On the existence of a complex structure on the six-sphere

Gábor Etesi

Budapest University of Technology and Economics
Department of Geometry
HUNGARY

Aarhus, 8-12 August 2011

History and known facts

Borel–Serre, 1951: among the spheres S^n ($n \geqq 1$) the two-sphere S^2 and the six-sphere S^6 are the only ones which can carry an almost complex structure.

The almost complex structure on S^2 is unique and integrable i.e., the two-sphere is a complex manifold ($S^2 \cong \mathbb{C}P^1$).

There exist uncountable many almost complex structures on S^6 but most of them are non-integrable. It has been unknown whether or not any of these almost complex structures are integrable i.e., whether or not the six-sphere is a complex manifold. It has rather been believed that it is *not* a complex manifold.

A hypothetical compact complex 3-manifold X homeomorphic to S^6 has many remarkable properties:

(i) (LeBrun, 1987; Zhou, 2006) The almost complex operator J on S^6 underlying X is not orthogonal with respect to the standard round metric on $S^6 \subset \mathbb{R}^7$;

(ii) (Gray, 1997; Ugarte, 2000) $h^{0,1}(X) = 1$ (in particular X is non-Kähler);

(iii) (Campana–Demailly–Peternell, 1998) $a(X) = 0$ i.e., the algebraic dimension of X is zero;

(iv) (Huckleberry–Kebekus–Peternell, 2000) X is a non-homogeneous complex manifold moreover all orbits of $\mathrm{Aut}_{\mathscr{O}}(X)$ are closed. From this it follows that there exists a 1-parameter family of "exotic $\mathbb{C}P^3$'s".

Lie groups as complex manifolds

A hierarchy of complex manifolds:

(i) Compact algebraic manifolds (projective varieties);

(ii) Compact non-algebraic manifolds (e.g. Hopf manifolds, Inoue surfaces, Calabi–Eckmann manifolds, Samelson manifolds, X);

(iii) Non-compact analytic spaces (e.g. Stein-manifolds).

Samelson, 1953: if G is a compact even dimensional Lie group then there exists an integrable almost complex structure J on G making G into a homogeneous complex manifold.

Definition

Let G be a compact even dimensional Lie group and let J be an integrable almost complex structure on G. Then a complex manifold Y is a Samelson manifold if its underlying almost complex manifold is (G, J).

Two facts about the exceptional compact Lie group G_2:

(i) It is an example of a Samelson manifold: there exist compact, complex, non-algebraic 7-manifolds Y (parameterized by $\mathbb{C}^+ \sqcup \mathbb{C}^-$) homeomorphic to G_2;

(ii) It is a (non-trivial) principal bundle over the six-sphere:

$$\pi : \ \mathrm{G}_2 \stackrel{\mathrm{SU}(3)}{\longrightarrow} S^6$$

(this follows from the existence of octonions or Cayley numbers).

The key idea to attack S^6 (motivated by quantum field theory):

(i) Take any $Y \cong (\mathrm{G}_2, J)$ and consider J as a Higgs field on G_2;

(ii) Consider a non-linear Higgs field theory with $\mathrm{SO}(14)$ gauge symmetry on G_2 with Lagrangian

$$\mathscr{E}(\Phi, g_\Phi) := \frac{1}{2}\int_{\mathrm{G}_2}\left(|N_\nabla\Phi|^2_{g_\Phi} + e^2|\Phi\Phi^* - \mathrm{Id}_{T\mathrm{G}_2}|^2_{g_\Phi}\right)\mathrm{d}V_y$$

where $N_\nabla\Phi$ is the Nijenhuis operator of the Higgs field Φ and g_Φ is a metric on G_2 for which Φ is orthogonal;

(iii) Let us try to construct an (integrable) almost complex structure on S^6 by picking the ground mode of the Fourier expansion of the vacuum $\Phi = J$ with respect to $\mathrm{SU}(3)$.

This ground mode is the average along the $\mathrm{SU}(3)$-fibers hence descends from G_2 to a tensor field on S^6.

Fourier expansion over principal bundles

Let G be a compact Lie group and $\pi : P \to M$ be a principal G-bundle. Assume that there exists a complex vector bundle $p : E \to P$ such that the G-action on P lifts to E i.e., E is a vector bundle over P with a G-action. Consider a section

$$s \in L^2(P; E).$$

We obtain a unitary representation of G on $L^2(P; E)$ by

$$(\gamma \cdot s)(y) := s(y\gamma)\gamma^{-1}, \quad \gamma \in G, \quad y \in P.$$

There exists an orthogonal decomposition into isotypical summands

$$L^2(P; E) \cong \bigoplus_{\rho \in \mathrm{Irr}(G;\mathbb{C})} W(V_\rho)$$

where $\rho : G \to \mathrm{Aut}\, V_\rho$ is a finite dimensional complex irreducible representation of G.

The orthogonal projections are given by fiberwise integration

$$\Pi_\rho s := \frac{1}{\mathrm{Vol}\, G}\int_G (\gamma \cdot s)\, \overline{\chi}_\rho(\gamma)\mathrm{d}\gamma$$

where χ_ρ is the character of ρ. Hence the Fourier expansion of $s \in L^2(P; E)$ is defined by the formal sum

$$s \sim \sum_{\rho \in \mathrm{Irr}(G;\mathbb{C})} \Pi_\rho s.$$

The ground mode $s_1 := \Pi_1 s \in L^2(P; E)$ given by the trivial representation $1 \in \mathrm{Irr}(G; \mathbb{C})$ descends to a section

$$\tilde{s}_1 \in L^2(P/G; E/G) \cong L^2(M; E/G).$$

An energy estimate

Now consider $P := \mathrm{G}_2$, $G := \mathrm{SU}(3)$ and $M := S^6$. Let $\Phi \in C^\infty(\mathrm{G}_2; \mathrm{End}(T\mathrm{G}_2))$ be the Higgs field equal to an integrable almost complex structure J on G_2. This Higgs field satisfies

$$\frac{1}{2}\int_{\mathrm{G}_2} \left(|N_\nabla \Phi|^2_{g_\Phi} + e^2|\Phi\Phi^* - \mathrm{Id}_{T\mathrm{G}_2}|^2_{g_\Phi}\right) \mathrm{d}V_y = 0 \qquad (1)$$

where $N_\nabla \Phi$ is the Nijenhuis operator of Φ and g_Φ is a metric on G_2 for which Φ is orthogonal.

The tangent bundle admits a splitting

$$T\mathrm{G}_2 \cong H \oplus W$$

where for all $x \in S^6$ we have $W|_{\pi^{-1}(x)} \cong T\mathrm{SU}(3)$ i.e., W is the *vertical bundle* with respect to $\pi : \mathrm{G}_2 \to S^6$ and H is a carefully selected *horizontal bundle* (given by the Lie algebra structure). This induces a splitting

$$\Phi = \begin{pmatrix} \eta & 0 \\ 0 & \phi \end{pmatrix}$$

where $\eta \in C^\infty(\mathrm{G}_2; \mathrm{End}\, W)$ and $\phi \in C^\infty(\mathrm{G}_2; \mathrm{End}\, H)$.

Consequently (1) and this splitting yields that along $H \subset T\mathrm{G}_2$

$$\frac{1}{2}\int\limits_{\mathrm{G}_2} \left(|N_\nabla \phi|^2_{g_\Phi} + e^2|\phi\phi^* - \mathrm{Id}_H|^2_{g_\Phi}\right) \mathrm{d}V_y = 0. \qquad (2)$$

Fourier expansion of $\phi \in C^\infty(\mathrm{G}_2; \mathrm{End} H) \subset C^\infty(\mathrm{G}_2; \mathrm{End}(H \otimes \mathbb{C}))$ with respect to $\mathrm{SU}(3)$:

Lift the right $\mathrm{SU}(3)$-action on G_2 to H as follows. Let $\beta \in C^\infty(\mathrm{G}_2; \mathrm{Aut} H)$ be an arbitrary automorphism. Given any $\gamma \in \mathrm{SU}(3)$, $y \in \mathrm{G}_2$ and $u(y) \in H_y$ consider the right $\mathrm{SU}(3)$-action on H of the form

$$(u(y))\gamma_{\beta,\pi_*} := (\pi_*|_{H_{y\gamma}} \beta_{y\gamma})^{-1} (\pi_*|_{H_y} \beta_y)(u(y)) \qquad (3)$$

where $\pi_*|_H : H \to TS^6$ is the derivative. With respect to this action we consider the Fourier expansion

$$\phi = \sum_{\rho \in \mathrm{Irr}(\mathrm{SU}(3);\mathbb{C})} \Pi_\rho \phi \qquad (4)$$

and the ground mode

$$\tilde{\phi}_1 \in C^\infty(\mathrm{G}_2/\mathrm{SU}(3); \mathrm{End}(H/\mathrm{SU}(3))) \cong C^\infty(S^6; \mathrm{End}\, TS^6).$$

Lemma

There exists a metric $h_{\tilde\phi_1}$ on S^6 such that $\tilde\phi_1 \in C^\infty(S^6; \mathrm{End}\, TS^6)$ has vanishing energy i.e.,

$$\mathscr{E}(\tilde\phi_1, h_{\tilde\phi_1}) := \frac{1}{2}\int_{S^6}\left(|N_\nabla\tilde\phi_1|^2_{h_{\tilde\phi_1}} + e^2|\tilde\phi_1\tilde\phi_1^* - \mathrm{Id}_{TS^6}|^2_{h_{\tilde\phi_1}}\right)\mathrm{d}V_x = 0. \tag{5}$$

Here $N_\nabla\tilde\phi_1$ is the Nijenhuis operator of $\tilde\phi_1$.

Proof. (sketch) The proof consists of three steps:

(i) $0 \leqq \mathscr{E}(\tilde\phi_1, h_{\tilde\phi_1})$ can be estimated via (2), (4) from above by

$$C_6\left|\int_{S^6}\left(\int_{\mathrm{SU}(3)} i_x^*\left(\sum_{\substack{\rho,\sigma\in\mathrm{Irr}(\mathrm{SU}(3);\mathbb{C})\\ \rho,\sigma\not\cong 1}}\langle\Pi_\rho\phi\,,\,\Pi_\sigma\phi\rangle_{g_\Phi}\right)\mathrm{d}\gamma\right)\mathrm{d}V_x\right|^{\frac{1}{2}}$$

where $i_x : \mathrm{SU}(3) \longrightarrow \pi^{-1}(x) \subset \mathrm{G}_2$ is an embedding;

(ii) The representation of $\mathrm{SU}(3)$ on $L^2(\mathrm{G}_2; H \otimes \mathbb{C})$ restricted to a subspace isomorphic to the standard 3 dimensional complex representation induces an $\mathrm{SU}(3)$-structure on TS^6 (valid only for S^6!). Hence there exists an almost complex operator $J : TS^6 \to TS^6$ and a function $f : S^6 \to \mathbb{R}$ such that

$$\tilde{\phi}_1 = fJ.$$

The previous estimate reduces to

$$0 \leqq \mathscr{E}(\tilde{\phi}_1, h_{\tilde{\phi}_1}) \leqq \sqrt{6}\, C_6 \left| \int\limits_{S^6} \left(1 - f^2(x)\right) \mathrm{d}V_x \right|^{\frac{1}{2}};$$

(iii) Recall that Fourier expansion depends on the right action (3) of $\mathrm{SU}(3)$ on H. Hence selecting the Fourier expansion appropriately we can achieve that $f = \pm 1$ yielding

$$\mathscr{E}(\tilde{\phi}_1, h_{\tilde{\phi}_1}) = 0.$$

This finishes the sketch of the proof of the lemma. $\diamond$

Because $\tilde{\phi}_1^2 = -\mathrm{Id}_{TS^6}$ and ∇ is torsion-free we obtain that $N_\nabla \tilde{\phi}_1 = N_{\tilde{\phi}_1}$ is just the Nijenhuis tensor of $\tilde{\phi}_1$. Hence by the Newlander–Nirenberg theorem the previous lemma implies that in fact $J = \pm\tilde{\phi}_1$ is an everywhere integrable almost complex structure on the six-sphere.

Theorem

There exists a complex manifold X which is homeomorphic to the six-sphere S^6.

Remark

Let Y be a Samelson manifold homeomorphic to G_2. By first sight (i.e., we cannot prove these in this moment) the constructed complex structure satisfies:

(i) The resulting complex structure on S^6 is apparently not orthogonal with respect to the standard metric;

(ii) We know that $h^{0,1}(Y) = 1$ (Pittie, 1986) hence $h^{0,1}(X) = 1$ is also expected;

(iii) We expect X to have low algebraic dimension because already Y is non-algebraic;

(iv) Homogeneity of Y is likely to be lost during Fourier expansion hence X is expected to be inhomogeneous.

The physics behind the construction

The broad concept of a spontaneous symmetry breaking:

Let M be an n-manifold and $G \subsetneqq \mathrm{GL}(n, \mathbb{R})$ be an abstract Lie group. Recall that

(i) A G-structure $\mathrm{Str}(G)$ on M is a principal sub-bundle of the frame bundle of M such that the fibers and the structure group of this bundle is G;

(ii) A G-structure is integrable or torsion-free if there exists a torsion-free connection ∇ on TM such that $\mathrm{Hol}\nabla \subsetneqq G$.

Starting setup (input data): Assume $(M, \mathrm{Str}(G), \nabla)$ is given such that $\mathrm{Hol}\nabla \subsetneqq G$ (probably ∇ is not torsion-free).
Note that G plays a double role here: it is both a structure group (for $\mathrm{Str}(G)$) and a holonomy group (of ∇).

Let $H \subsetneqq G$ be a subgroup.

Definition

A strong spontaneous symmetry breaking $G \to H$ is given in $(M, \mathrm{Str}(G), \nabla)$ if G as the *holonomy group* of ∇ is reducible to H i.e., $\mathrm{Hol}\nabla \subseteqq H$.

An example: The classical vacuum of a Higgs field theory

Let (M, g) be an oriented Riemannian manifold, $\dim_{\mathbb{R}} M = 2m$. Let $\Phi \in C^\infty(M; \mathrm{ad}M)$ be a section of the adjoint bundle (Higgs field) and ∇ be the Levi–Civita connection of g (gauge field). Consider the field theory with $\mathrm{SO}(2m)$ gauge symmetry

$$\mathscr{E}(\Phi, g) := \frac{1}{2}\int\limits_M (|\nabla\Phi|_g^2 + e^2|\Phi\Phi^* - \mathrm{Id}_{TM}|_g^2)\mathrm{d}V.$$

Hence we start with $(M, \mathrm{Str}(\mathrm{SO}(2m)), \nabla)$ and $\mathrm{Hol}\nabla \subsetneqq \mathrm{SO}(2m)$.

Classical vaccum: $\mathscr{E}(\Phi, g) = 0$ i.e.,

$$\begin{cases} 0 = & \nabla\Phi \\ 0 = & \Phi\Phi^* - \mathrm{Id}_{TM}. \end{cases}$$

If exists it gives rise to a Kähler structure (M, J, g) where $J = \Phi$ (because of the second vacuum equation). This is an (integrable) strong sp. symmetry breaking $\mathrm{SO}(2m) \to \mathrm{U}(m)$ because $\mathrm{Hol}\nabla \subseteqq \mathrm{U}(m) \subset \mathrm{SO}(2m)$ (because of the first vacuum equation).

We end up with $(M, \mathrm{Str}(U(m)), \nabla)$ and $\mathrm{Hol}\nabla \subseteqq \mathrm{U}(m)$.

Definition

A weak spontaneous symmetry breaking $G \rightarrow H$ is given in $(M, \mathrm{Str}(G), \nabla)$ if G as the *structure group* for $\mathrm{Str}(G)$ is reducible to H i.e., $\mathrm{Str}(G)$ reduces to $\mathrm{Str}(H)$.

Remark

(i) Probably for the connection $\mathrm{Hol}\nabla \nsubseteq H$;

(ii) A strong sp. symmetry breaking always implies the corresponding weak one but not the other way round.

An example: The classical vacuum of a non-linear Higgs field theory

Let (M, g) and $\Phi \in C^\infty(M; \mathrm{ad}M)$ and ∇ be as before and denote by $N_\nabla : C^\infty(M; \mathrm{ad}M) \to C^\infty(M; \mathrm{ad}M \otimes \Lambda^1 M)$ the Nijenhuis operator (i.e., for $J \in C^\infty(M; \mathrm{ad}M)$ simply $N_\nabla J = N_J$ is the Nijenhuis tensor of J). Consider the field theory with $\mathrm{SO}(2m)$ gauge symmetry

$$\mathscr{E}(\Phi, g) := \frac{1}{2}\int_M (|N_\nabla \Phi|^2_g + e^2|\Phi\Phi^* - \mathrm{Id}_{TM}|^2_g)\mathrm{d}V.$$

Hence we start with $(M, \mathrm{Str}(\mathrm{SO}(2m)), \nabla)$ and $\mathrm{Hol}\nabla \subseteqq \mathrm{SO}(2m)$.

Classical vaccum: $\mathscr{E}(\Phi, g) = 0$ i.e.,

$$\begin{cases} 0 = & N_{\nabla}\Phi \\ 0 = & \Phi\Phi^* - \mathrm{Id}_{TM}. \end{cases}$$

If exists it gives rise to a complex structure (M, J) where $J = \Phi$ (because of the second vacuum equation). This is an (integrable) weak sp. symmetry breaking $\mathrm{SO}(2m) \to \mathrm{U}(m)$ because $\mathrm{Str}(U(m)) \subset \mathrm{Str}(\mathrm{SO}(2m))$ (because of the first vacuum equation).

We end up with $(M, \mathrm{Str}(\mathrm{U}(m)), \nabla)$ but probably $\mathrm{Hol}\nabla \nsubseteq \mathrm{U}(m)$ (because probably $\nabla\Phi \neq 0$).

The strategy (very common in classical or quantum field theory)

(i) We seek a classical vacuum solution (5) in a non-linear Higgs field theory defined over S^6. This vacuum represents a weak sp. symmetry breaking corresponding to a complex structure on S^6. However finding this solution is not easy.

(ii) Therefore we first consider a solution of the problem in higher dimensions where the solution is easy. Due to Samelson the classical vacuum (1) of a non-linear Higgs field theory defined over G_2 exixts and represents a weak sp. symmetry breaking corresponding to a complex structure on G_2.

(iii) By dimensional reduction (Fourier expansion) we obtain a solution on S^6.

For further details please check:

arXiv: math.DG/0505634

or the more upgraded version

URL: http://www.math.bme.hu/~etesi/preprint.html