A Weil-Petersson metric on the Hitchin component

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Background

Let Γ be a torsion-free Gromov hyperbolic group and $\rho : \Gamma \to SL(m, \mathbb{R})$ a representation.
For $\gamma \in \Gamma$, we define its length by $L_\gamma(\rho) = \log(\Lambda(\rho(\gamma)))$ where $\Lambda(A)$ is the spectral radius of A.
Where defined, we let the *entropy* h_ρ of ρ be given by

$$R_T(\rho) := \{[\gamma] \mid L_\gamma(\rho) < T\} \qquad h_\rho := \lim_{T\to\infty} \frac{\log(\#R_T(\rho))}{T}$$

Then the *intersection* number of two representations is defined to be

$$I(\rho_0, \rho_1) := \lim_{T\to\infty} \left(\frac{1}{\#R_T(\rho_0)} \sum_{[\gamma]\in R_T(\rho_0)} \frac{L_\gamma(\rho_1)}{L_\gamma(\rho_0))} \right).$$

We also define the *renormalised intersection* as

$$J(\rho_0, \rho_1) := \frac{h_{\rho_1}}{h_{\rho_0}}.I(\rho_0, \rho_1).$$

Statements

Let $C(\Gamma, m)$ be the space of convex irreducible representations (definition later). This is an analytic variety.

Theorem

The entropy, and intersection are well-defined positive analytic functions on $C(\Gamma, m)$ invariant under $Out(\Gamma)$. Also $J(\rho_1, \rho_2) \geq 1$, and if $\alpha : [0,1] \to C(\Gamma, m)$ analytic, with $J_{\alpha,s}(t) = J(\alpha(s), \alpha(t))$ then

$$J'_{\alpha,s}(s) = 0 \qquad J''_{\alpha,s}(s) \geq 0$$

We define

$$L(\alpha) := \int_0^1 \sqrt{J''_{\alpha,s}(s)}ds$$

Statements

Theorem

The pseudo path metric above is a path metric on $C(\Gamma, m)$ and is invariant under $Out(\Gamma)$. Furthermore it is a Riemannian metric on the submanifold of generic representations.

Theorem

The above metric for the case $\Gamma = \pi_1(S)$ defines a mapping class group invariant Riemannian metric on the Hitchin component that is an extension of the Weil-Petersson metric on the Fuchsian representations.

Statements

Corollary

The deformation space of convex cocompact representations into $PSL(2,\mathbb{C})$ admits a path metric which is Riemannian outside the Fuchsian locus and restricts to Weil-Petersson on the Fuchsian locus.

The case of quasifuchsian space $QF(S)$ was done in an earlier paper.

Theorem

For Γ a finitely generated group and G rank 1 semi-simple, then the Hausdorff dimension of the limit set varies analytically

The quasifuchsian and Schottky case is due to Bowen-Ruelle, and generalized to function groups by Anderson-Rocha.

Outline

1. Describe the Pressure metric for metric Anosov flows.
2. Describe Convex Anosov Representations
3. Define geodesic flow $U\rho\Gamma$ for convex Anosov representation.
4. State it is a Hölder reparametrization of geodesic flow for group and it is a metric Anosov flow.
5. Describe thermodynamic mapping for representations and pullback of Pressure metric.
6. Equation describing degeneracy of metric.
7. Sketch proof that generic reps are non-degenerate points

Thermodynamics

Let X be a compact metric space with a Hölder continuous flow ϕ_t generating vector field $\frac{\partial}{\partial t}$.

Let O be set of periodic orbits for ϕ. For $a \in O$ let l_a be its length. Then l_a is the total mass of lebesgue measure δ_a on a.

Let $f \in C^h(X, \mathbb{R}_+)$ (positive, Hölder continuous). The reparametrization of ϕ by f is the flow ϕ^f generated by $f\frac{\partial}{\partial t}$. For flow ϕ^f the length of $a \in O$ is $l_a(f) = \int_X f\delta_a$.

If $f, g \in C^h(X, \mathbb{R})$ then $f \sim g$ (Livsic cohomologous) if there exists a function V such that

$$f(x) - g(x) = \left.\frac{\partial}{\partial t}\right|_{t=0} V(\phi_t(x))$$

Note if $f \sim g$ then $l_a(f) = l_a(g)$ for all $a \in O$.

Thermodynamics: Entropy

We define

$$R_T(f) = \{a \mid l_a(f) \leq T\}$$

and (if limit exists), we define the entropy to be

$$h_f = \lim_{T\to\infty} \frac{\log(\#R_T(f))}{T}$$

Let M_ϕ be the set of $\phi-$invariant probability measures on X.

Then we have a bijection from M_ϕ to M_{ϕ^f} by the map

$$\mu \to \frac{f\mu}{\int f\mu} = \widehat{f\mu}$$

Thermodynamics: Pressure

The pressure function is defined by

$$P(\phi, f) = \sup_{m \in M_\phi} \left(h(\phi, m) + \int_X f \ dm \right)$$

where $h(\phi, m)$ is the metric entropy of ϕ with respect to m.

If m gives equality to above then m is an *equilibrium state* for f.

For $f = 0$ an equilibrium state is called a *probability of maximal entropy*.

Thermodynamics: Metric Anosov Flows

A flow (X, ϕ) is *metric Anosov*, if the tangent bundle decomposes as a sum of ϕ invariant bundles

$$TX = E_+ \oplus E_0 \oplus E_-$$

such that E_0 is the direction of the flow, E_+ expands under the flow, and E_- contracts under the flow.

We assume from now on that ϕ is metric Anosov flow.

Theorem

(Livsic) Let $f \in C^h(X, \mathbb{R})$. Then $f \sim 0$ if and only if $l_a(f) = 0$ for all $a \in O$.

Thermodynamics: Uniqueness of equilibrium states

Theorem

(Bowen) There is a unique measure of maximal entropy μ_ϕ. It satisfies

$$\mu_\phi = \lim_{t\to\infty}\left(\frac{1}{\#R_T(1)}\sum_{a\in R_T(1)}\frac{\delta_a}{l_a}\right)$$

Furthermore if $f \in C^h(X,\mathbb{R}_+)$ then $0 < h_f < \infty$.

Theorem

(Bowen-Ruelle) Every $g \in C^h(X,\mathbb{R})$ has a unique equiibrium state m_g. If $m_f = m_g$ then $f - g$ is cohomologous to a constant. Also the Pressure function satisfies

$$P(\phi, g) = \lim_{T\to\infty}\frac{1}{T}\log\left(\sum_{a\in R_T(1)} e^{l_a(f)}\right)$$

Thermodynamics: Derivatives of Pressure

Theorem

(Ratner-Ruelle)

1) The function $t \to P(f+tg)$ *is analytic.*

2) $$P'(f)(g) := \left.\frac{dP(f+tg)}{dt}\right|_{t=0} = \int g \, dm_f$$

3) If $P'(f)(g) = 0$ *then*

$$P''(f)(g) := \left.\frac{d^2P(f+tg)}{dt^2}\right|_{t=0} = Var(g, m_f)$$

where $$Var(g,m) = \lim_{T\to\infty} \frac{1}{T} \int \left(\int_0^T g(\phi_s(x))ds \right)^2 \, dm$$

4) If $Var(g, m_f) = 0$ *then* $g \sim 0$.

Thermodynamics: Pressure Metric

Consider space of pressure zero functions

$$\mathcal{P}(X) = \{f \in C^h(X) \mid P(f) = 0\}/\sim$$

Then its tangent space is naturally identified as

$$T_{[f]}\mathcal{P}(X) = \{g \in C^h(X) \mid \int g\ dm_f = 0\}/\sim$$

The pressure metric $||.||$ is then defined by

$$||[g]||^2_{[f]} = -\frac{Var(g, m_f)}{\int f\ dm_f}.$$

By above, it is a positive definite metric on $\mathcal{P}(X)$.

Convex Anosov Representations: Convex Representations

A representation of $\rho : \Gamma \to SL(m, \mathbb{R})$ is *convex* if there exist ρ-equivariant Hölder maps ξ and ξ^* from $\partial_\infty\Gamma$ to $\mathbb{RP}^m$ and $\mathbb{RP}^{m*}$(= space of planes in $\mathbb{R}^m$) respectively, called *limit curves*, so that for $x, y \in \partial_\infty\Gamma, x \neq y$ we have

$$\xi(x) \oplus \xi^*(y) = \mathbb{R}^m.$$

Convex representations where introduced by Sambarino in his thesis.

They generalize hyperconvex representations which Labourie, Guichard show is equivalent to being a Hitchin representation.

A convex representation is Anosov if a certain associated bundle is contracting.

Note, a bundle E over a compact space X contracts under flow ϕ if for any metric $||.||$ on E, there is a $t_0 > 0$ such that $||\phi_{t_0}(v)|| \leq \frac{1}{2}||v||$ for all v.

Convex Anosov Representations: Hopf Parametrization

Let $\Gamma = \pi_1(M)$ where M is a n-hyperbolic manifold.

If $v \in T_p\mathbb{H}^n$ we parametrize by $(v_{-\infty}, v_{+\infty}, B_{v_\infty}(p, o))$ where $v_{\pm\infty}$ are the endpoints of the geodesic, $B_\xi(x, y)$ is the Horocyclic distance between x, y measured from ξ.

We identify $TM = (\mathbb{S}^{n-1}_\infty \times \mathbb{S}^{n-1}_\infty - \text{Diagonal}) \times \mathbb{R}$.

We can further identify $\partial_\infty\Gamma = \mathbb{S}^{n-1}_\infty$ and have

$$TM = (\partial_\infty\Gamma^{(2)} \times \mathbb{R})/\Gamma.$$

The geodesic flow is $\phi_t(x, y, s) = (x, y, s + t)$ and the action of Γ is

$$\gamma(x, y, t) = (\gamma.x, \gamma.y, t - B_y(o, \gamma^{-1}(o))$$

Same works for for $\Gamma = \pi_1(M)$, M negatively curved.

Convex Anosov Representations: Gromov Geodesic Flow

Gromov introduced a geodesic flow for a hyperbolic group which generalizes Hopf parametrizaton.

He describes a proper, cocompact action of Γ on $\widetilde{U_0\Gamma} = \partial_\infty\Gamma^{(2)} \times \mathbb{R}$ which commutes with translation on $\mathbb{R}$ and is the diagonal action on $\partial_\infty\Gamma^{(2)}$.

There is a metric on $\widetilde{U_0\Gamma}$, well-defined up to Hölder equivalence such that Γ acts by isometries and the $\mathbb{R}$-orbits are quasigeodesics.

We then define $U_0\Gamma = \widetilde{U_0\Gamma}/\Gamma$ and give it the flow ϕ_t induced by translation in $\mathbb{R}$. The pair $(U_0\Gamma, \phi)$ is the *geodesic flow* of the group Γ.

Given a representation ρ we define the associated bundle E_ρ over $U_0\Gamma$ by

$$E_\rho = (\widetilde{U_0\Gamma} \times \mathbb{R}^m)/\Gamma \qquad \text{where} \qquad \gamma(p, v) = (\gamma p, \rho(\gamma) v)$$

The flow ϕ_t lifts to a flow ψ_t on E_ρ by trivial action on $\mathbb{R}^m$.

For ρ convex then $E_\rho = E_1 \oplus E_2$ where

$$E_1([x, y, t]) = \xi(x) \qquad E_2([x, y, t]) = \xi^*(y).$$

We say a convex representation ρ is *Anosov* if the bundle $E_1 \otimes E_2^*$ is contracted by the flow ψ_t.

If ρ convex Anosov then E_1 is also contracting.

Theorem

(Guichard-Wienhard) All irreducible convex representations are Anosov.

Convex Anosov Representations: Hitchin Representations

Let $\Gamma = \pi_1(S)$, a closed surface group. Let $r_m : SL(2,\mathbb{R}) \to SL(m,\mathbb{R})$ be the irreducible representation given by action on the space of homogeneous $m-1$ polynomials $g(P(x,y)) = P(g(x,y))$.

Then a representation $\rho : \Gamma \to SL(m,\mathbb{R})$ is *Fuchsian* if $\rho = r_m \circ \rho_0$ where ρ_0 is a discrete faithful representation in $SL(2,\mathbb{R})$. A representation is *Hitchin* if it can be deformed into a Fuchsian representation.

Theorem

(Hitchin) The Hitchin component is diffeomorphic to a ball of dimension $-\chi(S)(m^2-1)$.

Theorem

(Labourie) All Hitchin representations are convex, irreducible and Anosov.

Geodesic Flow of a Representation

Let $\pi : F \rightarrow B$ be the bundle with $B = \mathbb{RP}^m \times \mathbb{RP}^{m*} - \{(U, V) \mid U \in V\}$ and fiber $M(U, V) = \{(u, v) \mid u \in U, v \in V, < v|u >= 1\}/((u, v) \sim (-u, -v))$. Then F is an $\mathbb{R}$-bundle.

We define the flow on F by

$$\phi_t(U, V, (u, v)) = (U, V, (e^t.u, e^{-t}.v))$$

Given a convex representation ρ we then define the bundle F_ρ over $\partial_\infty \Gamma^{(2)}$ by the pullback $F_\rho = (\xi, \xi^*)^*(E)$. The flow ϕ_t gives a flow on F_ρ and the action of Γ on $\partial_\infty \Gamma^{(2)}$ extends to an action on F_ρ. We let

$$U_\rho \Gamma = F_\rho / \Gamma$$

and let ϕ_t be the flow induced by $\mathbb{R}$-action.

The pair $(U_\rho \Gamma, \phi)$ is the *geodesic flow of the representation*.

Geodesic Flow is Hölder Reparametrization

Theorem

If ρ is convex Anosov then the action of Γ on F_ρ is proper and cocompact. Furthermore the flow (U_ρ, ϕ_t) is a Hölder reparametrization of the geodesic flow $U_0\Gamma$.

Theorem

If ρ is convex Anosov then the geodesic flow $U_\rho\Gamma$ is a metric Anosov flow.

Thermodynamic Mapping: Definition

Given $\rho \in C(\Gamma, m)$, the space $U_\rho\Gamma$ is a reparametrization of geodesic flow on $X = U_0\Gamma$ by some function f_ρ.

Theorem

(Bowen,Sambarino) Let $f \in C^h(X, \mathbb{R}_+)$ then

$$P(\phi, -h.f) = 0$$

if and only if $h = h_f$. Moreover the equilibrium state $m = m_{-h_f.f}$ satisifies $\widehat{f.m} = \mu_{\phi^f}$

We now define the *thermodynamic mapping* $\Psi : C(\Gamma, m) \to \mathcal{P}(X)$ by $\Psi(\rho) = -h(f_\rho).f_\rho$

We pullback the Pressure metric to $C(\Gamma, m)$ to obtain a pseudo-metric. For a smooth curve ρ_t we let $\Psi_t = \Psi(\rho_t)$. Then

$$||\dot{\rho}_0|| := ||\dot{\Psi}_0||$$

Thermodynamic Mapping: Pressure Metric Reformulation

We have $P(\Psi_t) = 0$ giving

$$P'(\Psi_t)\dot{\Psi}_t = 0 \qquad \text{(first derivative)}$$

$$P''(\Psi_t)\dot{\Psi}_t + P'(\Psi_t)\ddot{\Psi}_t = 0 \qquad \text{(second derivative)}$$

Thus

$$Var(\dot{\Psi}_t, m_t) = -\int \ddot{\Psi}_t \; dm_t$$

and

$$||\dot{\Psi}_0||^2 = -\frac{Var(\dot{\Psi}_0, m_0)}{\int \Psi_0 \; dm_0} = \frac{\int \ddot{\Psi}_0 \; dm_0}{\int \Psi_0 \; dm_0}$$

Then

$$||\dot{\Psi}_0||^2 = F''(0) \qquad \text{where} \qquad F(t) = \frac{\int \Psi_t \; dm_0}{\int \Psi_0 \; dm_0}$$

Thermodynamic Mapping: Intersection

We have

$$F(t) = \frac{h_t}{h_0}\frac{\int f_t \ dm_0}{\int f_0 \ dm_0}$$

By above

$$\mu_0 = \widehat{f_0 m_0} = \frac{f_0 m_0}{\int f_0 dm_0}$$

By Bowen,

$$\mu_0 = \lim_{T\to\infty}\left(\frac{1}{\#R_T(f)_0}\sum_{a\in R_T(f_0)}\frac{f_0.\delta_a}{l_a(f_0)}\right)$$

and

$$\frac{\int f_t \ dm_0}{\int f_0 \ dm_0} = \int \frac{f_t}{f_0} d\mu_0 = \lim_{T\to\infty}\left(\frac{1}{\#R_T(f_0)}\sum_{a\in R_T(f_0)}\frac{l_a(f_t)}{l_a(f_0)}\right) = I(f_0, f_t)$$

Degeneracy: Criterion for zero length

Theorem

$||v||_\rho = 0$ if and only if there exists a $k \in \mathbb{R}$ such that

$$l'_a(\rho)(v) = k.l_a(\rho) \qquad \text{for all } a \in O$$

Proof: Let ρ_t such that $\rho_0 = \rho$ and $v = \dot{\rho}_0$.
Then $||v||_\rho = 0$ if and only if $\dot{\Psi}_0 \sim 0$.
By Livsic, $\dot{\Psi}_0 \sim 0$ if and only if $l_a(\dot{\Psi}_0) = 0$ for all $a \in O$.
As $l_a(\Psi_t) = h(\rho_t) l_a(\rho_t)$, then

$$l_a(\dot{\Psi}_0) = h(\rho) l'_a(\rho)(v) + h'(\rho)(v) l_a(\rho) = 0$$

Therefore

$$l'_a(\rho)(v) = -\left(\frac{h'(\rho)(v)}{h(\rho)}\right) l_a(\rho) = k.l_a(\rho)$$

Degeneracy: Trace Identities

A function f is *log-type* K if $(\log f)'(\rho)(v) = K.(\log f)(\rho)$.

For $||v||_\rho = 0$ then $f(\rho) = \Lambda(\rho(g))$ is log-type K for all $g \in \Gamma$.

For $\rho(g) = A \in SL(m, \mathbb{R})$. We write

$$A = \Lambda(A)p(A) + m(A) + \lambda(A)q(A)$$

where $p(A)$ is projection onto maximal eigendirection, $q(A)$ is projection onto minimal eigendirection and $m(A)$ has spectral radius strictly smaller than $\Lambda(A)$.

Then for $g, h \in \Gamma$, let $A = \rho(g), B = \rho(h)$ consider

$$Trace(A^n B) \qquad Trace(p(A)B^n)$$

Then $Trace(p(A)B), Trace(p(A)p(B))$ are also log-type K.

Degeneracy: Generic Element

An element $A \in SL(m, \mathbb{R})$ is generic if its centralizer is a maximal torus.

A representation is *generic* if $\rho(\Gamma)$ contains a generic element.

For ρ generic, we choose g, h, coprime, such that $\rho(h)$ is generic. Let $A = \rho(g), B = \rho(h)$.

Then show that $K = 0$ by considering expansion of $Trace(p(A)B^n)$.

Thus function $f(\rho) = Trace(p(A)p(B))$ has $f'(\rho)(v) = 0$.

$Trace(p(A)p(B)) = b(A^-, B^-, B^+, A^+)$ where b is a generalized cross-ratio as defined by Labourie.

For ρ irreducible, these generate the tangent space giving that $v = 0$.