Exchange algebras, differential Galois theory and Poisson Lie groups

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The Setting

- The space $\mathcal{H}$ of Schroedinger operators on the circle

$$H = -\partial_x^2 - u, \qquad u \in C^\infty(S^1), \quad S^1 \simeq \mathbb{R}/2\pi\mathbb{Z},$$

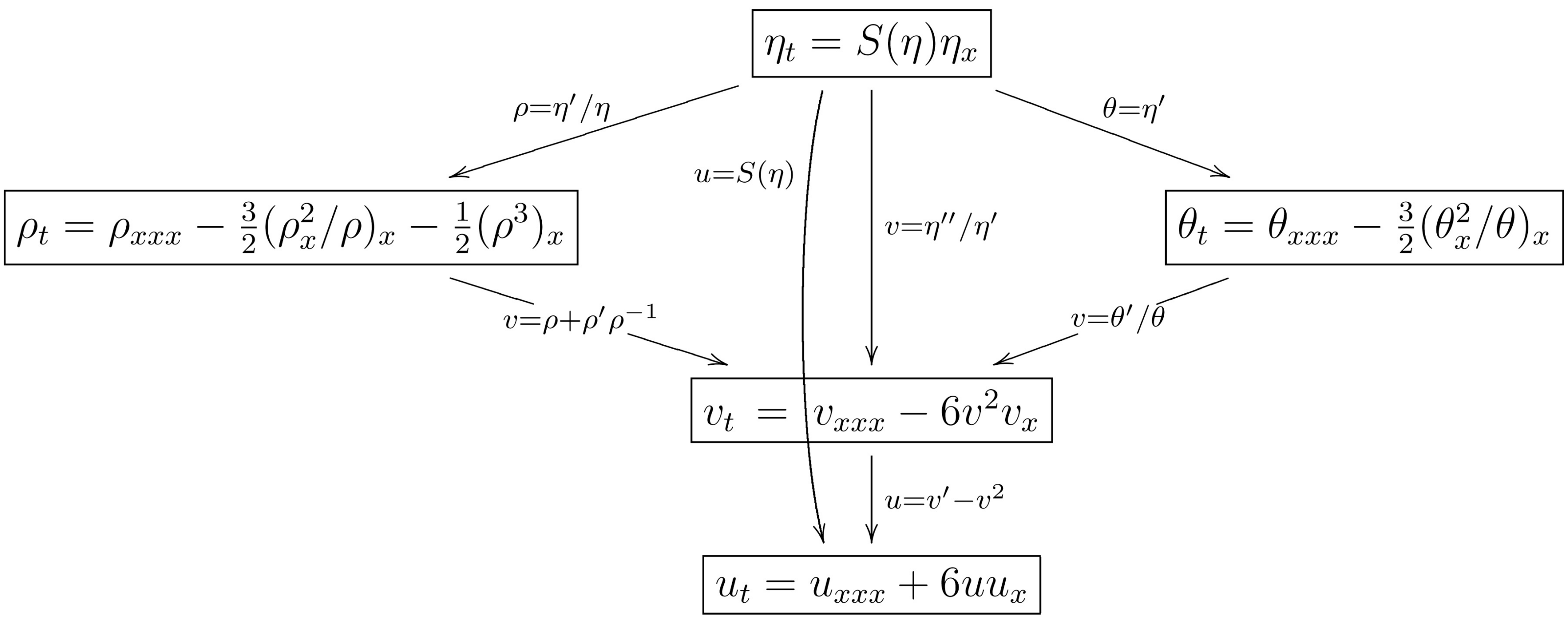
  is the phase space for the KdV hierarchy (with periodic boundary conditions). It carries a family of natural Poisson structures which play an important rôle in the Hamiltonian description of the KdV flows.
- The Virasoro Poisson operator:

$$l = \frac{1}{2}\partial_x^3 + u\partial_x + \partial_x u.$$

- Problem:
  - Extend the Poisson structure to the space of wave functions.

The tower of KdV-like equations

- A closely related practical question: The tower of KdV-like equations.



The tower of KdV-like equations–2

As we shall see, this diagram is naturally understood in terms of the differential Galois theory.
This is the point of view suggested by George Wilson in 1989.

- Problem: equip all phase spaces in this diagram with natural Poisson structures in such a way that all arrows become Poisson mappings.

General problem

- In a more abstract language, second order differential operators on the line are associated with *projective connections*; solutions of the Schroedinger equations are covariantly constant sections of the associated projective bundle.
  In a similar way, the space of flat linear connections on a circle carries a natural Poisson structure. The question is:
- Extend this Poisson structure to the space of covariantly constant sections.
  In both cases, obstructions are connected with nontrivial cohomology:
  - In linear case, it is the Maurer–Cartan cocycle.
  - In projective case it is the Gelfand–Fuchs cocycle.

Elementary theory

- The space $V = V_u$ of solutions of the Schroedinger equation
$$-\psi'' - u\psi = 0$$
is 2-dimensional.
- Any two solutions ϕ, ψ have constant wronskian $W = \phi\psi' - \phi'\psi$.
- An element $w \in V$ may be regarded as a quasi-periodic plane curve (such that $w \wedge w'$ is nowhere zero).
- Monodromy matrix:

$w(x+2\pi n) = w(x)M^n, \ n \in \mathbb{Z}$, where $w = (\phi, \psi)$ is a row vecto

- The group $G = SL(2)$ acts naturally on V (preserving the wronskian) by right multiplication.

Classical Theorem:

- Any pair of linearly independent solutions of the Schroedinger equation defines a non-degenerate quasi-periodic projective curve $\gamma : \mathbb{R} \to \mathbb{C}P_1$ such that

$$\gamma(x + 2\pi) = \gamma(x)M.$$

- Conversely, any non-degenerate quasi-periodic projective curve may be lifted to a non-degenerate curve in $\mathbb{C}^2$ such that its wronskian is equal to 1 and hence gives rise to a 2nd order differential equation.

Schwarzian Derivative

Fix an affine coordinate on $\mathbb{C}P_1$ in such a way that ∞ corresponds to the zeros of the second coordinate ψ of the point on the plane curve; with this choice γ is replaced with the affine curve $x \mapsto \eta(x) = \phi(x)/\psi(x)$. The potential u may be restored from η by the formula

$$u = \frac{1}{2}S(\eta), \quad \text{where} \quad S(\eta) = \frac{\eta'''}{\eta'} - \frac{3}{2}\left(\frac{\eta''}{\eta'}\right)^2$$

is the Schwarzian derivative.

- $S(u)$ is projective invariant: if $\tilde{\eta} = \frac{a\eta+c}{b\eta+d}$, then $S(\tilde{\eta}) = S(u)$.

Point of view of differential Galois theory

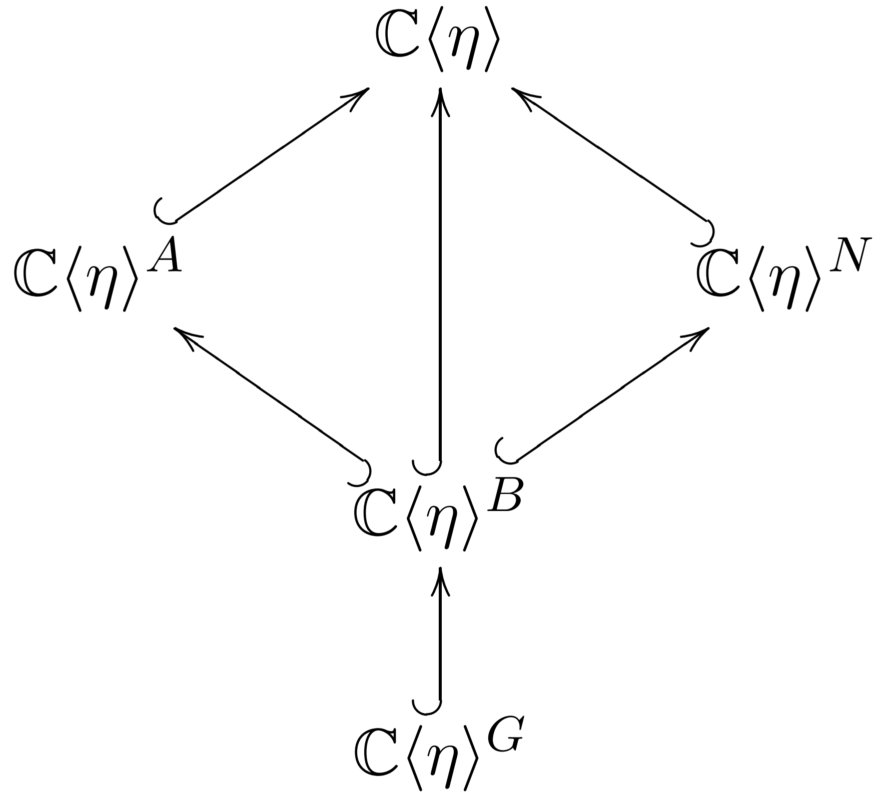
- We define the differential field $\mathbb{C}\langle\psi_1,\psi_2\rangle$ as a free algebra of rational functions in an infinite set of variables $\psi_1,\psi_2,\psi_1',\psi_2',\psi_1'',\psi_2'',\ldots$ with a formal derivation ∂ such that $\partial\psi_i^{(n)}=\psi_i^{(n+1)}$.
- A *differential automorphism* is an automorphism of $\mathbb{C}\langle\psi_1,\psi_2\rangle$ (as an algebra) which commutes with ∂. All differential automorphisms are induced by linear transformations $(\psi_1,\psi_2)\mapsto(\psi_1,\psi_2)\cdot g,\ g\in GL(2,\mathbb{C})$.
- Let (W) be the differential ideal in $\mathbb{C}\langle\psi_1,\psi_2\rangle$ generated by $\psi_1\psi_2'-\psi_1'\psi_2-1$. Automorphisms which preserve W belong to $G=SL(2)$.
- The differential subfield of G-invariants coincides with $\mathbb{C}\langle u\rangle$.

Intermediate differential fields

Let $Z = \{\pm 1\}$ be the center of G and N, A, $B = AN$ its standard subgroups (nilpotent, split Cartan & Borel). The subfields of invariants are freely generated differential algebras:

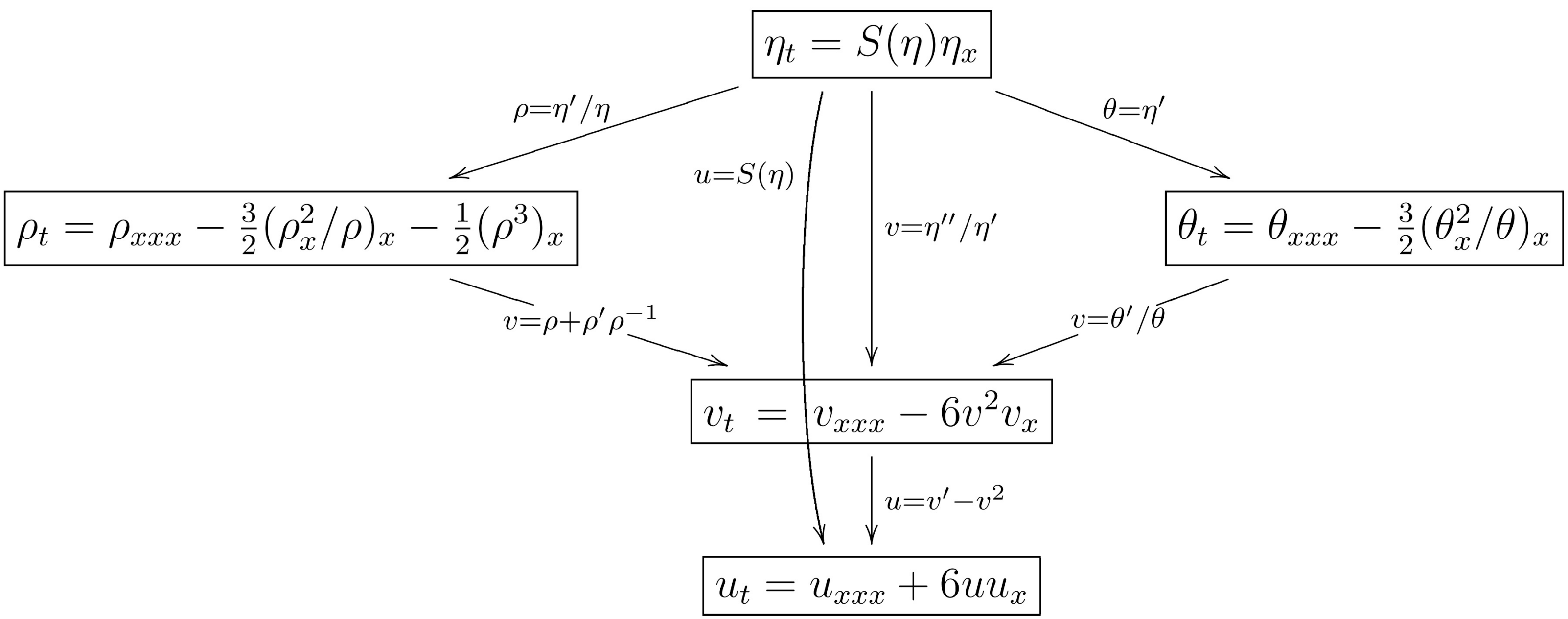
- $\mathbb{C}\langle \phi, \psi \rangle^Z = \mathbb{C}\langle \eta \rangle, \quad \eta = \phi/\psi$,
- $\mathbb{C}\langle \eta \rangle^A = \mathbb{C}\langle \rho \rangle, \quad \rho = \eta'/\eta$,
- $\mathbb{C}\langle \eta \rangle^N = \mathbb{C}\langle \theta \rangle, \quad \theta = \eta'$,
- $\mathbb{C}\langle \eta \rangle^B = \mathbb{C}\langle v \rangle, \quad v = \eta''/\eta'$,
- $\mathbb{C}\langle \eta \rangle^G = \mathbb{C}\langle u \rangle, \quad u = S(\eta)$.

Extension Tower



KdV–like equations once again

Basic classical observation: there is a family of compatible KdV-like flows on each level of the tower related by natural differential substitutions:



Main Problem:

- Equip all levels of the tower with Poisson structure so as to make all arrows Poisson mappings.
  This is a surprisingly non-trivial question.
  - Attempt of G.Wilson: Look at symplectic forms which can be pulled back.
  - Obstructions:
    - When monodromy is nontrivial, symplectic forms fail to be closed!
  - Our approach:
    - Guess the answer from natural covariance requirements.

Main result:

- The Poisson bracket on the (projectivized) space of wave functions is rigid and essentially unique; the Galois group automatically becomes a Poisson group with the standard Poisson structure; other possible Poisson structures on G are excluded.

A reminder on Poisson Lie groups

Let G be a Lie group with Lie algebra $\mathfrak{g}$. A Poisson structure on G is called multiplicative if the multiplication

$$m\colon G \times G \to G$$

is a Poisson mapping. A Lie group equipped with a multiplicative Poisson bracket is called a *Poisson Lie group*. An action $G \times \mathcal{M} \to \mathcal{M}$ of a Poisson group on a Poisson manifold $\mathcal{M}$ is called a *Poisson action* if this mapping is Poisson; in other words, for $F, H \in \mathrm{Fun}(\mathcal{M})$, their Poisson bracket at the transformed point $g \cdot m \in \mathcal{M}$ may be computed as follows:

$$\{F, H\}_{\mathcal{M}}(g \cdot m) = \left\{\hat{F}(m, \cdot\,), \hat{H}(m, \cdot\,)\right\}_G (g) + \left\{\hat{F}(\,\cdot\,, g), \hat{H}(\,\cdot\,, g)\right\}_{\mathcal{M}}$$

In that case we shall also say that the Poisson bracket on $\mathcal{M}$ is G-*covariant*.

- Important fact:
  - Multiplicative Poisson bracket on G gives rise to the structure of a Lie algebra on the dual space $\mathfrak{g}^*$;
  - The dual of the commutator map $[\ ,\ ]\colon \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^*$ is a 1-cocycle on $\mathfrak{g}$.

  A pair $(\mathfrak{g}, \mathfrak{g}^*)$ with these properties is called a *Lie bialgebra*.
- Fundamental Theorem (Drinfeld):
  - Multiplicative Poisson bracket is completely determined by its linearization.

Hence there is an equivalence of two categories:

- Category of Poisson Lie groups (morphisms $=$ Lie group homomorphisms which are also Poisson mappings)
- Category of Lie bialgebras (morphisms $=$ homomorphisms of Lie algebras such that their duals are homomorphisms of the dual algebras).
- Important tool:
  - Restrict Poisson action to subgroups.
    - Obvious possibility:
      - Restrict to Poisson subgroups.
    - More flexible option:
      - Restrict to admissible subgroups.

Admissible subgroups and Poisson Reducti

- Definition:
  - A subgroup $H \subset G$ of a Poisson Lie group G is called admissible if the subalgebra of H-invariants $\mathrm{Fun}(\mathcal{M})^H \subset \mathrm{Fun}(\mathcal{M})$ is closed with respect to the Poisson bracket.
- Admissibility criterion:
  - $H \subset G$ is admissible if and only if $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ is a Lie subalgebra;
  - $H \subset G$ is a Poisson subgroup if and only if $\mathfrak{h}^\perp$ is an ideal in $\mathfrak{g}^*$.
- Poisson Reduction:
  - If H is admissible, $\mathrm{Fun}(\mathcal{M})^H \simeq \mathrm{Fun}(\mathcal{M}/H)$ inherits the natural Poisson structure.

The case of $SL(2)$

- The group $G = SL(2, \mathbb{C})$ carries a family of natural Poisson structures called the Sklyanin brackets which make it a Poisson Lie group.
- These Poisson structures are parameterized by the choice of a classical r-matrix $r \in \mathfrak{g} \wedge \mathfrak{g}$.
  In usual tensor notation we have

$$\{g_1,\, g_2\} = [r,\, g_1\, g_2], \tag{1}$$

- Important fact:
  - Up to natural equivalence there exist three types of classical r-matrices:

    (a) $r = 0$;
    (b) $r = h \wedge f$
    (c) $r = \epsilon\, e \wedge f$, where ϵ is a scaling parameter.

Case (a) gives trivial bracket; case (b) (the so called triangular r-matrix) is degenerate. Case (c) (alias, quasitriangular or factorizable case) is generic.

- Explicit formulae in case (c):
  - Let $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then

$$\{\alpha, \beta\} = \epsilon\alpha\beta, \quad \{\alpha, \gamma\} = \epsilon\alpha\gamma,$$
$$\{\beta, \delta\} = \epsilon\beta\delta, \quad \{\gamma, \delta\} = \epsilon\gamma\delta,$$
$$\{\beta, \gamma\} = 0, \quad \{\alpha, \delta\} = 2\epsilon\beta\gamma.$$

- Important:
  - $\det g = \alpha\delta - \beta\gamma$ is a Casimir function and hence the Poisson bracket is well defined on the coordinate ring of $SL(2)$ and even of $PSL(2)$.)

Yang-Baxter tensor

$$[[r,r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \in \wedge^3 \mathfrak{g}$$

This tensor naturally arises in the check of the Jacobi identity.

- Important to notice:
  - For $\mathfrak{g} = sl(2)$ the Yang–Baxter equation does not impose any restrictions on r, since $\wedge^3 \mathfrak{g} \simeq \mathbb{C}$ is trivial.
- However, there are still two different cases to distinguish.
  - $[r,r] = 0$ in cases (a) and (b).
  - $[r,r] = -\epsilon^2 \neq 0$ in case (c) (r standard quasitriangular r-matrix).

The Dual Group

- The dual Lie algebra $\mathfrak{g}^*$ associated with the standard r-matrix is

$$\mathfrak{g}^* = \{(X_+, X_-) \in \mathfrak{b}_+ \oplus \mathfrak{b}_- \mid \operatorname{diag} X_+ + \operatorname{diag} X_- = 0\}\,.$$

- The dual Lie group is

$$G^* = \{(b_+, b_-) \in B_+ \times B_- | \operatorname{diag} b_+ \cdot \operatorname{diag} b_- = I\}\,.$$

- Important fact:
  - The dual group is also a Poisson Lie group.
- The Poisson bracket on G^* may be described in terms of Poisson bracket relations for matrix coefficients of (b_+, b_-). There is another way based on the Gauss decomposition mapping $G^* \to G(b_+, b_-) \mapsto b_+ b_-^{-1}$:

The dual Poisson structure

- Important assertion:
  - The mapping

$$G^* \to G : (b_+, b_-) \mapsto M = b_+ b_-^{-1}$$

    maps G^* onto an open dense subset in G; the induced Poisson structure extends smoothly to the entire manifold G.
  - Explicitly we have

$$\{M_1, M_2\} = M_1 M_2 r + r M_1 M_2 - M_2 r_+ M_1 - M_1 r_- M_2, \tag{2}$$

    where $r_\pm = r \pm \epsilon t$ and $t \in \mathfrak{g} \otimes \mathfrak{g}$ is the tensor Casimir element.

Back to the study of wave functions:

- Starting point: The space $\mathcal{W}$ of *all* quasi-periodic plane curves,

$$\mathcal{W} = \{(w = (\phi, \psi), M) \mid w(x + 2\pi) = w(x)M\} \,.$$

- Scaling group: $\mathcal{C} := C^\infty(\mathbb{R}/2\pi\mathbb{Z}, \mathcal{R}^\times)$.
- Constraint set:

$$\mathcal{W}' = \{w \in \mathcal{W}; W(w) = 1\} \,.$$

$\mathcal{W}'$ is the cross-section of the scaling action; it may be identified with the space $\mathcal{V}$ of wave functions.
Hence $\mathcal{V}$ has a twofold description:

  - as a quotient space
  - as a subspace $\mathcal{W}' \subset \mathcal{W}$.

Covariance and exchange bracket

- Another action on $\mathcal{W}$: the natural action of $G = SL(2)$.
- Covariance axiom:
  $\mathcal{C} \times \mathcal{W} \to \mathcal{W}$ and $G \times \mathcal{W} \to \mathcal{W}$ are Poisson maps.
  This presumes that both $\mathcal{C}$ and G may carry Poisson bracket (possibly trivial) which makes them Poisson Lie groups; their action is then Poisson action.
- Lemma 1.
  - Suppose that the bracket on $\mathcal{W}$ is covariant with respect to the action of $\mathcal{C}$. Then the Poisson structure on $\mathcal{C}$ is trivial and, writing $w = (\phi, \psi)$, the bracket of evaluation functionals has the form:

$$\{w_1(x), w_2(y)\} = w_1(x) w_2(y) R(x, y), \tag{3}$$

Exchange matrix

The “exchange matrix” $R(x, y) \in \mathrm{Mat}(4)$ is given by

$$R(x,y) = \begin{pmatrix} A(x-y) & 0 & 0 & 0 \\ 0 & B(x-y) & -C(y-x) & 0 \\ 0 & C(x-y) & -B(y-x) & 0 \\ 0 & 0 & 0 & D(x-y) \end{pmatrix}.$$

- Let us drop temporarily the Jacobi identity and consider all Poisson brackets of this type which are G-covariant.
  - First the case of G-invariant brackets:

Jacobi Identity: case of trivial bracket on G

- Lemma 2.
  - Assume that the Poisson bracket (3) is right-G-invariant (hence G carries trivial bracket); then the exchange matrix has the structure

$$R_0(x,y) = a(x-y)I + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c(x-y) & -c(x-y) & 0 \\ 0 & c(x-y) & -c(x-y) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

    where a and c are arbitrary odd functions.

Jacobi Identity: nontrivial case

Now assume that G carries a *nontrivial* Poisson structure.

- Lemma 3.
  - Assume that the Poisson bracket (3) is right-G-covariant; then the exchange matrix has the structure

$$R_r(x,y) = R_0(x,y) + r, \tag{4}$$

- Recall the classification of r-matrices: either

(1) $[[r,r]] = 0$ ($r = 0$ or r degenerate, cases (a)& (b);

(2) or $[[r,r]] = -\epsilon^2 \neq 0$ (r *quasitriangular*).

In case (2) the Yang–Baxter tensor $[[r,r]]$ gives an extra term to the Jacobi identity for the exchange bracket.

- Lemma 4.
  - The exchange bracket (3) with exchange matrix (4) satisfies the Jacobi identity if and only if

$$c(x-y)c(y-z)+c(y-z)c(z-x)+c(z-x)c(x-y)=0$$

in case (1) and

$$c(x-y)c(y-z)+c(y-z)c(z-x)+c(z-x)c(x-y)=-\epsilon^2$$

in case (2).

Solution of the functional equation

- Solution:
  $c_\lambda(x-y) = \epsilon \coth \lambda(x-y)$ in case (2).
- Important limiting case $\lambda \to \infty$: $c(x-y) = \epsilon\,\mathrm{sign}(x-y)$.
- In case(1) the solution is $c(x-y) = \frac{\lambda}{x-y}$.
- Poisson bracket relations for the projectivized wave functions:

$$\{\eta(x), \eta(y)\} = \epsilon \left(\eta(x)^2 - \eta(y)^2\right) - c(x-y)\left(\eta(x) - \eta(y)\right)^2 . \tag{5}$$

- Proposition.
  - Formula (5) defines a family of G-covariant Poisson brackets on the space of “projective curves” η. Notice that a drops out, but there is still some freedom in the choice of c.

Wronskian constraint

- One more key ingredient:
  - the wronskian constraint.

It fixes c completely and allows to choose a in a natural way as well.

- We have

$$\{W(x), \phi(y)\} = (c(x-y) - 2a(x,y))W(x)\phi(y) \\ - c'(x-y)\phi(x)[\phi(x)\psi(y) - \psi(x)\phi(y)]. \quad (6)$$

A similar formula holds for $\{W(x), \psi(y)\}$.

Crucial observation

- Proposition.
  - The constraint $W = 1$ is compatible with the Poisson brackets for the scaling invariant η if and only if the last term in (6) is identically zero; this is possible if and only if $c'(x-y)$ is a multiple of $\delta(x-y)$, i.e., if $c(x-y)$ is a multiple of $\mathrm{sign}(x-y)$.
- Without restricting the generality, we can now assume that r is the *standard quasitriangular r-matrix*, $r = e \wedge f$. Other possible choices differ by rescaling and conjugation.
- Standard fact:
  - With this choice, B, A, N, together with the opposite Borel subgroup, give the complete list of *admissible subgroups* of G.

Spontaneous symmetry breaking

- Important conclusion:
  - Differential Galois group spontaneously becomes a Poisson group and the choice of its Poisson structure is essentially unique.

Poisson bracket relations for the wronskia

- We have:

$$\{W(x), W(y)\} = (\mathrm{sign}(x-y) - 2a(x-y))W(x)W(y), \quad (7)$$

or, equivalently

$$\{\log W(x), \log W(y)\} = (\mathrm{sign}(x-y) - 2a(x-y)). \quad (8)$$

- Proposition.
  - Assume that a is so chosen that

$$\mathrm{sign}(x-y) - 2a(x-y) = \delta'(x-y).$$

(In other words, $a(x-y)$ is the distribution kernel of the operator $\frac{1}{2}\left(\partial^{-1} - \partial\right)$.) Then:

Wronskian as the moment map

(i) The logarithms of wronskians form a Heisenberg Lie algebra, the central extension of the abelian Lie algebra of $\mathcal{C}$.

(ii) Let $\mathcal{C}' = \mathcal{C}/\mathbb{C}^*$ be the quotient of the scaling group over the subgroup of constants; $\log W$ is the moment map for the action of $\mathcal{C}'$ on $\mathcal{W}$.

- The resulting picture:
  - The scaling action is Hamiltonian with moment map $\mu = \log W$.
  - $\mathcal{V}$ arises as a result of Hamiltonian reduction with respect to $\mathcal{C}$ over the zero level of μ.
  - The constraint set $\log W = 0$ is (almost) non-degenerate (i.e., this is a 2^{nd} class constraint, according to Dirac).

Poisson brackets for the monodromy

- Remark.
  - The projective invariants $\eta(x)$ commute with the wronskian and hence their Poisson brackets are not affected by the constraint.
- Theorem.
  -

$$\{w(x)_1, M_2\} = w(x)_1 \big[M_2 r_+ - r_- M_2 \big],$$
$$\{M_1, M_2\} = M_1 M_2 r + r M_1 M_2 - M_2 r_+ M_1 - M_1 r_- M_2. \tag{9}$$

- The Poisson bracket for the monodromy is precisely the Poisson bracket of the dual Poisson group G^*. In other words:

Monodromy as a non-abelian moment ma

- Theorem.
  - The ‘forgetting map’ $\mu : (w, M) \mapsto M$ is a Poisson morphism from $\mathcal{W}$ into the dual group G^*.
- Assertion
  - The mapping μ is the non-abelian moment map associated with the right action of G on $\mathcal{W}$
- Reminder:
  - A nonabelian moment map associated with a Poisson group action $G \times \mathcal{M} \to \mathcal{M}$ is a mapping to the dual Poisson group, $\mu : \mathcal{M} \to G^*$; in our case, μ is simply described if the dual group is modeled on G via the Gauss factorization map.

Basic Poisson bracket relations

- Basic Poisson bracket relations for differential Galois invariants:
  - $\{\eta(x), \eta(y)\} = \eta(x)^2 - \eta(y)^2 - \text{sign}(x-y)\,(\eta(x) - \eta(y))^2$.
  - For $\theta = \eta'$ we have

$$\{\theta(x), \theta(y)\} = 2\,\text{sign}(x-y)\theta(x)\theta(y).$$

  - For $v = \frac{1}{2}\eta''/\eta' = \frac{1}{2}\theta'/\theta$ we have

$$\{v(x), v(y)\} = \frac{1}{2}\delta'(x-y).$$

  - For $u = \frac{1}{2}v' - v^2 = S(\eta)$ we have

$$\{u(x), u(y)\} = \frac{1}{2}\delta'''(x-y) + \delta'(x-y)\big[u(x) + u(y)\big]. \quad (10)$$

Nonlocal bracket for A-invariants

The last formula (10) reproduces the Virasoro bracket (which has not been assumed *a priori*!)

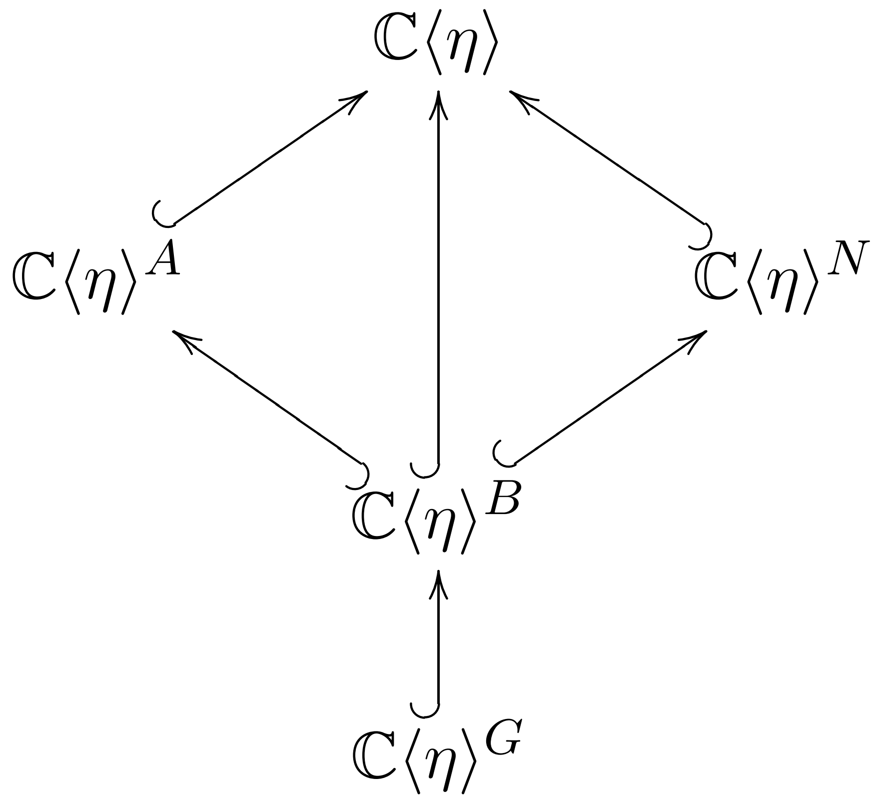
- Poisson bracket relations (38) – (10) listed above are algebraic. Since the basic Poisson bracket relations (5) are nonlocal, this need not always be the case. This is what happens in the case of A-invariants:
  - The differential subalgebra of A-invariants in $\mathbb{C}\langle\eta\rangle$ is generated by $\rho = \eta'/\eta$.
  - The Poisson brackets for ρ have the form

$$\{\rho(x),\, \rho(y)\} =$$

$$2\rho(x)\rho(y)\left[\sinh\int_x^y \rho(s)\,ds + \operatorname{sign}(x-y)\cosh\int_x^y \rho(s)\,ds\right].$$

Back to the extension tower

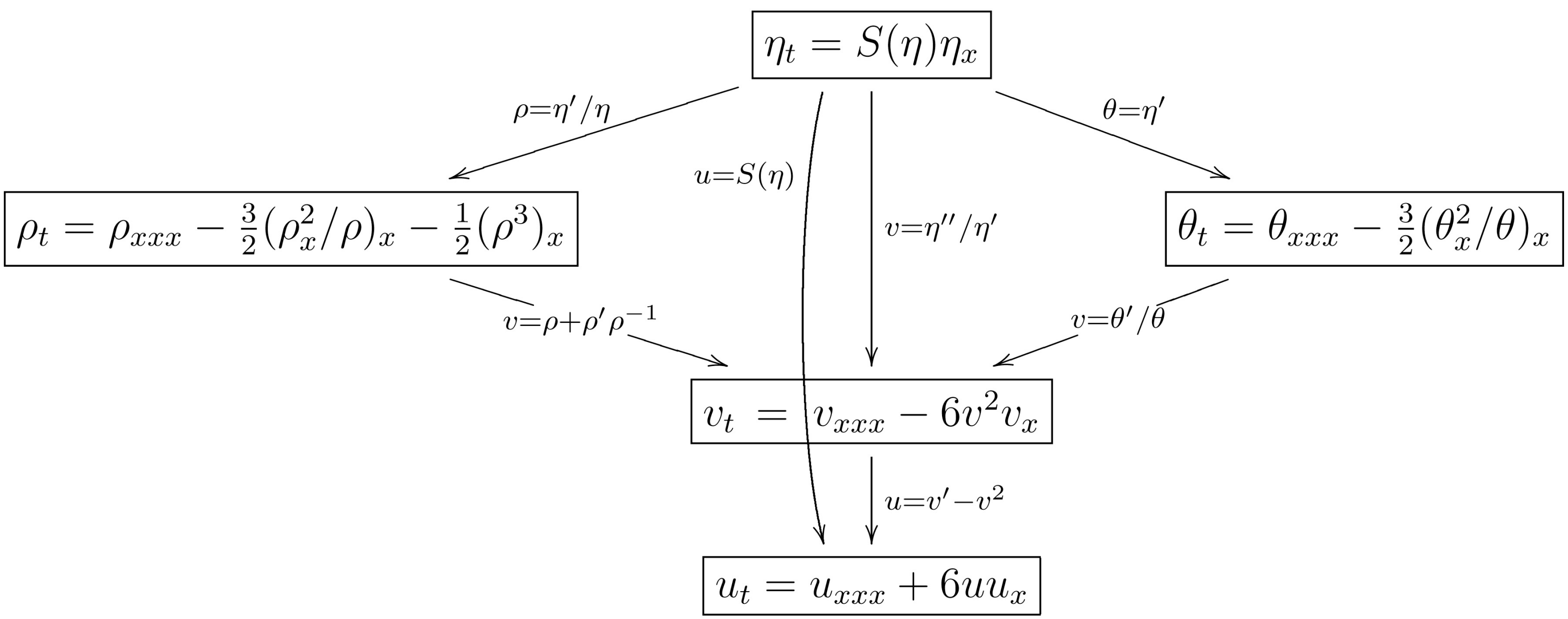
- Theorem
  - All arrows in the extension tower



are Poisson algebra homomorphisms.

Back to the extension tower–2

- Theorem
  - All arrows in the commutative diagram



are Poisson mappings.

- Theorem (second part)
  - The KdV-like equations listed in the diagram are generated by the standard Hamiltonian

$$H = \int u^2 \, dx.$$

    (which lives on the bottom level of the extension tower but can be pulled back to all upper levels)
  - A similar assertion holds for all higher KdV flows.
  - All flows factorize over those lying beneath.

Discrete case

- Second order difference equations on the one-dimensional lattice with periodic potential

$$\phi_{n+2} + u_n \phi_{n+1} + \phi_n = 0, \quad u_{n+N} = u_n. \tag{11}$$

- In operator form

$$\left(\tau^2 + u\,\tau + 1\right)\phi = 0, \tag{12}$$

where τ is the shift operator, $(\tau\phi)_n = \phi_{n+1}$.

- Elementary Theory:
  - For a given u, the space of its solutions is two-dimensional;
  - Any two solutions ϕ, ψ have constant wronskian $W = \phi_n \psi_{n-1} - \phi_{n-1}\psi_n$.

Projective picture

- Projective description:
  - An (ordered) projective configuration is a map $\gamma : \mathbb{Z} \to \mathbb{C}P_1$.
  - A configuration is non-degenerate if $\gamma_n \neq \gamma_{n+1}$. for all n.
  - A plane configuration is a map $w : \mathbb{Z} \to \mathbb{C}^2$; it is non-degenerate if $w_n \wedge w_{n+1} \neq 0$.
  - We denote w_n by the row vector (ϕ_n, ψ_n).
- Theorem.
  - Any pair of independent solutions of the discrete Schroedinger equation defines a non-degenerate quasi-periodic projective configuration $\gamma : \mathbb{Z} \to \mathbb{C}P_1$ such that $\gamma_{n+N} = \gamma_n \cdot M$, where M is the monodromy.

- Theorem (suite).
  - Any two projective configurations associated with a given discrete Schroedinger equation are related by a global projective transformation.
  - Any non-degenerate quasi-periodic projective configuration may be lifted to a non-degenerate plane configuration such that its wronskian

$$W[w]_n := \phi_n \psi_{n-1} - \psi_n \phi_{n-1} = 1.$$

- We replace the projective line with its affine model putting $\eta_n = \phi_n / \psi_n$. The group $G = SL(2)$ is the (difference) Galois group of equation (11).
- G acts on η_n by fractional linear transformations.

Schwarzian vs cross-ratio

- The obvious analogue of the Schwarzian is the cross-ratio

$$s_n[\eta] := [\eta_n, \eta_{n+1}, \eta_{n+2}, \eta_{n+3}] = \frac{\eta_n - \eta_{n+2}}{\eta_n - \eta_{n+1}} \cdot \frac{\eta_{n+1} - \eta_{n+3}}{\eta_{n+2} - \eta_{n+3}};$$

- Curious fact:
  - The potential u cannot be directly restored from η. Instead, we have:
    - $s_n = u_n u_{n+1}$.
  - If the lattice length is *odd*, the potential still may be uniquely restored from s_n by solving a quadratic equation. Hence in this case $\mathbb{C}\langle u\rangle$ is a *quadratic extension* of $\mathbb{C}\langle \eta\rangle^G$.

Covariant Poisson brackets

Denote by $\mathcal{W}$ the space of all plane quasi-periodic configurations and by $\mathcal{C}$ the discrete scaling group.

- Proposition.
  - Assume that the Poisson structure on $\mathcal{W}$ is covariant with respect to the right action of G and to the natural action of the scaling group. Then it has the form

$$\{w^1_m(x), w^2_n(y)\} = w^1_m(x) w^2_n(y) R(m-n),$$

where

$$R(k) = R_0(k) + r, \quad R_0(k) = a_k I + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c_k & -c_k & 0 \\ 0 & c_k & -c_k & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

- Proposition (suite).
  - a_k is an arbitrary odd function;
  - c_k is an odd function which satisfies

$$c_{n-m}c_{m-k} + c_{m-k}c_{k-n} + c_{k-n}c_{n-m} = \alpha, \tag{13}$$

    where $\alpha = 0$ when $r = 0$ or r triangular and $\alpha = -\epsilon^2$ for r quasitriangular. (In the sequel, we set $\epsilon = 1$).

  The space $\mathcal{V} \subset \mathcal{W}$ of discrete wave functions is defined by the constraint $W[w] = 1$.

- Poisson bracket relations for the Wronskian:

$$\{W_n, \phi_m\} = (a_{n-m} + a_{n-1-m} - c_{n-m})W_n\phi_m + (c_{n-m} - c_{n-m-1})(\phi_n\phi_{n-1}\psi_m - \phi_n\psi_{n-1}\phi_m). \tag{14}$$

- Proposition.
  - Scaling invariants η_n commute with the wronskian if and only if the second term in (14) is proportional to $W_n\phi_m$; hence

$$c_{n-m} - c_{n-m-1} = \big(\delta_{nm} + \delta_{n,m+1}\big). \tag{15}$$

- Important fact:
  - This recurrence relation is again solved by the sign function.
- Hence we get:

$$\{\eta_n, \eta_m\} = \eta_n^2 - \eta_m^2 - \mathrm{sign}(n-m)\big(\eta_n - \eta_m\big)^2. \tag{16}$$

- Remark.
  - This is in fact a Poisson subalgebra of the continuous algebra.
  - Of course, it is not true that the wave functions of the discrete equation are the values of the wave functions for the continuous equation!
- Poisson bracket relations for Galois invariants.
  - We have:
    - $\mathbb{C}(\eta)^N = \mathbb{C}(\theta)$, where $\theta_m := \eta_{m+1} - \eta_m$
    - $\mathbb{C}(\eta)^B = \mathbb{C}(\lambda)$, where

$$\lambda_m := \frac{\eta_{m+2} - \eta_{m+1}}{\eta_{m+1} - \eta_m}.$$

Poisson bracket relations

- We get: $\{\theta_m, \theta_n\} = -2\,\mathrm{sign}(m-n)\theta_m\theta_n$, $\{\lambda_m, \lambda_n\} = 2\big(\delta_{m+1,n} - \delta_{m,n+1}\big)\lambda_m\lambda_n$.
- A natural interpretation of the variables λ_n is connected with the Miura transform for the discrete Schroedinger equation.
  - Assume that the difference operator (12) is factorized,

$$\tau^2 + u\,\tau + 1 = (\tau + v)(\tau + v^{-1}). \tag{17}$$

  - The potentials u, v are related by the difference Miura map,

$$u_n = v_n + v_{n+1}^{-1}. \tag{18}$$

Poisson bracket relations–2

- Assume that ψ satisfies the first order equation $(\tau + v^{-1})\psi = 0$. Let ϕ be the second solution of this equation such that $W(\phi, \psi) = 1$ and $\eta = \phi/\psi$; then

$$\eta_{n+1} - \eta_n = \frac{1}{\psi_n \psi_{n+1}}.$$

Clearly, $v_n = -\psi_n/\psi_{n+1}$ and hence

$$v_n v_{n+1} = \frac{\psi_n}{\psi_{n+2}} = \frac{\psi_{n+1}\psi_n}{\psi_{n+2}\psi_{n+1}} = \frac{\eta_{n+2} - \eta_{n+1}}{\eta_{n+1} - \eta_n} = \lambda_n \qquad (19)$$

- Thus λ_n is the product of two neighbouring potentials in the factorized Schroedinger operator.

Poisson bracket relations–3

- Remark.
  - The potentials v_n again are not rational Galois invariants of B and belong to a quadratic extension of $\mathbb{C}(\lambda)$.
- We have:

$$s_n = u_n u_{n+1} = \frac{(1+\lambda_n)(1+\lambda_{n+1})}{\lambda_{n+1}}. \tag{20}$$

- Poisson bracket relations for λ_n and s_n:

$$\begin{aligned}\{\lambda_m, \lambda_n\} &= (\delta_{m+1,n} - \delta_{m,n+1})\lambda_m \lambda_n, \\ \{s_m, s_n\} &= \big(\delta_{m+1,n} - \delta_{m,n+1}\big)(s_m + s_n - s_m s_n) \\ &\quad + s_m s_n \big(s_{m+1}^{-1}\delta_{m+2,n} - s_{n+1}^{-1}\delta_{m,n+2}\big).\end{aligned} \tag{21}$$

Discrete Virasoro algebra

- Proposition.
  - Let $\Phi_n = (-1)^n \operatorname{sign} n,\ n \neq 0,\ \Phi_0 = 0$. Then

$$\{v_n, v_m\} = 2\Phi_{n-m} v_n v_m, \tag{22}$$

$$\{u_n, u_m\} = 2\Phi_{n-m} u_n u_m + 2(\delta_{m+1,n} - \delta_{m,n+1}). \tag{23}$$

- Remark
  - Formula (22) coincides with the lattice Virasoro algebra introduced by Frenkel, Reshetikin & myself, while (21) coincides with the Faddeev–Takhtajan version of the lattice Virasoro algebra. Structure constants Φ_{n-m} arise in paper of Frenkel, Reshetikin & myself in the framework of the discrete Drinfeld–Sokolov theory.

Possible Extensions:

1. Energy dependent case;
2. q-difference case;
3. Higher order equations.