# *Extremal length geometry on Teichmüller space*

Hideki Miyachi

Department of Mathematics, Osaka University

August , 2012

Progress in Low-dimensional topology: Teichmüller theory and 3-manifold groups,
Centre for Quantum geometry of Moduli spaces, Science & Technology, Århus University

# Outline

# Geometry on Teichmüller space

The geometry of the Teichmüller space in terms of the Teichmüller distance is well-studied.

- (Teichmüller) Teichmüller space with the Teichmüller distance is complete and uniquely geodesic.
- (Royden) The Teichmüller distance is a Finsler distance, and coincides with the Kobayashi distance under the natural complex structure.

The geometry of the Teichmüller distance is also studied as metric space.

- (Masur) The Teichmüller space is not negatively curved in the sense of Busemann. Expecially, it is not a $\mathrm{CAT}(0)$-space.
- (Masur-Wolf, McCauthy-Papadopoulos, Ivanov) The Teichmüller space is not Gromov hyperbolic.

## Extremal length geometry after Kerckhoff, Gardiner and Masur

- S. Kerckhoff studies asymptotic geometry on extremal length of simple closed curves.
- Kerckhoff found an excellent formula on the Teichmüller distance, which tells us that the Teichmüller distance is represented by the ratio of the extremal length.
- “Extremal length geometry on the Teichmüller space” is nothing but the geometry on Teichmüller space via extremal length.

  However, from Kerckhoff’s formula, we often call the geometry on the Teichmuller distance the extremel length geometry.

## Extremal length geometry after Kerckhoff, Gardiner and Masur

- F. Gardiner and H. Masur formulated a "natural" compactification for the extremal length geometry of the Teichmüller space by assembling asymptotic behavior of the extremal lengths of simple closed curves, which we call the Gardiner-Masur compactification.

Our naive questions of this study are

(1) How natural?

(2) What can we know about the geometry of the Teichmüller distance from the compactification?

To answer these questions, we attempt to develop

"Thurston theory on extremal length geometry"

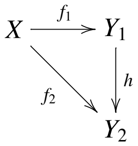
Namely, we will discuss extremal length geometry on Teichmüller space via intersection number.

# Teichmüller theory

Let $T_{g,m}$ be the Teichmüller space of Riemann surfaces of type $(g,m)$ with $2g-2+m>0$. Namely,

$$T_{g,m} = \{(Y,f) \mid f : X \to Y \text{ qc}\}/\sim$$

where $X$ is a base Riemann surface. Two marked Riemann surfaces $(Y_1,f_1)$ and $(Y_2,f_2)$ are equivalent if there is a conformal mapping $h : Y_1 \to Y_2$ such that $h \circ f_1$ is homotopic to $f_2$.




The Teichmüller distance is a distance on $T_{g,m}$ defined by

$$d_T((Y_1,f_1),(Y_2,f_2)) = \frac{1}{2}\log\inf\{K(h) \mid h \text{ qc homotipic to } f_2^{-1}\circ f_1\}$$

where $K(h)$ is the maximal dilatation of $h$.

# The space of measured foliations

Let $\mathcal{S}$ be the set of homotopy classes of non-trivial and non-peripheral simple closed curved on $X$. We consider the embedding

$$\mathbb{R}_+ \otimes \mathcal{S} \ni t\alpha \mapsto [\mathcal{S} \ni \beta \mapsto t\, i(\beta, \alpha)] \in \mathbb{R}_+^{\mathcal{S}},$$

where $\mathbb{R} = \{t \in \mathbb{R} \mid t \geq 0\}$. Then, the closure $\mathcal{MF} \subset \mathbb{R}_+^{\mathcal{S}}$ of the image is called the space of measured foliations. The quotient space

$$\mathcal{PMF} = (\mathcal{MF} - \{0\})/\mathbb{R}_+ \subset P\mathbb{R}^{\mathcal{S}} = (\mathbb{R}_+^{\mathcal{S}} - \{0\})/\mathbb{R}_{>0}$$

under the action $\mathbb{R}_{>0} \times \mathbb{R}_+^{\mathcal{S}} \ni (t, F) \to tF \in \mathbb{R}_+^{\mathcal{S}}$ is called the space of projective measured foliations.

It is known the following (Thurston).

- $\mathcal{MF}$ is homeomorphic to $\mathbb{R}^{6g-6+2m}$.
- $\mathcal{PMF}$ is homeomorphic to $S^{6g-7+2m}$.

# Extremal length

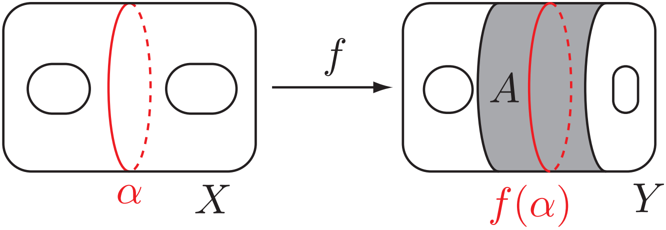
Let $\alpha \in \mathcal{S}$. The extremal length of $\alpha$ on $y = (Y, f) \in T_{g,m}$ is defined by

$$\mathrm{Ext}_y(\alpha) = \inf\{1/\mathrm{Mod}(A) \mid \text{the core of } A \text{ is homotopic to } f(\alpha)\}$$

where

$$\mathrm{Mod}(A) = \frac{1}{2\pi}\log r$$

when $A \cong \{1 < |z| < r\}$.



# Properties of extremal length and Kerckhoff's formula

- Kerckhoff showed that

$$\mathbb{R}_+ \otimes \mathcal{S} \ni t\alpha \mapsto \mathrm{Ext}_y(t\alpha) := t^2\mathrm{Ext}_y(\alpha)$$

extends continuously on $\mathcal{MF}$.

- The extremal length has the distortion property.

$$e^{-2d_T(y_1,y_2)}\mathrm{Ext}_{y_1}(F) \le \mathrm{Ext}_{y_2}(F) \le e^{2d_T(y_1,y_2)}\mathrm{Ext}_{y_1}(F)$$

for $F \in \mathcal{MF}$ and $y_1, y_2 \in T_{g,m}$.

**Theorem 1 (Kerckhoff's formula)**

*The Teichmüller distance is represented by the ratio of the extremal length:*

$$d_T(y_1, y_2) = \frac{1}{2}\log\sup_{\alpha\in\mathcal{S}}\frac{\mathrm{Ext}_{y_1}(\alpha)}{\mathrm{Ext}_{y_2}(\alpha)}.$$

# Gardiner-Masur compactification

Consider a mapping

$$\Phi_{GM} : T_{g,m} \ni y \mapsto [\mathcal{S} \ni \alpha \mapsto \mathrm{Ext}_y(\alpha)^{1/2}] \in P\mathbb{R}_+^{\mathcal{S}}.$$

Kerckhoff's formula asserts that $\Phi_{GM}$ is injective:
Suppose $\Phi_{GM}(y_1) = \Phi_{GM}(y_2)$. Then, there is a constant $c > 0$ such that $\mathrm{Ext}_{y_2}(\alpha) = c\mathrm{Ext}_{y_1}(\alpha)$ for all $\alpha \in \mathcal{S}$. Hence, we have

$$\begin{aligned}\frac{1}{2}\log c &= \frac{1}{2}\log\sup_{\alpha\in\mathcal{S}}\frac{\mathrm{Ext}_{y_2}(\alpha)}{\mathrm{Ext}_{y_1}(\alpha)} \\ &= d_T(y_2, y_1) = d_T(y_1, y_2) \\ &= \frac{1}{2}\log\sup_{\alpha\in\mathcal{S}}\frac{\mathrm{Ext}_{y_1}(\alpha)}{\mathrm{Ext}_{y_2}(\alpha)} = -\frac{1}{2}\log c,\end{aligned}$$

and hence $c = 1$. This means that $d_T(y_1, y_2) = 0$ and $y_1 = y_2$.

# Gardiner-Masur compactification

Gardiner and Masur showed that the closure of the image

$$\mathrm{cl}_{GM}(T_{g,m}) := \overline{\Phi_{GM}(T_{g,m})} \subset P\mathbb{R}_+^{\mathcal{S}}$$

is compact. This compactification of $T_{g,m}$ is called the Gardiner-Masur compactification. We call the boundary

$$\partial_{GM} T_{g,m} = \mathrm{cl}_{GM}(T_{g,m}) - \Phi_{GM}(T_{g,m})$$

the Gardiner-Masur boundary.

The following is a fundamental observation due to Gardiner and Masur.

- $\mathcal{PMF} \subset \partial_{GM} T_{g,m}$ as subsets of $P\mathbb{R}_+^{\mathcal{S}}$. If $3g-3+m \geq 2$, $\mathcal{PMF}$ is a proper subset of $\partial_{GM} T_{g,m}$.

# Extremal length geometry via intersection number

Consider cones

$$\begin{aligned} C_{GM} &= \mathrm{proj}^{-1}(\mathrm{cl}_{GM}(T_{g,m})) \cup \{0\} \subset \mathbb{R}_+^{\mathcal{S}} \\ \mathcal{T}_{GM} &= \mathrm{proj}^{-1}(\Phi_{GM}(T_{g,m})) \cup \{0\} \subset \mathbb{R}_+^{\mathcal{S}} \\ \tilde{\partial}_{GM} &= \mathrm{proj}^{-1}(\partial_{GM} T_{g,m}) \cup \{0\} \subset \mathbb{R}_+^{\mathcal{S}} \end{aligned}$$

where $\mathrm{proj}\colon \mathbb{R}_+^{\mathcal{S}} - \{0\} \to P\mathbb{R}_+^{\mathcal{S}}$ is the projection. Notice that $\mathcal{MF} \subset \tilde{\partial}_{GM}$ since $\mathcal{PMF} \subset \partial_{GM} T_{g,m}$.

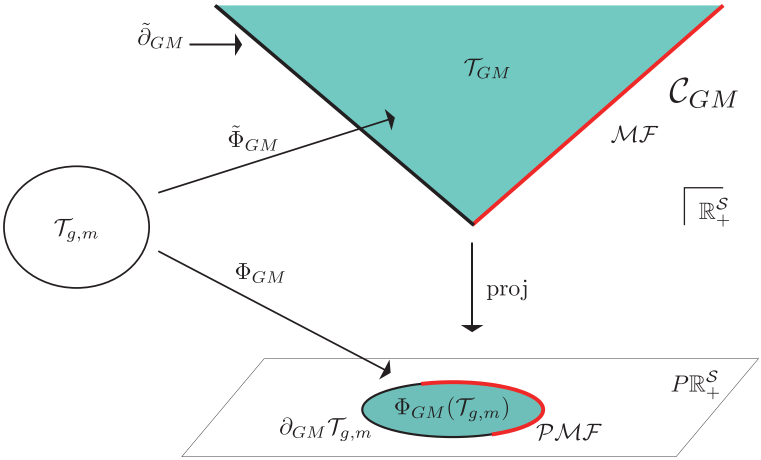
The Gardiner-Masur embedding

$$\Phi_{GM}\colon T_{g,m} \ni y \mapsto [\mathcal{S} \ni \alpha \mapsto \mathrm{Ext}_y(\alpha)^{1/2}] \in \Phi_{GM}(T_{g,m}) \subset P\mathbb{R}_+^{\mathcal{S}}$$

has a canonical lift

$$\tilde{\Phi}_{GM}\colon T_{g,m} \ni y \mapsto [\mathcal{S} \ni \alpha \mapsto \mathrm{Ext}_y(\alpha)^{1/2}] \in C_{GM} \subset \mathbb{R}_+^{\mathcal{S}}.$$

# Cones


$\tilde{\partial}_{GM}$
$\mathcal{T}_{GM}$
$\mathcal{C}_{GM}$
$\mathcal{MF}$
$\tilde{\Phi}_{GM}$
$\mathcal{T}_{g,m}$
$\mathbb{R}_+^{\mathcal{S}}$
$\Phi_{GM}$
proj
$P\mathbb{R}_+^{\mathcal{S}}$
$\Phi_{GM}(\mathcal{T}_{g,m})$
$\partial_{GM}\mathcal{T}_{g,m}$
$\mathcal{PMF}$

## Theorem 2 (Unification of extremal length geometry)

*There is a unique continuous function*

$$i(\cdot,\cdot)\colon \mathcal{C}_{GM}\times\mathcal{C}_{GM}\to\mathbb{R}_+$$

*with the following properties.*

(i) *For any* $y\in T_{g,m}$ *and* $\alpha\in\mathcal{S}$,

$$i(\tilde{\Phi}_{GM}(y),\alpha)=\mathrm{Ext}_y(\alpha)^{1/2}\quad \text{for all } \alpha\in\mathcal{S}.$$

(ii) *For* $\mathfrak{a},\mathfrak{b}\in\mathcal{C}_{GM}$ *and* $t,s\geq 0$, $i(t\mathfrak{a},s\mathfrak{b})=ts\,i(\mathfrak{b},\mathfrak{a})$.

(iii) *For any* $y,z\in T_{g,m}$,

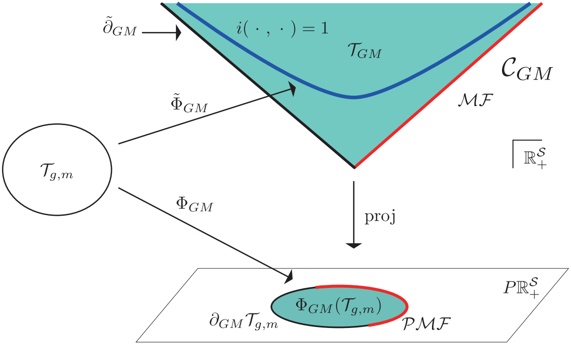
$$i(\tilde{\Phi}_{GM}(y),\tilde{\Phi}_{GM}(z))=\exp(d_T(y,z)).$$

*In particular, we have* $i(\tilde{\Phi}_{GM}(y),\tilde{\Phi}_{GM}(y))=1$ *for* $y\in T_{g,m}$.

(iv) *For* $F,G\in\mathcal{MF}\subset\mathcal{C}_{GM}$, *the value* $i(F,G)$ *is equal to the (original) geometric intersection number* $F$ *and* $G$

# Cones and Hyperbolid model

$$\exists !\, i(\,\cdot\,,\,\cdot\,)\colon C_{GM} \times C_{GM} \to \mathbb{R}_+$$




From this picture, our hyperboloid model might be expected to be a kind of a counter part of the Bonahon's realization of Teichmüller space into the space of geodesic currents.

Fix $x_0 \in T_{g,m}$. We define a mapping

$$\tilde{\Psi}_{GM}\colon T_{g,m} \ni y \mapsto [\mathcal{S} \ni \alpha \mapsto \mathcal{E}_y(\alpha)] \in C_{GM} \subset \mathbb{R}_+^{\mathcal{S}}$$

where $K_y = \exp(2d_T(x_0, y))$ and $\mathcal{E}_y(F) = (\mathrm{Ext}_y(F)/K_y)^{1/2}$. By definition,

$$\mathrm{proj} \circ \tilde{\Psi}_{GM} = \Phi_{GM} \quad \text{on } T_{g,m}.$$

**Theorem 3 (Embedding with basepoint)**

*The embedding $\tilde{\Psi}_{GM}$ extends homeomorphically to $\mathrm{cl}_{GM}(T_{g,m})$ onto its image.*

Notice that

$$\begin{aligned} i(\tilde{\Psi}_{GM}(y_1), \tilde{\Psi}_{GM}(y_2)) &= i\left(K_{y_1}^{-1/2}\tilde{\Phi}_{GM}(y_1), K_{y_2}^{-1/2}\tilde{\Phi}_{GM}(y_2)\right) \\ &= \exp(-d_T(x_0, y_1) - d_T(x_0, y_2) + d_T(y_1, y_2)) \\ &= \exp(-2\langle y_1 \,|\, y_2\rangle_{x_0}). \end{aligned}$$

where

$$\langle y_1 \,|\, y_2\rangle_{x_0} = \frac{1}{2}(d_T(x_0, y_1) + d_T(x_0, y_2) - d_T(y_1, y_2))$$

is the Gromov product.

Since $\tilde{\Psi}_{GM}$ admits a continuous extension to $\mathrm{cl}_{GM}(T_{g,m})$, we have the following.

**Corollary 4 (Extension of the Gromov product)**

*For any $x_0 \in T_{g,m}$, the Gromov product*

$$\langle \cdot \mid \cdot \rangle_{x_0} : T_{g,m} \times T_{g,m} \to \mathbb{R}_+$$

*extends continuously to $\mathrm{cl}_{GM}(T_{g,m}) \times \mathrm{cl}_{GM}(T_{g,m})$ with values in $\mathbb{R}_+ \cup \{\infty\}$.*

Moreover, we can see that

$$\exp(-2\langle [F] \mid [G] \rangle_{x_0}) = \frac{i(F,G)}{\mathrm{Ext}_{x_0}(F)^{1/2}\mathrm{Ext}_{x_0}(G)^{1/2}}$$

where $[F], [G] \in \mathcal{PMF} \subset \partial_{GM} T_{g,m}$.

# Sketch of the proof

A rough sketch of the proof is as follows:

(1) Construct systems of "nice neighborhoods" of points of $C_{GM}$ to show that the family $\{\mathcal{E}_y\}_{y \in T_{g,m}}$ is equicontinuous on $\mathcal{MF}$, where we recall

$$\mathcal{E}_y(F) = \left\{ \frac{\mathrm{Ext}_y(F)}{K_y} \right\}^{1/2} = e^{-d_T(x_0,y)}\mathrm{Ext}_y(F)^{1/2}.$$

(2) Then, the family $\{\mathcal{E}_y\}_{y \in T_{g,m}}$ is a normal family as a family of continuous functions on $\mathcal{MF}$. Hence, we have a family $\{\mathcal{E}_p \mid p \in \mathrm{cl}_{GM}(T_{g,m})\}$ of continuous functions. We can see that

$$\tilde{\Psi}_{GM} \colon \mathrm{cl}_{GM}(T_{g,m}) \ni p \mapsto [\mathcal{S} \ni \alpha \mapsto \mathcal{E}_p(\alpha)] \in C_{GM} \subset \mathbb{R}_+^{\mathcal{S}}$$

is continuous, which implies Theorem 3.

Notice that $\mathrm{proj} \circ \tilde{\Psi}_{GM} = \Phi_{GM} \colon \mathrm{cl}_{GM}(T_{g,m}) \to \mathrm{cl}_{GM}(T_{g,m})$ is a homeomorphism. Hence, any element in $\mathcal{C}_{GM}$ is written as $t \cdot \tilde{\Psi}_{GM}(p)$ with $t \geq 0$ and $p \in \mathrm{cl}_{GM}(T_{g,m})$.

(3) We define the intersection number on $\mathcal{C}_{GM} \times \mathcal{MF}$ by

$$i(t \cdot \tilde{\Psi}_{GM}(p), F) = t\mathcal{E}_p(F).$$

We can see that the intersection number coincides with the original intersection number on $\mathcal{MF} \times \mathcal{MF}$ .

(4) We define the extremal length of $\mathfrak{a} \in \mathcal{C}_{GM}$ on $\mathfrak{y} \in \mathcal{T}_{GM}$ by

$$\mathrm{Ext}_{\mathfrak{y}}(\mathfrak{a}) = t^2 \sup_{F \in \mathcal{MF}} \frac{i(\mathfrak{a}, F)}{\mathrm{Ext}_y(F)^{1/2}}$$

where $\mathfrak{y} = t\tilde{\Psi}_{GM}(y)$ for $t \geq 0$ and $y \in T_{g,m}$. Notice from Minsky's inequality and Gardiner-Masur's work, when $\mathfrak{a} \in \mathcal{MF}$, the above extremal length coincides with the original extremal length.

(5) We re-define intersection number on $\mathcal{T}_{GM} \times \mathcal{C}_{GM}$ and a function on $\mathcal{C}_{GM}$ by

$$i(\mathfrak{y}, \mathfrak{a}) = \mathcal{E}_{\mathfrak{y}}(\mathfrak{a}) = \left\{ \frac{\mathrm{Ext}_{\mathfrak{y}}(\mathfrak{a})}{K_y} \right\}^{1/2}$$

where $\mathfrak{y} = t\tilde{\Psi}_{GM}(y)$ ($t \geq 0$ and $y \in T_{g,m}$).

(6) We can check that

$$i(\tilde{\Psi}_{GM}(y), \tilde{\Psi}_{GM}(z)) = \exp(-2\langle y \,|\, z \rangle_{x_0})$$

for $y, z \in T_{g,m}$, which implies our intersection number on $\mathcal{T}_{GM} \times \mathcal{T}_{GM}$ is symmetric.

(7) As before, we construct 'nice neighborhoods" for points in $\mathcal{C}_{GM}$ to show that for any fixed $R > 0$, the family

$$\{\mathcal{E}_{\mathfrak{y}} \mid \mathfrak{y} = t\tilde{\Psi}_{GM}(y) \text{ for } y \in T_{g,m},\ 0 \leq t \leq R\}$$

is a normal family as a family of continuous functions on $\mathcal{C}_{GM}$, and we can see that the intersection number defined in (5) extends on $\mathcal{C}_{GM} \times \mathcal{C}_{GM}$ with desired properties.

# Characterization of isometries

As an application, we can see the following.

**Theorem 5 (Royden, Earle, Kra, Ivanov, Markovic)**

*When $(g,m) \neq (1,2)$, any isometry on $(T_{g,m}, d_T)$ is induced from a homeomorphism on the base surface. Furthermore, when the isometry is holomorphic, the corresponding homeomorphism is orientation-preserving.*

The latter part of the theorem in our proof is due to a joint work with R. Mineyama.

Our proof is somewhat modeled on the Ivanov's proof. However, since we use the boundary of Teichmüller space, we can obtain a rigidity result of more coarser mappings than isometries to obtain the characterization.

Our proof of the characterization of holomorphic automorphisms is rather technical. We omit the detail here.

The idea is as follows :

1. We consider the Maskit embedding of Teichmüller space. By applying H. Shiga and H. Tanigawa's argument, we can analyze the behavior of geodesic rays in the Gardiner-Masur compactification to show that the restriction of the induced homeomorphism to any subsurface of type $(1,1)$ or $(0,4)$ is orientation-preserving.
2. We also apply F. Luo's characterization for orientability to show that the induced homeomorphism is orientation-preserving.

# Rigidity theorem

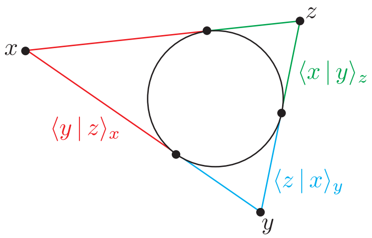
A mapping $\omega\colon (M, d_M) \to (M, d_M)$ is called a mapping of bounded distortion for triangles if there are constants $D_1, D_2 > 0$ such that

$$\frac{1}{D_1}\langle x \,|\, y\rangle_z - D_2 \leq \langle \omega(x) \,|\, \omega(y)\rangle_{\omega(z)} \leq D_1\langle x \,|\, y\rangle_z + D_2 \quad \text{for } x, y, z \in M$$

A mapping $\omega'\colon (M, d_M) \to (M, d_M)$ is said to be a quasi-inverse of $\omega$ if

$$\sup_{x\in M}\{d_M(x, \omega \circ \omega'(x)), d_M(x, \omega' \circ \omega(x)), \} \leq D_3$$

for some $D_3 > 0$.



## Theorem 6 (Rigidity)

*Let $\omega\colon T_{g,m} \to T_{g,m}$ be a mapping of bounded distortion for triangles. Suppose that*

- *$\omega$ admits a quasi-inverse $\omega'$.*
- *Both $\omega$ and $\omega'$ extend continuously to $\mathrm{cl}_{GM}(T_{g,m})$.*

*Then, $\omega$ preserves $\mathcal{S} \subset \mathcal{PMF} \subset \partial_{GM}T_{g,m}$ and induces an automorphism on the complex of curves of the base surface $X$.*

Notice that

- (L. Liu and W. Su) Any isometry on $T_{g,m}$ extends homeomorphically to $\mathrm{cl}_{GM}(T_{g,m})$.
- (Clear) Any isometry on $T_{g,m}$ is a mapping of bounded distortion for triangles with a quasi-inverse.

Hence, from the above theorem, any isometry on $T_{g,m}$ induces an automorphism on the complex of curves, and hence induces a homeomorphism on the base surface if $(g, m) \neq (1, 2)$.

# Action of $\mathrm{MCG}^*(X)$

We assume that $X$ has an anti-conformal involution $j_0$. Then, $h \in \mathrm{MCG}^*(X)$ acts on $T_{g,m}$ by

$$h_*(Y, f) = \begin{cases} (Y, f \circ h^{-1}) & (h \text{ is ori. pres.}) \\ (Y^*, r_Y \circ f \circ j_0 \circ \psi_h^{-1}) & (h \text{ is ori. rev.}) \end{cases}$$

where

- $Y^*$ is the mirror image of $Y$,
- $r_Y : Y \to Y^*$ is the anti-conformal mapping induced by the identity mapping on the underlying surface of $Y$, and
- $\psi_h : X \to X$ is an orientation preserving homeomorphism satisfying $h = \psi_h \circ j_0$ (in homotopy sense).

We can see

- The action of $h \in \mathrm{MCG}^*(X)$ is isometric on $(T_{g,m}, d_T)$.
- The extension of the action of $h \in \mathrm{MCG}^*(X)$ to $\partial_{GM} T_{g,m}$ preserves $\mathcal{S}$, and it coincides with $\alpha \mapsto h(\alpha)$ on $\mathcal{S}$.

# Characterization of isometries from Rigidity theorem

Let $\omega \in \mathrm{Isom}(T_{g,m}, d_T)$. A difficult part in the proof is to show the surjectivity of the canonical homomorphism $\mathrm{MCG}^*(X) \to \mathrm{Isom}(T_{g,m}, d_T)$.

(1) From Rigidity theorem, $\omega$ induces an automorphism of the complex of curves of $X$.

(2) From the topological assumption of $X$, such automorphism is induced from a homeomorphism $h$ on $X$ (Ivanov, Korkmaz, Luo).

(3) From the density of $\mathcal{S}$ in $\mathcal{PMF}$, $\omega \circ h_*^{-1}$ is the identity on $\mathcal{PMF} \subset \partial_{GM} T_{g,m}$.

(4) Then, $\omega \circ h_*^{-1}$ is the identity on $T_{g,m}$ and hence $\omega = h_*$.

**Remark 1**

The proof (especially, of step (4)) is modeled on the Ivanov's proof. However, the situations are different. In fact, he discusses with the "exponential maps", while we discuss with the GM compactification.

# Sketch of the proof of Rigidity theorem

Define the null space of $\mathfrak{a} \in \mathcal{C}_{GM}$ by

$$\mathcal{N}(\mathfrak{a}) = \{\mathfrak{b} \in \mathcal{C}_{GM} \mid i(\mathfrak{a}, \mathfrak{b}) = 0\}.$$

We first notice the following characterization via null set.

**Proposition 4.1 (Characterization of points in $\mathcal{PMF}_0$ via $\mathcal{MF}$)**

*For $[G] \in \mathcal{PMF}$, the following hold.*

- *(Ivanov)* $\mathcal{N}(G) \cap \mathcal{MF}$ *is of codimension* $1$ *if and only if* $[G] \in \mathcal{S}$.
- *(By definition)* $\mathcal{N}(G) \cap \mathcal{MF}$ *is of dimension* $1$ *if and only if* $[G] \in \mathcal{PMF}^{UE}$. *In this case,* $\mathcal{N}(G) \cap \mathcal{MF} = \{tG \mid t \geq 0\}$.

# Basic property of the intersection number

Our intersection number satisfies the following expected property.

**Proposition 4.2 (Basic property)**

- *For* $\mathfrak{a} \in C_{GM}$, $\mathcal{N}(\mathfrak{a}) = \{0\}$ *if and only if* $\mathfrak{a} \in \mathcal{T}_{GM}$.
- *If* $\mathfrak{a} \in \tilde{\partial}_{GM}$, $\mathfrak{a} \in \mathcal{N}(\mathfrak{a})$. *Namely,* $i(\mathfrak{a}, \mathfrak{a}) = 0$ *for* $\mathfrak{a} \in \tilde{\partial}_{GM}$.
- *For any* $\mathfrak{a} \in \tilde{\partial}_{GM}$, $\mathcal{N}(\mathfrak{a}) \cap \mathcal{MF} \neq \{0\}$.

**Proposition 4.3 (Characterization of u.e. in $C_{GM}$)**

*For* $\mathfrak{a} \in C_{GM} - \{0\}$, *the following are equivalent.*

(1) *There is a* $\mathfrak{b} \in C_{GM} - \{0\}$ *with* $\mathcal{N}(\mathfrak{a}) = \{t\mathfrak{b} \mid t \geq 0\}$.

(2) $\mathcal{N}(\mathfrak{a}) = \{t\mathfrak{a} \mid t \geq 0\}$

(3) $\mathfrak{a} \in \mathcal{MF}$ *and* $\mathfrak{a}$ *is a uniquely ergodic measured foliation.*

(4) $\mathcal{N}(\mathfrak{a})$ *contains a uniquely ergodic measured foliation.*

# Sketch of the proof of Rigidity theorem

We identify $\mathrm{cl}_{GM}(T_{g,m})$ with the image under $\tilde{\Psi}_{GM}$. Extends $\omega$ on $C_{GM}$ by $\omega(t\mathfrak{a}) = t\omega(\mathfrak{a})$ ($\mathfrak{a} \in \tilde{\Psi}_{GM}(\mathrm{cl}_{GM}(T_{g,m}))$). From the assumption,

$$\frac{1}{D_1}\langle x \,|\, y\rangle_z - D_2 \le \langle \omega(x) \,|\, \omega(y)\rangle_{\omega(z)} \le D_1\langle x \,|\, y\rangle_z + D_2$$

$$\sup_{x \in T_{g,m}} \{d_T(x, \omega \circ \omega'(x)), d_T(x, \omega' \circ \omega(x))\} \le D_3$$

for all $x, y, z \in T_{g,m}$. Notice that $i(\tilde{\Psi}_{GM}(y), \tilde{\Psi}_{GM}(z)) = \exp(-2\langle y \,|\, z\rangle_{x_0})$.

(1) We can see

$$e^{-2D_2} J(p,q)\, i(p,q)^{D_1} \le i(\omega(p), \omega(q)) \le e^{2D_2} J(p,q)\, i(p,q)^{1/D_1}$$

and

$$e^{-2D_3}\, i(p,q) \le i(p, \omega \circ \omega'(q)) \le e^{2D_3}\, i(p,q)$$

for $p, q \in \mathrm{cl}_{GM}(T_{g,m})$ with $J(p,q) \neq 0$.

(2) Applying the inequalities above, $\omega \circ \omega'(\mathcal{N}(\mathfrak{a})) \subset \mathcal{N}(\mathfrak{a})$

From (2) before, we have

$$\omega \circ \omega'(\mathcal{N}(\mathfrak{a})) \subset \mathcal{N}(\mathfrak{a}).$$

(3) Let $G$ be u.e. Then, $\mathcal{N}(G) = \{tG \mid t \geq 0\}$. Therefore,

$$\omega \circ \omega'(G) \in \omega \circ \omega'(\mathcal{N}(G)) = \mathcal{N}(G) = \{tG \mid t \geq 0\},$$

and hence, $\omega \circ \omega'([G]) = [G]$ for $[G] \in \mathcal{PMF}^{UE}$.

From the density of u.e. foliaitons in $\mathcal{MF}$ (Consider the stable foliations of p.A's), we have

$$\omega \circ \omega'([G]) = \omega' \circ \omega([G]) = [G]$$

for all $[G] \in \mathcal{PMF}$.

In particular, $\mathcal{MF}$ are contained in both $\omega(\tilde{\partial}_{GM})$ and $\omega'(\tilde{\partial}_{GM})$.

From (1) above, we have

$$e^{-2D_2} J(p,q)\, i(p,q)^{2D_1} \leq i(\omega(p),\omega(q)) \leq e^{2D_2} J(p,q)\, i(p,q)^{2/D_1}.$$

$$\mathcal{MF} \subset \omega(\tilde{\partial}_{GM}), \quad \mathcal{MF} \subset \omega'(\tilde{\partial}_{GM}).$$

(4) We deduce $\omega(\mathcal{PMF}) = \omega'(\mathcal{PMF}) = \mathcal{PMF}$. Indeed, let $G \in \mathcal{MF}$ be u.e., and take $F \in \mathcal{N}(\omega(G)) \cap \mathcal{MF}$. Let $\mathfrak{a} \in \tilde{\partial}_{GM}$ with $F = \omega(\mathfrak{a})$. Then,

$$i(\omega(\mathfrak{a}), \omega(G)) = i(F, \omega(G)) = 0.$$

Hence $i(\mathfrak{a}, G) = 0$ and $\mathfrak{a} = tG$ for some $t > 0$.

This means that $\omega(G) = t^{-1}F \in \mathcal{MF}$ and $\omega(\mathcal{PMF}) \subset \mathcal{PMF}$. Hence $\omega$ is a homeomorphism on $\mathcal{PMF}$.

Notice from (1) that

$$e^{-2D_2} J(p,q)\, i(p,q)^{D_1} \leq i(\omega(p),\omega(q)) \leq e^{2D_2} J(p,q)\, i(p,q)^{1/D_1}$$

and

$$e^{-2D_3}\, i(p,q) \leq i(p,\omega\circ\omega'(q)) \leq e^{2D_3}\, i(p,q)$$

for $p,q\in \mathrm{cl}_{GM}(T_{g,m})$ with $J(p,q)\neq 0$.

(5) $\omega$ preserves the null spaces in $\mathcal{MF}$ and $\omega(\mathcal{S})=\mathcal{S}$ from the characterization of simple closed curves via the codimension of the null spaces.

(6) By applying the inequalities above again, we have $i(\omega(\alpha),\omega(\beta))=0$ if $i(\alpha,\beta)=0$ for $\alpha,\beta\in\mathcal{S}$ and $\omega$ induces an automorphism of the complex of curves.

However, one of the most important problem is still open .......

**Problem 1 (Geometric characterization)**

What do geometric objects correspond to points in the Gardiner-Masur boundary?

Needless to say, in the case of the Thurston compactification, points in the boundary correspond to projective measured foliations. Geometric objects, projective measured foliations, are very useful and powerful tools.

All suggestions and comments are welcome !

Thank you for your attention (^o^)

Tak til alle ! - for alt hvad I har gjort for os her i Århus.