Differential Equations and Stable Homotopy

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Outline

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Introduction: Differential equations

Stable Homotopy

Differential Equations and Stable Homotopy

Remarks on the Proof

A Glimpse on the global Picture

Introduction

Differential equation

$$\phi(x, x', x'', \dots) = y$$

Analyst's task

Find solutions, i.e. functions x satisfying the differential equation.

Topologist's way to study equations

- Look for spaces X, Y such that equation fits into map

$$\begin{aligned} \phi : X &\to Y \\ x &\mapsto \phi(x, x', x'', \dots) \end{aligned}$$

- Study map (X, Y)

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Example

Fundamental Theorem of Algebra

A polynomial equation $P(z) = 0$ has a complex solution $z \in \mathbb{C}$.

Topologists proof

- Consider Riemann sphere $S^2 = \mathbb{C} \cup \infty$
- Polynomial extends to $P \in \text{map}^*(S^2, S^2)$
- $\deg : \text{map}^*(S^2, S^2) \to \mathbb{Z}$ classifies components
- If $f : S^2 \to S^2$ is not surjective, then $\deg f = 0$.

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Equations from a Topologists View

pros and cons

Advantages

- Similar equations are dealt with in one go
- Focus on generic phenomena
- Equations can be deformed to easy solvable ones
- Good for (non-)existence proofs.

Disadvantages

- No explicit solutions
- No standard method to find X, Y realizing equation $\phi = y$.
- Understanding $\mathrm{map}(X, Y)$ harder than solving $\phi = y$

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What's the problem

Generic situation

- H, H' Sobolev spaces of sections of bundles over manifold
- $\phi : H' \to H$ a nonlinear partial differential equation
- solution sets are compact

Problem

- Find suitable spaces X and Y containing H' and H such that
  - ϕ extends to $\overline{\phi} : X \to Y$
  - $\mathrm{map}(X, Y)$ has non-trivial topology
- study $\mathrm{map}(X, Y)$, i.e. study the space of all such PDE's

Usual suspects

What about 1-point completions of Hilbert spaces?

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Stable Homotopy

spheres

Definition (Hilbert sphere)

- Given: Hilbert space H
- $S^H := H \cup \infty$
- Complements of bounded $B \subset H$ are neighborhoods of ∞.

Notation: $S^n = S^{\mathbb{R}^n}$

Fact (Hilbert sphere is sphere)

Homeomorphism:

$$\begin{aligned} S^H &\to \{\|x\| = 1\} \subset H \times \mathbb{R} \\ h &\mapsto \frac{1}{\|h\|^2 + 1}\left(2h, \|h\|^2 - 1\right) \end{aligned}$$

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Stable Homotopy

Extending maps to point at infinity

Suppose: H', H finite dimensional, $U' \subset H'$ open subset, $\phi : U' \to H$ continuous.

- ϕ extends to $\overline{\phi} : S^{H'} \to S^H$ with $\overline{\phi}\left(S^{H'} \setminus U'\right) = \infty$ if and only if ϕ is proper.
- Defines bijection of $\mathrm{map}^\bullet\left(S^{H'}, S^H\right)$ with set

$$\left\{\left(U', \phi\right) \mid U' \subset H' \text{ open}, \phi : U' \to H \text{ proper}\right\}$$

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Suspension

H', H, V Hilbert spaces, $U' \subset H'$ open subset

- $\phi : U' \to H$ proper $\Rightarrow \phi \times \mathrm{id}_V : U' \times V \to H \times V$ is proper.
- defines in finite dimensional case suspension

$$\begin{array}{rcl} \Sigma^V : \mathrm{map}^\bullet \left(S^{H'}, S^H \right) & \to & \mathrm{map}^\bullet \left(S^{H' \times V}, S^{H \times V} \right) \\ \overline{\phi} & \mapsto & \overline{\phi \times \mathrm{id}_V} \end{array}$$

Notation

$\Omega^n \Sigma^n \left(S^k \right) := \mathrm{map}^\bullet \left(S^n, S^{k+n} \right)$ with compact-open topology.

- suspension $\Sigma : \Omega^n \Sigma^n \left(S^k \right) \hookrightarrow \Omega^{n+1} \Sigma^{n+1} \left(S^k \right)$ embedding.
- topologists view $\Omega^n \Sigma^n \left(S^k \right)$ as dolled up version of sphere S^k.

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Theorem (Freudenthal suspension)

The suspension map $\Omega^n\Sigma^n\left(S^k\right) \hookrightarrow \Omega^{n+1}\Sigma^{n+1}\left(S^k\right)$ is $(n+2k-1)$-connected.

Definitions

- $\Omega^\infty\Sigma^\infty\left(S^k\right) = \lim_{n\to\infty}\Omega^n\Sigma^n\left(S^k\right)$
- The set of components $\pi_k^S = \pi_0\left(\Omega^\infty\Sigma^\infty\left(S^{-k}\right)\right)$ is abelian group, called the k-th stable stem

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Stable Homotopy

Fact

$\Omega^\infty \Sigma^\infty := \cup_k \Omega^\infty \Sigma^\infty \left(S^k\right)$ something like a ring:

- Addition: $\phi_i : U_i' \to H$ proper maps, then

$$\left(U_1', \phi_1\right) + \left(U_2', \phi_2\right) := \left(U_1' \sqcup U_2', \phi_1 \sqcup \phi_2\right)$$

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- $\Omega^\infty \Sigma^\infty$ is initial object of all cohomology theories, of fundamental importance to topology.

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How to generalize to ∞ Dimensions?

Aim

- Concepts should apply to differential equations, i.e. maps between function spaces.
- In particular, to Seiberg-Witten monopole map
- Well known fact: For K-oriented 4-manifold with $b_1 = 0$, one can associate to monopole map an element in a stable stem.

Problems

What happens for H', H infinite dimensional Hilbert spaces?

- Hilbert sphere S^H is contractible, hence also $\mathrm{map}^\bullet\left(S^{H'}, S^H\right)$.
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From Hilbert to Fréchet (and back)

Why abandon Hilbert space setup?

Solving PDEs involves jumping between Sobolev completions.

- Such jumps don't fit with Hilbert spheres
- Orthogonal structure not preserved by differential operators
- Results, however, can be stated purely in Hilbert space setting.

Setup

- F', F Fréchet spaces
- $l : F' \to F$ linear Fredholm operator
- ν, ν' continuous norms on F, F'

Main example

elliptic linear differential operator on C^∞-sections over manifold

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elliptic linear differential operator on $\mathcal{C}^\infty$-sections over manifold

Definition of set $Q(l)$

$$Q(l) = \left\{ \begin{array}{c} (U', \phi) \mid U' \subset F' \text{ open, } \phi : U' \to F \text{ continuous} \\ \text{satisfying (C) and (P)} \end{array} \right\}$$

Suppose $B \subset F$ is ν-bounded and closed, then

(C) the preimage $\phi^{-1}(B)$ is ν'-bounded and closed in F'

(P) $\overline{(\phi - l)(\phi^{-1}(B))} \subset F$ is compact.

Remarks

1. Condition (C) $\iff$ ϕ extends to 1-point completion
2. Condition (P) $\implies$ ϕ is proper

Differential Equations and Stable Homotopy

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Differential Equations and Stable Homotopy

Topology on $Q(l)$

The topology is generated by sets of the form

$$W_b(B', B) = \left\{ \phi \mid \overline{(\phi - l)(B')} \subset B^\circ \right\}$$
$$W_u(B', B) = \left\{ \phi \mid B'^\circ \supset \phi^{-1}(\overline{B}) \right\}$$

for ν'-bounded $B' \subset F'$ and ν-bounded $B \subset F$.

Remarks

Suppose F', F are Hilbert spaces.

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Differential Equations and Stable Homotopy

Theorem
$Q(l)$ is weakly homotopy equivalent to $\Omega^{\infty}\Sigma^{\infty}\left(S^{-\mathrm{ind} l}\right)$.

Corollary
A PDE in $Q(l)$ represents an element in the stable stem $\pi^{st}_{\mathrm{ind} l}$. If it is non-trivial, then the solution space is not empty.

Special aspects go back to ideas of A. Schwarz (1950's)

Outline

One-Point Completions

$Q(I)$ is mapping space, but of which spaces?

Definition

Thom space $Th_\nu(F) = F \cup \infty$. Neighborhoods of ∞ are complements of ν-bounded sets.

Fact

$\tilde{\nu}$ another continuous norm on F, then id_F extends continuously

$$Th_\nu(F) \to Th_{\tilde{\nu}}(F)$$

if and only if $\nu \leq \lambda\tilde{\nu}$ for some $\lambda > 0$.

Fact

F is metrizable, $Th_\nu(F)$ not.

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One-Point Completions

Sobolev norms on $\mathcal{C}^\infty$-sections

- ν_k Sobolev L^2_k-norm on $\Gamma(\xi)$ space of sections of bundle.
- identity on $\Gamma(\xi)$ extends to bijective, continuous, but not invertible maps

$$\ldots \longrightarrow Th_{\nu_k}(\Gamma(\xi)) \longrightarrow Th_{\nu_{k+1}}(\Gamma(\xi)) \longrightarrow \ldots.$$

- these do not extend to Sobolev completions: identity extends

$$\ldots \longrightarrow L^2_{k+1}(\Gamma(\xi)) \longrightarrow L^2_k(\Gamma(\xi)) \longrightarrow \ldots$$

Functorial Properties

$$Q(l) \subset \mathrm{map}^{\bullet}\left(Th_{\nu'}\left(F'\right), Th_{\nu}(F)\right)$$

as set, but not as topological space!

Theorem

The following maps are continuous:

1. *evaluation* $\mathrm{ev} : Q(l) \times Th_{\nu'}(F') \to Th_{\nu}(F)$ *and*
2. *composition* $Q(l) \times Q(l') \to Q(l \circ l'), (\Phi, \Phi') \mapsto \Phi \circ \Phi'$.

Pseudo-Inverse to l

Definition

A linear operator $m : F \to F'$ *is pseudo-inverse* to l if

- lm and ml are projections to finite co-dimensional subspaces
- such that

$$\dim(\ker(ml)) - \dim(\ker(lm)) = \mathrm{ind}(l).$$

Fact

The set of pseudo-inverses is a directed set.

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Thickened Graphs of I

Definition

For a pseudo-inverse m to I, the *m-thickened graph*

$$\Gamma_m(I) \subset Th_{\nu'}(F') \times Th_{\nu}(F)$$

of I is the closure of the set

$$\left\{ (f', f) \mid ImI(f') = Im(f) \right\}.$$

Definition

Subspace of $Q(I)$:

$$Q_m(I) = \{\Phi \mid \mathrm{graph}(\Phi) \subset \Gamma_m(I)\}$$

maps contained by m-thickened graph of I.

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Main Observation

For pseudo-inverse m to l, set $V = \ker lm$ and $V' = \ker ml$.

Theorem

The map

$$\begin{array}{rcl} j_m : map^{\bullet}\left(S^{V'}, S^{V}\right) & \rightarrow & Q_m(l) \\ f & \mapsto & \left(h' \mapsto lml\left(h'\right) + f\left(\left(1 - ml\right)h'\right)\right) \end{array}$$

indeed is a closed embedding. The image is a deformation retract.

Remainder of Proof

- constructions are sufficiently functorial
- induction over the set of all pseudo-inverses
- point set topology

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The space Q

In the Hilbert space situation $H = H'$, we can say more:

Definition of space Q

$$Q = \left\{ \begin{array}{c} (U', \phi, \overline{I}) \mid U' \subset F' \text{ open}, \ \phi : U' \to F \text{ continuous} \\ \text{satisfying (C) and (P)}, \overline{I} \text{ a unit in Calkin-algebra} \end{array} \right\}$$

Suppose $B \subset F$ is ν-bounded and closed, then

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(P) $\overline{(\phi - I)(\phi^{-1}(B))} \subset F$ is compact for some $I \in \overline{I}$.

Remarks

1. Calkin algebra is algebra of bounded linear endomorphisms of H modulo ideal of compact operators.
2. Space of units in Calkin algebra is $\mathbb{Z} \times BO$ (Atiyah-Jänich).
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The fibration for Q

Result

Q admits a fibration $Q \to \mathbb{Z} \times BO$.

- The fiber over the component $\{k\} \times BO$ is weakly homotopy equivalent to $\Omega^\infty \Sigma^\infty \left(S^{-k}\right)$.
- In particular, the total space over the component $\{1\} \times BO$ is the quotient G/O.
- Here $O \to G$ is the j-homomorphism, $G \subset \Omega^\infty \Sigma^\infty \left(S^0\right)$ the components of degree ± 1.