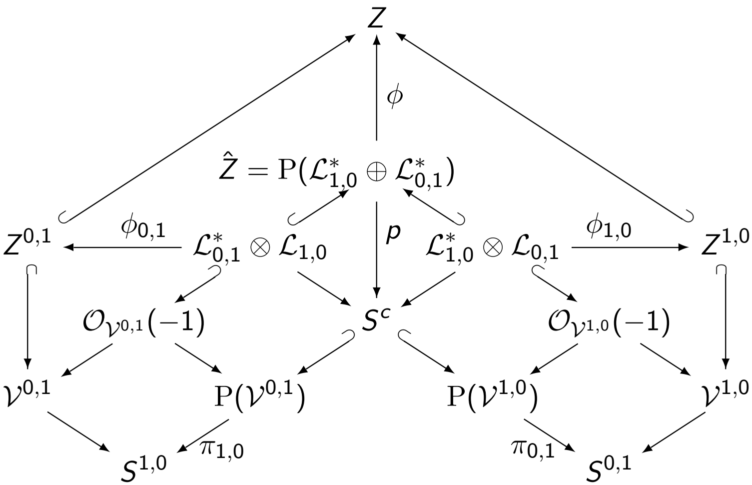
C. A and B are mutually inverse up to local isomorphism.

2. A picture of the construction of Z from S



Hooked arrows are open embeddings.
Other arrows are fibrations or (open embeddings of) blow-downs.

2. Twistor theory of complexified quaternionic manifolds

Let Z be a holomorphic $(2n+1)$-manifold containing twistor lines. Kodaira moduli space is a holomorphic $4n$-manifold M^c, with incidence relation (twistor correspondence)

$$\begin{array}{ccc} & F_M := \{(z,u) \in Z \times M^c : z \in u\} & \\ \swarrow^{\pi_Z} & & \searrow^{\pi_{M^c}} \\ Z & & M^c, \end{array}$$

where $u \in M^c$ is the twistor line $\pi_Z(\pi_{M^c}^{-1}(u)) \subseteq Z$.

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where $u \in M^c$ is the twistor line $\pi_Z(\pi_{M^c}^{-1}(u)) \subseteq Z$.

Normal bundle to twistor lines defines a bundle $\mathcal{N} \to F_M$. Locally over M^c, $\mathcal{N} \cong \pi_{M^c}^* \mathcal{E} \otimes \pi_Z^* \mathcal{O}_Z(1)$ where

- $\mathcal{E}$ is a rank $2n$ bundle on M^c
- $\mathcal{O}_Z(1)$ is a line bundle on Z of degree 1 on each twistor line.

By Kodaira, $T_u M^c \cong H^0(u, \mathcal{N}|_u) \cong \mathcal{E}_u \otimes \mathcal{H}_u$, $\mathcal{H}_u = H^0(u, \mathcal{O}_Z(1))$. Thus $TM^c \cong \mathcal{E} \otimes \mathcal{H}$; say $X \in TM^c$ is *null* if decomposable.

2. α-submanifolds

The fibre of F_M over $z \in Z$ projects to a submanifold α_z of M^c called an α-*submanifold*.

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Twistor lines through $z \in u$ are determined by their tangent space at z, so α_z is isomorphic to an open submanifold of $\mathrm{P}(T_z Z)$, and has a canonical flat projective structure.
Also have (complexified) quaternionic connections: torsion-free tensor product connections $D^{\mathcal{E}} \otimes D^{\mathcal{H}}$.

Prop. For any α-submanifold α_z in M^c, any quaternionic connection induces an affine connection on α_z compatible with its canonical flat projective structure.

2. Why are constructions mutually inverse?

Let Q be a $U(1)$-invariant quaternionic structure on a nbhd M^{4m} of a fixed submanifold S^{2m} with no triholomorphic points.

- Weight space decomposition shows (S, J) is (maximal) totally complex submanifold of M, with J a section of $Q|_S$.
- $U(1)$ action lifts to holomorphic action on twistor space Z, generated by a vector field vanishing on sections $\pm J$ of $Z|_S$, denoted $S^{1,0}$ and $S^{0,1}$.

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- Let $\phi\colon \hat{Z} \to Z$ be the blow-up of Z along $S^{1,0} \cup S^{0,1}$, with exceptional divisor $\underline{0} \cup \underline{\infty}$.
- $\phi^{-1}(Z|_S)$ has a neighbourhood foliated by a $2n$-dimensional moduli space S^c of rational curves with trivial normal bundle.
- May assume $\hat{Z}$ is a P^1-bundle and S^c is an open nbhd of "diagonal" in $S^{1,0} \times S^{0,1}$.

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- May assume $\hat{Z}$ is a P^1-bundle and S^c is an open nbhd of "diagonal" in $S^{1,0} \times S^{0,1}$.
- Lift of $U(1)$ action to $\hat{Z}$ shows $\hat{Z} \setminus (\underline{0} \cup \underline{\infty})$ is a holomorphic principal $\mathbb{C}^\times$-bundle over S^c, hence $\hat{Z} \cong \mathrm{P}(\mathcal{L}^*_{0,1} \otimes \mathcal{L}^*_{1,0})$.
- By Proposition, induced c-projective structure has type $(1, 1)$.

3. Examples: complex grassmannians

Totally complex submanifolds S of quaternionic symmetric spaces (M, Q) fixed by a $U(1)$ action have been classified by Wolf: have many examples where (M, Q) is not even locally hypercomplex.

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Example. Complex grassmannian $M = \mathrm{Gr}_2(\mathbb{C}^{n+2})$ is a quaternionic symmetric space with twistor space $Z = F_{1,n+1}(\mathbb{C}^{n+2})$, the the flag manifold of pairs $B \leq W \leq \mathbb{C}^{n+2}$ with $\dim B = 1$ and $\dim W = n+1$. The real structure on Z sends the flag $B \leq W$ to $W^\perp \leq B^\perp$.

Then $M^c \cong \{(U, V) \in \mathrm{Gr}_2(\mathbb{C}^{n+2}) \times \mathrm{Gr}_n(\mathbb{C}^{n+2}) : \mathbb{C}^{n+2} = U \oplus V\}$, and a fixed decomposition $\mathbb{C}^{n+2} = A \oplus \tilde{A}$, with $\dim A = 1$ and $\dim \tilde{A} = n+1$, determines a submanifold $S^c = \{(U, V) \in M^c : A \leq U,\ V \leq \tilde{A}\}$ of M^c.

3. Examples: complex grassmannians

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We find that S^c is an open submanifold of $\mathrm{P}(\tilde{A}) \times \mathrm{P}(\tilde{A}^*)$ and $\hat{Z} \cong \mathrm{P}(\mathcal{O}_{\tilde{A}}(-1) \oplus \mathcal{O})|_{S^c} \cong \mathrm{P}(\mathcal{O} \oplus \mathcal{O}_{\tilde{A}^*}(-1))|_{S^c}$.

3. Swann bundles and twisted Armstrong cones

Any quaternionic $4m$-manifold (M, Q) has an associated hypercomplex cone $\tilde{M}$ of dimension $4(m+1)$ fibering over it, called the *Swann bundle*: it is the $\mathbb{C}^\times$ bundle over the twistor space Z of (M, Q) associated to the $m+1$ root of the anticanonical bundle K_Z^{-1}.

Question. If (M, Q) is constructed from a c-projective manifold S of type $(1,1)$ and a line bundle $\mathcal{L}$ of type $(1,1)$, how can we construct $\tilde{M}$ from S ?

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Any c-projective $2m$ manifold S has a complex affine $2(m+1)$-manifold fibering over it as the $\mathbb{C}^\times$ bundle associated to the $m+1$ root of the anticanonical bundle K_S^{-1} (Armstrong).

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To answer the question, we twist Armstrong's construction by $\mathcal{L}$.

Prop. If (M, Q) is constructed from S and $\mathcal{L}$, then its Swann bundle is constructed from the twisted Armstrong cone of $(S, \mathcal{L})$.

3. Four dimensions

A c-projective structure on a complex surface S is the same thing as a Möbius structure (C, 1998). It automatically has type $(1, 1)$. Any such S, together with a line bundle $\mathcal{L}$ with connection, gives rise to a self-dual conformal 4-manifold M with a $U(1)$ action having S as a component of the fixed point set.

By Jones–Tod (1985) and LeBrun (1990), the quotient $M/U(1)$ is (locally, near S) an *asymptotically hyperbolic Einstein–Weyl 3-manifold* with conformal infinity S (i.e., Möbius infinity $S, \mathcal{L}$).

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This is a minitwistor version of the H-space construction. The flat model is hyperbolic 3-space $\mathcal{H}^3$, which is a $U(1)$ quotient of the embedding of $\mathbb{C}P^1 \cong S^2$ in $\mathbb{H}P^1 \cong S^4$.

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However, the following conjecture remains open.

Conjecture (LeBrun, 1990). If B is an asymptotically hyperbolic Einstein–Weyl 3-manifold on the interior of a compact manifold $\overline{B}$ with conformal infinity $\partial\overline{B}$, then B is $\mathcal{H}^3$, with the Einstein–Weyl structure of the hyperbolic metric.