# Perspectives on $G_2-$instantons

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# Preliminaries

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E.g.: Riemannian manifold $(X, g)$ of dimension $n \geq 4$,
$SU(n)-$bundle $E \to X$, $A$ connection on $E$.

The *Yang-Mills functional*:

$$YM(A) \doteq \|F_A\|^2 = \int_X \langle F_A \wedge *F_A \rangle_{\mathfrak{su}(n)},$$

induces the (Euler-Lagrange) *Yang-Mills equation*

$$d_A^* F_A = 0.$$

$n = 4: \quad \Omega^2 = \Omega^2_+ \oplus \Omega^2_-, \quad F_A = \pm * F_A$ (*SD* or *ASD*) sols.
$n > 4: \quad$ ?!

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Tian et al.: a *closed* $(n-4)-$form $\Theta$ on $X$ generalises (A)SD:

$$F_A \wedge \Theta = - * F_A \qquad [\Theta - \text{instanton}].$$

**¿How to find closed tensors?**

*Fundamental holonomy theorem*:

$$\exists S \in \Gamma(\mathcal{T}) \quad \text{s.t.} \quad \nabla S = 0$$
$$\Updownarrow$$
$$\exists x \in X, S_x \in \mathcal{T}_x \quad \text{s.t.} \quad \text{Hol}(g).S_x = S_x$$

with $\mathcal{T} \doteq (\bigotimes^{\bullet} TX \otimes \bigotimes^{\bullet} T^* X)$.

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$(M^n, g)$ simply-connected Riemannian manifold, $g$ irreducible and nonsymmetric; then exactly one of the following holds:

1. Hol $(g) = SO(n)$
2. $n = 2m, m \geq 2$ : Hol $(g) = U(m) \subset SO(2m)$
3. $\mathbf{n = 2m, m \geq 2}$ : Hol $(\mathbf{g}) = \mathbf{SU(m) \subset SO(2m)}$
4. $n = 4m, m \geq 2$ : Hol $(g) = Sp(m) \subset SO(4m)$
5. $n = 4m, m \geq 2$ : Hol $(g) = Sp(m)\, Sp(1) \subset SO(4m)$
6. $\mathbf{n = 7, m \geq 2}$ : Hol $(\mathbf{g}) = \mathbf{G_2 \subset SO(7)}$
7. $n = 8, m \geq 2$ : Hol $(g) = \text{Spin}(7) \subset SO(8)$ .

We will be interested in the interplay between the instances:

Hol $(g) = SU(3)$ (Calabi-Yau $3-$folds): $\omega^{1,1}, \Omega^{3,0}$
Hol $(g) \subseteq G_2$ ($G_2-$manifolds): $\varphi^3$ (and $*\varphi^4$)

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$$\varphi_{\mathbf{0}} = \left(\mathbf{e^{12}} - \mathbf{e^{34}}\right)\mathbf{e^{5}} + \left(\mathbf{e^{13}} - \mathbf{e^{42}}\right)\mathbf{e^{6}} + \left(\mathbf{e^{14}} - \mathbf{e^{23}}\right)\mathbf{e^{7}} + \mathbf{e^{567}}$$

$\{e^i\}_{i=1,\dots,7}$ canonical basis of $\left(\mathbb{R}^7\right)^*$, $e^{ij} = e^i e^j \doteq e^i \wedge e^j$ etc.

$$\boxed{G_2 \doteq \{g \in GL(7) \mid g^*\varphi_0 = \varphi_0\}}$$

**Definition:** A $G_2-$*structure* on $M^7$ is a form $\varphi \in \Omega^3(M)$ s.t.,

$$\varphi_p = f_p^*(\varphi_0)$$

for some frame $f_p : T_pM \to \mathbb{R}^7, \quad \forall p \in M$.

If $\nabla\varphi = 0$ (torsion-free), $\left(M^7, \varphi\right)$ is a $G_2$-*manifold*; then we have

$$d\varphi = 0, \quad d *_\varphi \varphi = 0 \quad \text{and} \quad \mathrm{Hol}(\varphi) \subseteq G_2.$$

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# On the 7-torus... a Nahm transform? (I)

Usual Nahm transform:

$$\begin{array}{ccc} (E,\nabla) & & (\hat{E},\hat{\nabla}) \\ \downarrow & \leftrightarrow & \downarrow \\ T^4 & & \hat{T}^4 \end{array}$$

$$\hat{T}^4 = \left(\mathbb{R}^4\right)^* / \Lambda^*, \qquad \Lambda^* \doteq \left\{\xi \in (\mathbb{R}^4)^* \mid \xi(z) \in \mathbb{Z},\ z \in \Lambda\right\}.$$

Interestingly:

$$F^+_{\nabla} = 0 \quad \Rightarrow \quad F^+_{\hat{\nabla}} = 0 \quad !!!$$

and...

$$\begin{aligned} \operatorname{rk} \hat{E} &= c_2(E), \\ c_1(\hat{E}) &= c_1(E), \\ c_2(\hat{E}) &= \operatorname{rk} E \end{aligned}$$

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Ingredients:

- notion of instanton

$$\Omega^2(\mathfrak{g}) = \Omega^2_+(\mathfrak{g}) \oplus \Omega^2_-(\mathfrak{g})$$

- decomposition of Dirac spinors

$$D_\nabla = \begin{pmatrix} 0 & D^+_\nabla \\ D^-_\nabla & 0 \end{pmatrix} = D^*_\nabla$$

- Weitzenböck formula

$$D^+_\nabla D^-_\nabla = \nabla^*\nabla + F^+_\nabla$$

**¿ Can we set up an analogous construction for $T^7$?**

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**Definition** A $G_2-$*fibred torus* is a triplet $(\eta, L, \alpha)$ in which:

- $\eta$ is a metric on the vector space $\mathbb{R}^4$;
- $L$ is a lattice in the subspace $\Lambda^2_+(\mathbb{R}^4, \eta)$ of $2-$forms SD w.r.t. $\eta$;
- $\alpha : \mathbb{R}^4 \to \Lambda^2_+(\mathbb{R}^4, \eta)$ is a linear map.

Set $V \doteq \mathbb{R}^4 \oplus \Lambda^2_+$ and form the torus $\mathbb{T} = V/\tilde{L}$, with the lattice

$$\tilde{L} \doteq \left\{(\mu, \nu + \alpha\mu) \,|\, \mu \in \mathbb{Z}^4,\ \nu \in L\right\} \subset V.$$

**Proposition** Let $\mathbb{T}$ be a $G_2-$fibred torus; if $[A] \in \mathcal{M}^+$ is a self-dual connection on $E \to T^4$, then its lift $[\tilde{A}]$ by the fibration map $f : \mathbb{T} \to T^4$ is a $G_2-$instanton on $\tilde{E}$.

**¿Which instantons on such $\tilde{E} \to \mathbb{T}$, if any, are *not* pull-backs from self-dual connections over $T^4$?**

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Under a deformation of $G_2-$structure:

$$\varphi \to \varphi + \phi, \qquad *\varphi \to *\varphi + \xi_\phi, \qquad \xi_\phi \doteq *_\varphi \phi \in \Omega^4(\mathbb{T}),$$

the error $\xi_\phi \in \Omega^4(\mathbb{T})$ has four orthogonal components:

$$\Lambda^4(\mathbb{R}^4 \oplus \Lambda^2_+) = \underbrace{\Lambda^4(\mathbb{R}^4)}_{(I)} \oplus \underbrace{\Lambda^3(\mathbb{R}^4) \otimes \Lambda^1(\Lambda^2_+)}_{(II)} \oplus \underbrace{\Lambda^2(\mathbb{R}^4) \otimes \Lambda^2(\Lambda^2_+)}_{(III)} \oplus \underbrace{\Lambda^1(\mathbb{R}^4) \otimes \Lambda^3(\Lambda^2_+)}_{(IV)}$$

**(I)...(III)** correspond to various deformations of the fibred structure $(\eta, L, \alpha)$.
**(IV)** parametrises deformations **transverse to all fibred structures**.

**¿Do $(\varphi + \phi)-$instantons exist *at all*?**

NB.: There is a Chern-Simons formalism...

$$\rho(b)_A = \int_{\mathbb{T}} \mathrm{tr}\,(F_A \wedge b_A) \wedge *\varphi \qquad :o)$$

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$(W, \omega)$ Kähler manifold, $\mathcal{E} \to W$ holomorphic vector bundle:

$$\left\{ \begin{array}{c} \text{Hermitian metrics} \\ H \text{ on } \mathcal{E} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{unitary (Chern) connections} \\ A = A_H \text{ on } \mathcal{E} \end{array} \right\};$$

in particular, $F_{A_H} \in \Omega^{1,1}(\mathfrak{g})$. Then $H$ is *Hermitian Yang-Mills (HYM)* if the curvature has vanishing $\omega-$trace:

$$\hat{F}_A \doteq (F_A, \omega) = 0.$$

**Proposition:** A HYM connection $A$ on a hol. v.b. $\mathcal{E} \to W$ over a CY $3-$fold $W$ lifts to a $G_2-$instanton on $p_1^*\mathcal{E} \to M = W \times S^1$, where

$$\begin{aligned} \varphi &= \omega \wedge d\theta + \operatorname{Im}\Omega, \\ *\varphi &= \tfrac{1}{2}\omega \wedge \omega - \operatorname{Re}\Omega \wedge d\theta. \end{aligned}$$

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# Kovalev's construction

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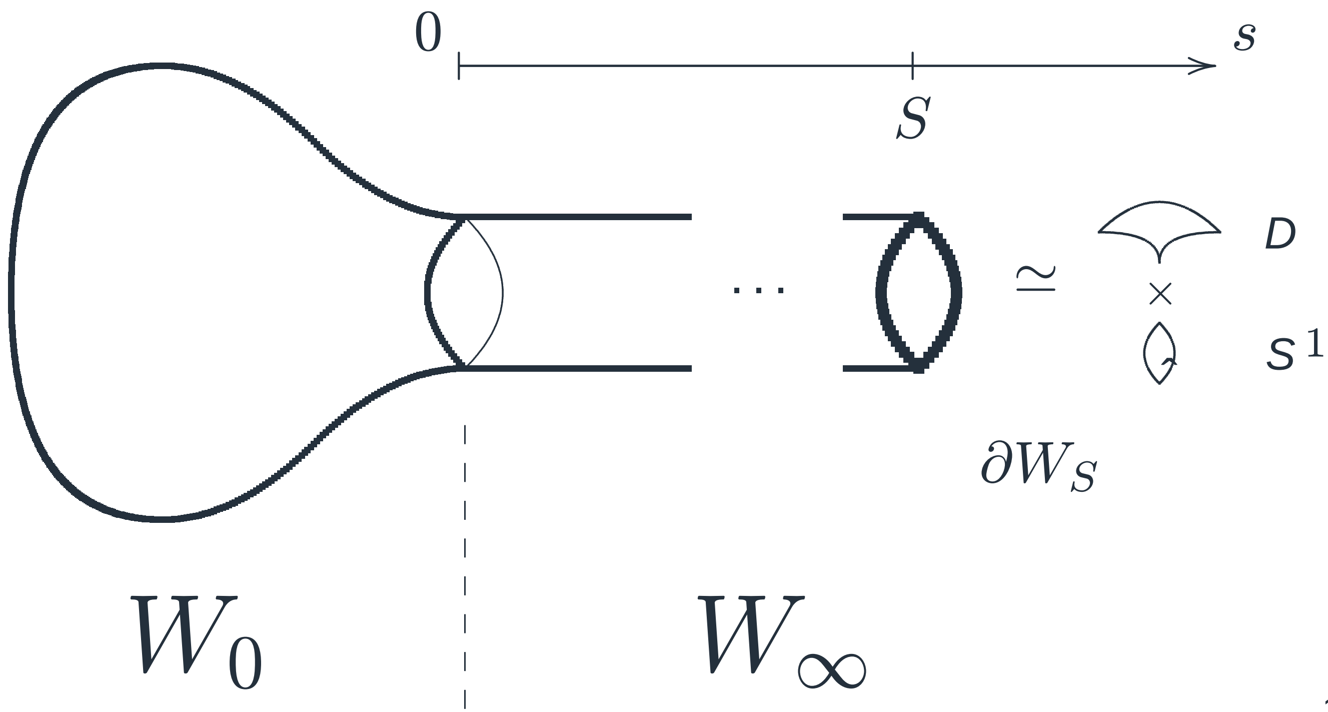
$(\bar{W}^3, \bar{\omega}, I)$ compact, simply-connected, Kähler with:

- $\exists\, K3-$surface $D \in |-K_{\bar{W}}|$ with $\mathcal{N}_{D/\bar{W}}$ (hol.) trivial;
- The complement $W = \bar{W} \setminus D$ has finite $\pi_1(W)$.

Think of $W$ as $W = W_0 \cup W_\infty$, where $W_0$ is compact with boundary and

$$\partial W_0 \simeq D \times S^1, \quad W_\infty \simeq \left(D \times S^1 \times \mathbb{R}_+\right).$$



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**Theorem [Calabi-Yau-Tian-Kovalev]:** For $W = \bar{W} \setminus D$ as above:

1. $W$ admits a complete Ricci-flat Kähler structure $\omega$;
2. Hol $(\omega) = SU(3)$, i.e. $W$ is Calabi-Yau;
3. along the tubular end $D \times S^1_\alpha \times (\mathbb{R}_+)_s$, the Kähler form $\omega$ and the holomorphic volume form $\Omega$ are exponentially asymptotic[1] to those of the product Ricci-flat Kähler metric on $D$:

$$\begin{aligned} \omega|_{W_\infty} &= \kappa_I + ds \wedge d\alpha + d\psi \\ \Omega|_{W_\infty} &= (ds + \mathbf{i} d\alpha) \wedge (\kappa_J + \mathbf{i}\kappa_K) + d\Psi. \end{aligned}$$

We say $(W, \omega)$ is an *exponentially asymptotically cylindrical Calabi-Yau (EACCY)* $3-$fold.

NB.: $\kappa_I$, $\kappa_J$ and $\kappa_K$ (hyper-)Kähler forms on the K3 surface $D$.

[1] with d$\psi, d\Psi = O\left(e^{-s}\right)$.

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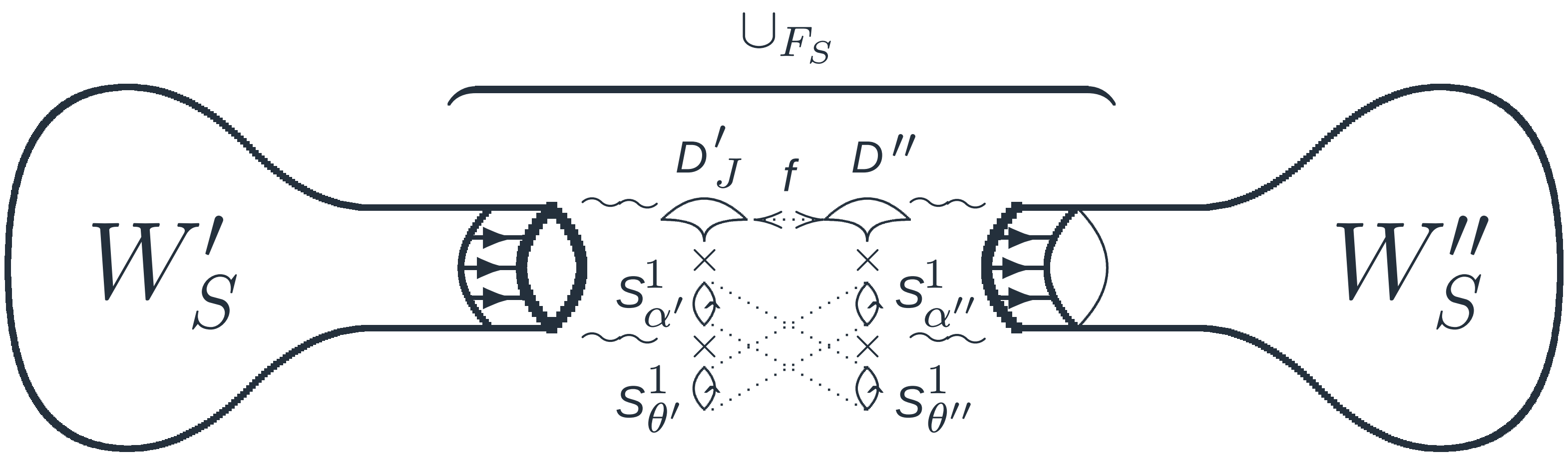
Given a suitable pair of 3-folds $W'$ and $W''$ as above, one obtains a compact oriented $7-$manifold

$$M_S = \left(W'_S \times S^1\right) \cup_{F_S} \left(W''_S \times S^1\right) \doteq W' \widetilde{\#}_S W''.$$

'Stretching the neck', one equips $M_S$ with a $G_2-$structure satisfying exactly

$$\text{Hol}\left(\varphi\right) = G_2.$$




**¿A gluing formula for the associated vertex algebras?**

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# Examples

Kovalev produces asymptotically cylindrical $3-$folds from several types of initial Fano varieties

$$\bar{W} = \mathrm{Bl}_S V$$

with:

(i) $V = \mathbb{P}^3$
(ii) $V \subset \mathbb{P}^4$, $\deg(V) = 2$ [complete intersection]
(iii) $V \subset \mathbb{P}^4$, $\deg(V) = 3$. [complete intersection]
(iv) $V = \mathbb{P}^2 \times \mathbb{P}^1$.
(v) $V \xrightarrow[D]{2:1} \mathbb{P}^3$, $\deg D = 4$.
(vi) $V = X_{22} \hookrightarrow \mathbb{P}^{13} \quad (g = 12)$.

NB.: Will not discuss (v) in this talk.

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# The Hermitian Yang-Mils problem

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**Goal:** to solve the HYM problem $\hat{F}_H = 0$ on suitable bundles over these EACCYs, *ergo* $G_2$—instantons on the pull-back over $W \times S^1$.

**Strategy:** consider first the 'nonlinear heat flow'

$$(\dagger) \begin{cases} H^{-1}\dfrac{\partial H}{\partial t} &= -2\mathbf{i}\hat{F}_H \\ H(0) &= H_0 \end{cases} \quad \text{on} \quad W_S \times [0, T[$$

over a truncation, with (Dirichlet) boundary condition

$$H \mid_{\partial W_S} = H_0 \mid_{\partial W_S}$$

where $H_0$ is a fixed *reference (Hermitian) metric* on $\mathcal{E} \to W$ with 'good' asymptotic behaviour. Then

$$H = \lim_{t<T\to\infty} \left( \lim_{S\to\infty} H_S(t) \right).$$

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**Definition:** A bundle $\mathcal{E} \to W$ is *stable at infinity* (or *asymptotically stable*) if it is the restriction of a hol.v.b. $\mathcal{E} \to \bar{W}$ satisfying:

- $\mathcal{E}$ is irreducible;
- $\mathcal{E}|_D$ is stable, hence also $\mathcal{E}|_{D_z}$ for $|z| < \delta$.

**Definition:** A *reference metric* $H_0$ on an asymptotically stable bundle $\mathcal{E} \to W$ is (the restriction of) a smooth Hermitian metric on $\mathcal{E} \to \bar{W}$ such that:

- $H_0|_{D_z}$ are the HYM metrics on $\mathcal{E}|_{D_z}$, $0 \leq |z| < \delta$;
- $H_0$ has finite energy: $\left\| \hat{F}_{H_0} \right\|_{L^2(W,\omega)} < \infty$.

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**Theorem:** Let $\mathcal{E} \to W$ be stable at infinity, equipped with a reference metric $H_0$, over an asymptotically cylindrical $SU(3)-$manifold $W$ as given by the Calabi-Yau-Tian-Kovalev theorem, and let $\{H_t\}_{t\in]0,\infty[}$ be the $1-$parameter family of Hermitian metrics on $\mathcal{E}$ solving the evolution equation $(\dagger)$ over $W$;

then the limit $H = \lim_{t\to\infty} H_t$ exists and is a smooth Hermitian Yang-Mills metric on $\mathcal{E}$, exponentially asymptotic in all derivatives to $H_0$ along the tubular end:

$$\boxed{\hat{F}_H = 0, \quad H \xrightarrow[S\to\infty]{C^\infty} H_0.}$$

...

so we have a $G_2-$instanton on $p_1^*\mathcal{E} \to W \times S^1$!!!

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Generalised Hoppe criterion (I)

Generalised Hoppe criterion (I)

Instanton monads over polycyclic varieties

Instanton monads over polycyclic varieties (II)

Canonical diagram

Asymptotic stability over a ‘bi-cyclic’ product

# Asymptotically stable bundles over Kovalev’s examples

# Stable bundles as monad cohomologies (I)

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Recall that a *monad* on $X$ is a complex of locally free sheaves

$$\mathrm{M}_\bullet \quad : \quad M_0 \stackrel{\alpha}{\longrightarrow} M_1 \stackrel{\beta}{\longrightarrow} M_2$$

such that $\beta$ is locally right-invertible, $\alpha$ is locally left-invertible. The (locally free) sheaf $E := \ker \beta / \alpha$ is called the *cohomology* of $\mathrm{M}_\bullet$.

Let $X^3$ be a projective $3-$fold, so that the cohomology of

$$0 \quad \to \quad \mathcal{O}_X(-1)^{\oplus c} \quad \stackrel{\alpha}{\to} \quad \mathcal{O}_X^{\oplus 2+2c} \quad \stackrel{\beta}{\to} \quad \mathcal{O}_X(1)^{\oplus c} \quad \to \quad 0 \qquad (1)$$

for $c \geq 1$ is a stable hol. v.b. $E \to X$ of rank $2$ and $c_1(E) = 0$ [Jardim]. Then we know that $E$ has no holomorphic sections:

**Lemma 1** *If $\deg(E) = 0$ and $E$ is stable, then $h^0(E) = 0$.*

**¿How about *asymptotically* stable?**

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Specialising Hoppe's criterion to a cyclic divisor $D \subset X$:

**Criterion [Hoppe]:** If $r = 2, \deg(E) = 0$ and Pic $D = \mathbb{Z}$, then

$$E|_D \text{ is stable} \quad \Leftrightarrow \quad h^0\left(E|_D\right) = 0.$$

**¿How about arbitrary rank and degree, and larger Picard group?**

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# Canonical diagram in the cyclic case

Denote $\deg D \doteq d > 0$ and $K \doteq \ker \beta$; then the above data fit in the following *canonical diagram*:

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & \mathcal{O}_X\left(-(d+1)\right)^{\oplus c} & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \to & K(-d) & \to & \mathcal{O}_X(-d)^{\oplus r+2c} & \to & \mathcal{O}_X(-(d-1))^{\oplus c} & \to & 0 \\
 & & \downarrow & & & & & & \\
0 & \to & E(-d) & \to & E & \to & E|_D & \to & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

**Proposition:** Let $E \to X^3$ be the vector bundle, over a nonsingular cyclic Fano variety, arising from an instanton monad of the form $(1)$; if $D \subset X$ is a cyclic divisor, then $E|_D$ is stable.

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The *Proposition* examples of asymptotically stable bundles over Kovalev's spaces obtained from types (i), (ii) and (iii), since 3-d complete intersections are cylic in $\mathbb{P}^n, n \geq 4$.

E.g., the *null-correlation bundle* over $\mathbb{P}^3$ [type (i)]:

$$X = \mathbb{P}^3, \qquad D \in |-K_{\mathbb{P}^3}| = |\mathcal{O}(4)|, \qquad r = 2, \quad c = 1$$

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
 & & \mathcal{O}(-5) & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \to & K(-4) & \to & \mathcal{O}(-4)^{\oplus 4} & \to & \mathcal{O}(-3) & \to & 0 \\
 & & \downarrow & & & & & & \\
0 & \to & E(-4) & \to & E & \to & E|_D & \to & 0 \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

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A projective. variety $X^n$ is *polycyclic* if Pic $(X) \simeq \mathbb{Z}^l$.
Notation:

$$\mathcal{O}_X(p) = \mathcal{O}_X(p_1, \ldots, p_l) \doteq \Upsilon_1^{\otimes p_1} \otimes \cdots \otimes \Upsilon_l^{\otimes p_l}$$

($\Upsilon_i \to X, 1 \leq i \leq l$: generators, $p \in \mathbb{Z}^l$: vector of multiplicities.)

Given any other $E \to X$, its (poly)twist is denoted

$$E(p) = E(p_1, \ldots, p_l) \doteq E \otimes \mathcal{O}_X(p_1, \ldots, p_l).$$

For a torsion-free coherent sheaf $F$ of rank $s$ over $X$, we have

$$\det F \doteq \det\left(\wedge^s F\right)^{\vee\vee} = \mathcal{O}_X(p_1, \ldots, p_l),$$

where $[c_1(F)] = p_1[h_1] + \cdots + p_l[h_l]$, for $[h_i] := c_1(\Upsilon_i) \in H^2(X, \mathbb{Z})$.

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For fixed polarisation $L \to X$, set $\deg_L F \doteq c_1(F) \cdot L^{n-1}$ and

$$\delta_L(p_1, \dots, p_l) := \deg_L \mathcal{O}_X(p_1, \dots, p_l).$$

Set $\mu_L(F) = \frac{\deg_L F}{\text{rank } F}$ (the $L-$slope), $k_E := \lceil \mu_L(E)/d \rceil$ and

$$E_{\mathrm{L-norm}} := E(-k_E, 0, \dots, 0).$$

**Proposition 2 (Generalised Hoppe criterion)** *Let $E \to X$ be a rank $r \geq 2$ holomorphic vector bundle over a polycyclic variety $X$ of Picard rank $l$. If $H^0\left((\Lambda^s E)_{\mathrm{L-norm}}(\vec{p})\right) = 0$ for every $1 \leq s \leq r-1$ and every $\vec{p} \in \mathbb{Z}^l$ such that $\delta_L(\vec{p}) \leq 0$, then $E$ is $L$-stable.*

NB.: $L$-stable: every $F \hookrightarrow E$ satisfies $\mu_L(F) < \mu_L(E)$.

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**Theorem 3** *Let $X = \mathbb{P}^n \times \mathbb{P}^m$ with $m \leq n$, and set $L := \mathcal{O}_X(1,1)$. Then monads of the form*

$$0 \to \mathcal{O}_X(-1,0)^{\oplus a} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus b} \oplus \mathcal{O}_X(-1,1)^{\oplus c} \xrightarrow{\beta} \mathcal{O}_X(0,1)^{\oplus a} \to 0 \tag{2}$$

*with $a, b, c \in \mathbb{N}$ such that $r := b + c - 2a = 2$, verify:*

- *the kernel bundle $K := \ker\beta$ is a $L$-stable bundle of rank $a+2$ with*

$$\deg_L(K) = \frac{n(n+1)\cdots(n+m+1)}{m!}\left(\left(\frac{m}{n}-1\right)c - \frac{m}{n}a\right);$$

- *for $c \geq a$, $E := \ker\beta/\alpha$ is a $L$-stable bundle of rank $2$ with*

$$\deg_L(E) = \frac{n(n+1)\cdots(n+m+1)}{m!}\left(1-\frac{m}{n}\right)(a-c).$$

**¿How about larger Cartesian products, or any polycyclic $X$?**

# Instanton monads over polycyclic varieties (II)


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The corresponding canonical diagram is:

$$
\begin{array}{ccccccc}
 & 0 & & & & & \\
 & \downarrow & & & & & \\
 & \mathcal{O}_X^{\oplus c}(-1,0)(-d) & & & & & \\
 & \downarrow & & & & & \\
0 \to & K(-d) & \to & \left\{\mathcal{O}_X^{\oplus b} \oplus \mathcal{O}_X^{\oplus c}(-1,1)\right\}(-d) & \to & \mathcal{O}_X^{\oplus c}(0,-1))(-d) & \to 0 \\
 & \downarrow & & & & & \\
0 \to & E(-d) & \to & E & \to & E|_D & \to 0 \\
 & \downarrow & & & & & \\
 & 0 & & & & &
\end{array}
$$

and again the claim follows by calculating the relevant cohomologies.

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# Instanton monads over polycyclic varieties (III)

**Example:** Now let $X = \mathbb{P}^2 \times \mathbb{P}^1$, with homogeneous coordinates $[x_0 : x_1 : x_2]$ and $[y_0 : y_1]$ on $\mathbb{P}^2$ and $\mathbb{P}^1$, respectively, and consider the explicit monad

$$0 \to \mathcal{O}_X \xrightarrow{\alpha} \mathcal{O}_X(1,0)^{\oplus 2} \oplus \mathcal{O}_X(0,1)^{\oplus 2} \xrightarrow{\beta} \mathcal{O}_X(1,1) \to 0$$

given by

$$\alpha = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \oplus \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} y_0 & y_1 & -x_0 & -x_1 \end{pmatrix}.$$

Twisting by $\mathcal{O}_X(-1,0)$ one obtains precisely an instanton monad of the form above, with $a = 1, b = c = 2$, thus indeed $b + c - 2a = 2$.

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# Asymptotic stability over a ‘bi-cyclic’ product

A divisor $D$ has *positive polydegree* if $\deg D = d = (d_1, d_2)$ with $d_1, d_2 \geq 0$.

**Proposition 4** *Let $E \to X^3 = X_1 \times X_2$ be a (stable, rank 2) vector bundle, over a nonsingular polycyclic Fano variety, arising from an instanton monad of the form above, with $r = 2$; if $D \subset X$ is a polycyclic divisor of positive polydegree, then $E|_D$ is stable.*

This covers type (iv) in Kovalev’s list of examples.

**¿How about double covers of type (v)?**

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# Gluing of $G_2-$instantons

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Now want a $G_2-$instanton $A_S = A'\tilde{\#}_S A''$ over the *compact* $G_2-$manifold $M_S$:

$$\begin{array}{c} \mathcal{E} \doteq (p^*\mathcal{E}')\,\tilde{\#}_S\ (p^*\mathcal{E}'') \\ \downarrow \\ M_S = \left(W'_S \times S^1\right)\tilde{\#}_S\left(W''_S \times S^1\right). \end{array}$$

Due to the asymptotics of the HYM problem, one can arrange

$$\left\|F^+_{A_S}\right\|_{L^2(M_S)} = O\left(e^{-S}\right)\ldots$$

**Theorem** *Let* $\mathcal{E}^{(i)} \to W^{(i)}$, $i = 1, 2$, *be asymptotically stable bundles with same structure group* $G$, *with* acyclic *reference metrics* $H_0^{(i)}$, *admitting a* $G-$*isomorphism* $g : \mathcal{E}'|_{D'_J} \tilde{\to} \mathcal{E}''|_{D''}$.
*There exists* $S_0 > 0$ *such that the bundle* $\mathcal{E}_S \to M_S \doteq W'\tilde{\#}_S W''$ *admits a* $G_2-$*instanton, for every* $S \geq S_0$.

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**¿To which extend are *all* $G_2-$instantons obtained in this fashion?**

Restriction to $D$ induces two maps into $\mathcal{M}_D$, whose images are 'morally' Lagrangian submanifolds.

**¿Can we make this rigorous, and characterise the final moduli space $\mathcal{M}_{W'\tilde{\#}_S W''}$ as a Lagrangian intersection, and 'count' points?**

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Thank you!