Quantum Integrable Spin Systems

and

Generalized Schur - Weyl duality

Petr P.Kulish

- Integrable systems and representation theory
- Schur -Weyl duality of $sl(2)$ and $\mathcal{S}_N$; IrReps on $\mathcal{H} = \bigotimes_1^N \mathbb{C}^2$ (mult. free decomposition of $\mathcal{H}$)
- Heisenberg $XXX_{\frac{1}{2}}$ spin chain. (bound. cond)

$$sl(2) - \text{inv.} \quad H = \sum_{k=1}^{N} (\vec{\sigma}_k, \vec{\sigma}_{k+1}) = \sum_k (2\mathcal{P}_{k,k+1} - I) \in \mathbb{C}[\mathcal{S}_N]$$

QISM — $3d$ algebra Yangian $\mathcal{J}(sl(2))$

- Generalizations: $\begin{cases} \text{anisotropy} & XXZ_{\frac{1}{2}} \\ \text{higher rank} & sl(N),\ so... \\ \text{other IrReps} & XXX_{s=1,\frac{3}{2}} \end{cases}$
- $XXZ_{\frac{1}{2}}$ spin chain $sl_q(2)$-invar. and Hecke $\mathfrak{H}_N(q)$ algebra $(TL_N(q))$
- $\mathcal{U}_q(n)$-inv. spin chains $\mathcal{H} = \bigotimes_1^N \mathbb{C}^n$, $(TL_N(q))$

  The same spectrum and the structure of representation ring.
- $\mathcal{U}_q(so(2n+1))$ spin chain $\mathcal{U}_q(so(3))$ and $BMW_N(q, 1/q^2)$

Schur - Weyl duality (Examples)

- $XXX_{\frac{1}{2}}$, $\mathcal{U}(sl(2))$ (symm. alg) $\mathbb{C}[\mathcal{S}_N]$ on $\mathcal{H} = \bigotimes_1^N \mathbb{C}^2$,

$$H \simeq \sum_{k=1}^{N-1} \mathcal{P}_{k,k+1}$$

- Suthserland model. $\mathcal{U}(sl(n))$; $\mathbb{C}[\mathcal{S}_N]$ on $\mathcal{H} = \bigotimes_1^N \mathbb{C}^n$,

- $XXZ_{\frac{1}{2}}$, $\mathcal{U}_q(sl(2))$; $\mathfrak{H}_N(q)$ on $\mathcal{H} = \bigotimes_1^N \mathbb{C}^2$,

$$H \simeq \sum_k \check{R}_{k,k+1}(q) \in (\mathrm{TL}_N(q))$$

- $XXZ_{TL}(h)$, $\mathcal{U}_q(n)$; $\mathrm{TL}_N(q)$ on $\mathcal{H} = \bigotimes_1^N \mathbb{C}^n$,

  (param. $B, B^{-1} \in \mathrm{Mat}(\mathbb{C}^n)$), $H \simeq \sum_k X_{k,k+1}$ (rank 1)

- XXZ_1, $\mathcal{U}_q(O(3))$; BMW-alg $W_N(q, 1/q^2)$ on $\mathcal{H} = \bigotimes_1^N \mathbb{C}^3$,

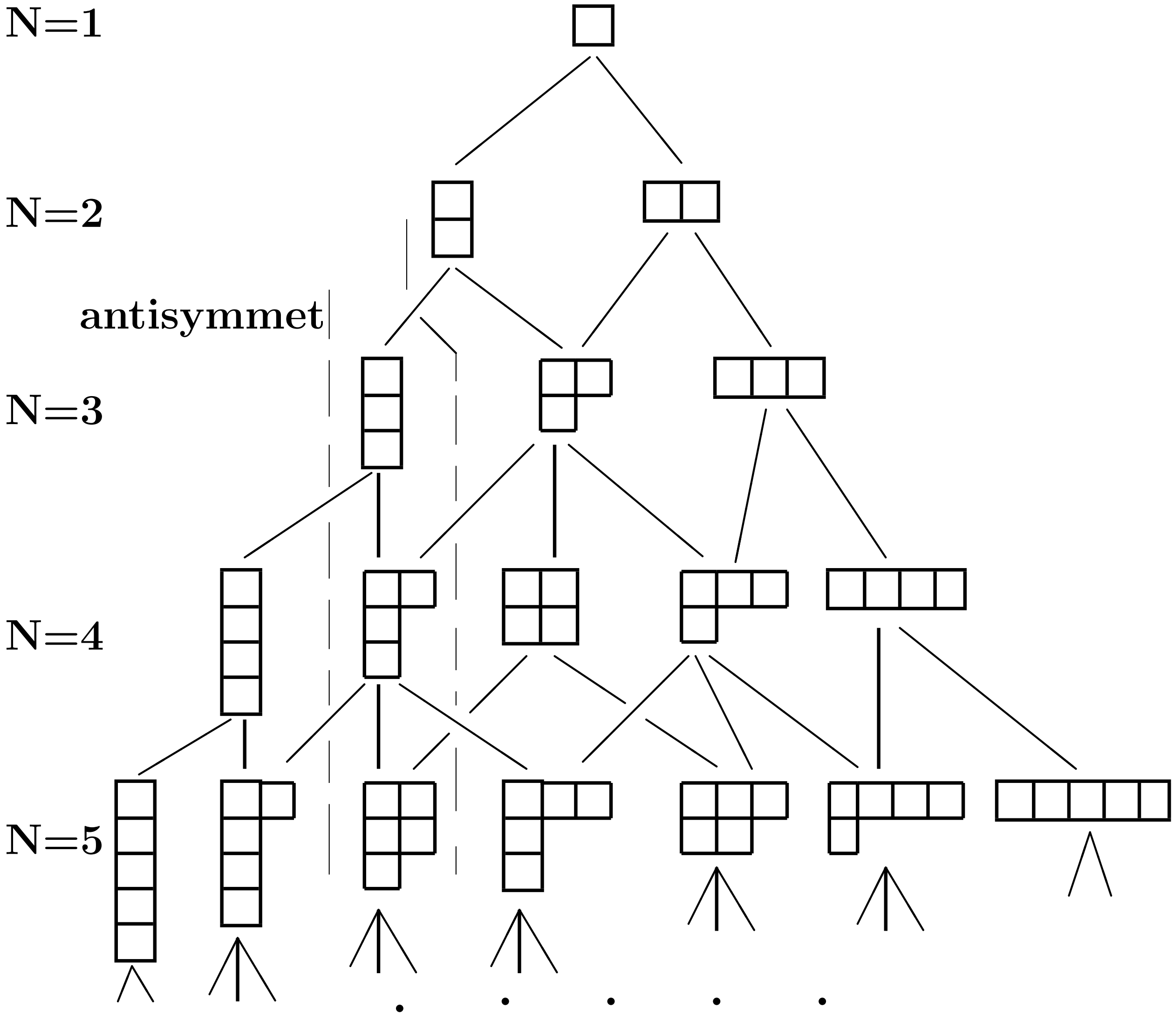
- $q - A_2^{(2)}$-model, $\mathcal{U}_q(O(3))$; $W_N(q, 1/q^2)$ on $\mathcal{H} = \bigotimes_1^N \mathbb{C}^3$,

$\mathcal{U}$ — symmetry alg. $\mathcal{J}_N$ — central alg.

$$\mathcal{H} = \bigotimes_1^N V = \bigoplus_\lambda V_\lambda \otimes W^\lambda, \qquad H \simeq \sum_1^{N-1} h_k = \bigoplus_\lambda H\Big|_{W^\lambda}$$

Bratteli diagram for $\mathcal{S}_N$

$$\mathbb{C} \simeq \mathcal{S}_1 \subset \mathcal{S}_2 \subset \ldots \subset \mathcal{S}_N$$

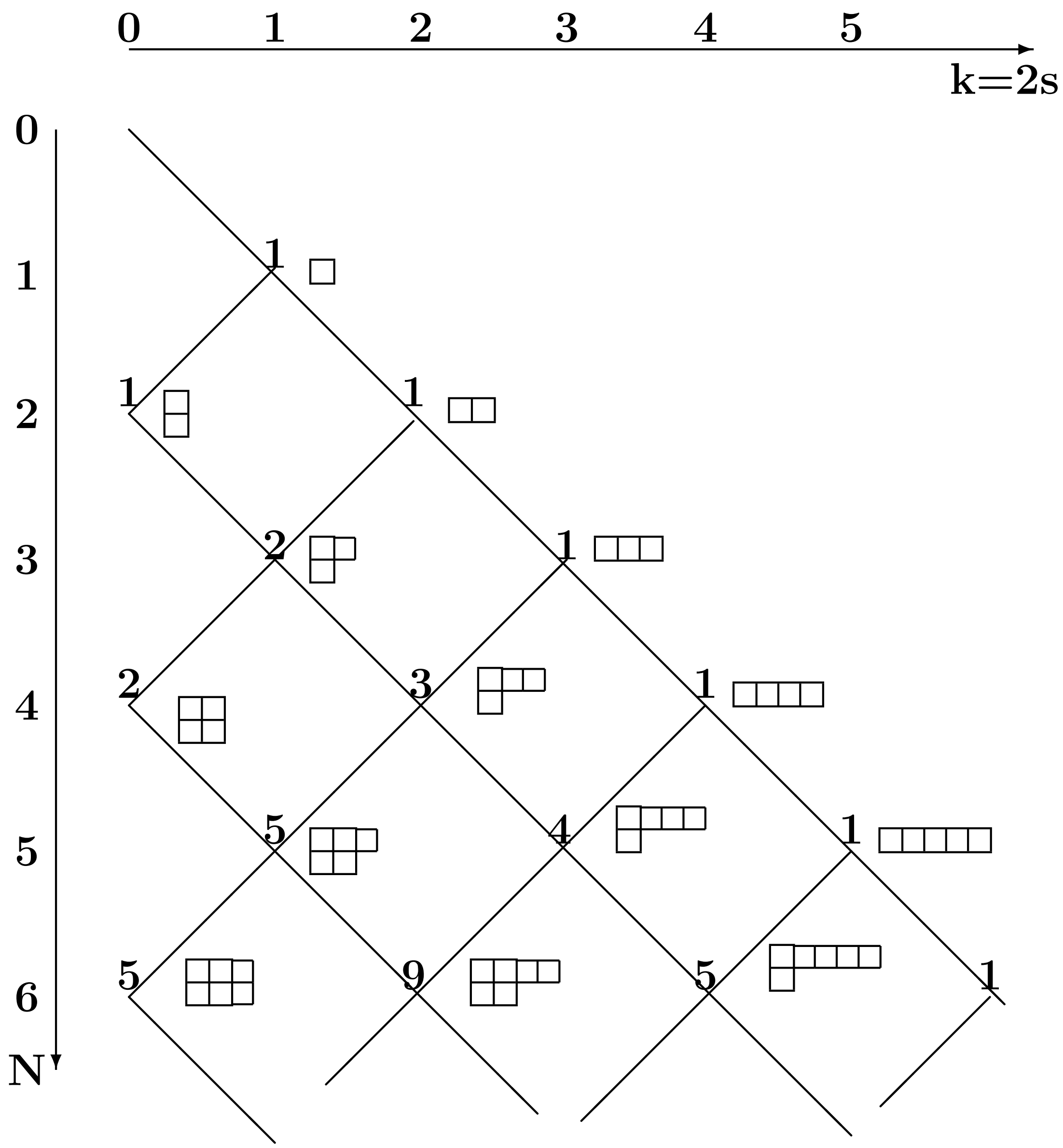


Realization on $\mathcal{H} = \bigotimes_1^N \mathbb{C}^2$ only TWO ROW Y-diagrams

Realization on $\mathcal{H} = \bigotimes_1^N \mathbb{C}^3$ only THREE ROW Y-diagrams

Schur - Weyl duality $sl(2)$-case

$$\mathcal{H} = \bigotimes_{1}^{N} \mathbb{C}^2$$

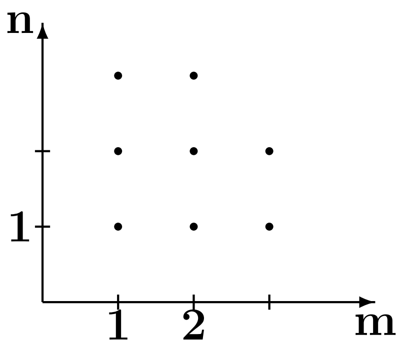


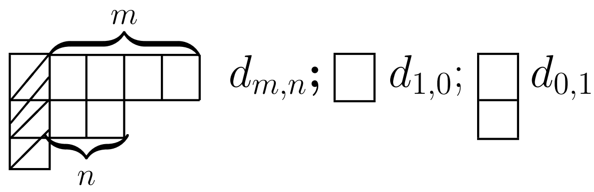
Number in the corners — multiplicity

$\lambda \vdash N, \quad (\lambda_1, \lambda_2 | \lambda_1 + \lambda_2 = N), \quad n = \lambda_2, \quad C_N^n - C_N^{n-1}$

(Catalan numbers $n = N/2$) $\quad p_k \cdot p_1 = p_{k+1} + p_{k-1}$

$sl(3)$ case





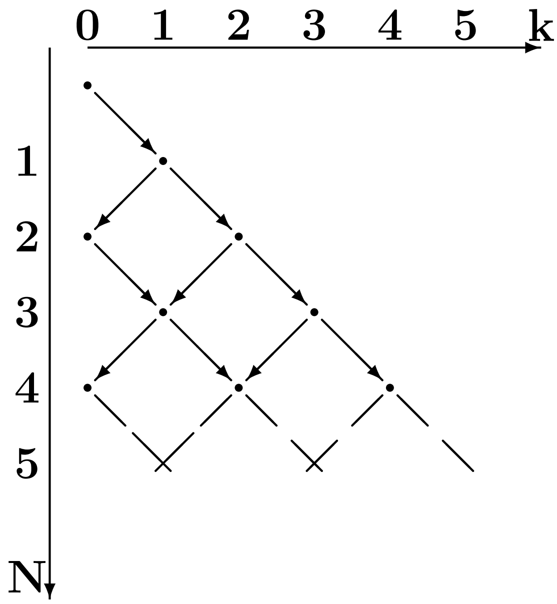
$$\begin{cases} d_{m,n} \cdot d_{1,0} = d_{m+1,n} + d_{m,n+1} + d_{m-1,n} \\ d_{m,n} \cdot d_{0,1} = d_{m+1,n} + d_{m,n+1} + d_{m-1,n} \end{cases}$$

$$sl_q(3) \sim d_{1,0} = d_{0,1} = 3 \sim \mathbb{C}^3; \quad d_{1,0} = d_{0,1} = n > 3 \sim ?$$

Tensor category of $sl(2)$ finite dim. reps

$$sl(2), \quad \left\{S^z, X^+, X^-\right\}, \quad s = 0, \frac{1}{2}, 1, \ldots;$$

$$\text{IrReps}\, V_{l,s} \quad \text{dim} V_k = k+1 \quad 2s = k \in \mathbb{Z}_{\geq 0}$$



$$V_1 \otimes V_1 = V_0 \oplus V_2;$$

$$V_1 \otimes V_k = V_{k-1} \oplus V_{k+1};$$

$$V_1^{\otimes N} = \bigoplus_{k=0,1}^{N} V_k \otimes \mathbb{C}^{\nu_k};$$

$$V_l \otimes V_k = \bigoplus_{|l-k|}^{l+k} V_m$$

Multiplicity free "C-G"-decomposition.

This "ring of representations" correspond to some quantum algebra $\mathfrak{b}_n$ if one starts from $V_0 \simeq \mathbb{C}^1, \quad V_1 \simeq \mathbb{C}^n$ (e.g. $\text{dim} V_2 = n^2 - 1$)

$$n p_k = p_{k+1} + p_{k-1}, \quad p_{-1} = 0,\ p_0 = 1 \qquad \text{dim} V_k = p_k(n).$$

p_k — Chebyshev polynomial of the 2-nd kind.

V_k as corepresentations of dual Hopf algebra and FRT-formalism

$$R_{1\,2} T_1 T_2 = T_2 T_1 R_{1\,2}$$

Girardeau $\geq$ 1990; Dubois-Viollet, Bichon, Etingof, Ostrik,...

Coboundary Twists and the Jordanian deformation

Coboundary twists of a Hopf algebra $\mathcal{U}$:

$$\mathcal{F}^{(cob)} = (u \otimes u)\Delta(u^{-1}).$$

The twisted Hopf algebra has coproduct

$$\widetilde{\Delta} = \mathcal{F}^{(cob)}\Delta(\mathcal{F}^{(cob)})^{-1},$$

and is isomorphic as algbra to the original one. The universal R-matrix of $\mathcal{U}$ is transformed with this twist just by the similarity transformation

$$\mathcal{R} \to \mathrm{Ad}(u \otimes u)\mathcal{R}.$$

$$\mathcal{F}^{(cob)}(q,t) = (u \otimes u)\Delta(u^{-1}) \in \mathcal{U}_q(sl(2)) \otimes \mathcal{U}_q(sl(2))$$

Note that $\mathcal{F}$ is nonsingular in the limit $q \to 1$, while the corresponding element $u(q,t)$ is singular. This coboundary twist in the $q \to 1$ limit is no more a coboundary and leads to the jordanian twist $\mathcal{F}^{(j)}$.

An element $u(q,t)$ with these properties is $(X^+ \in \mathcal{U}_q(sl(2)))$

$$u(q,t) = \exp_{q^2}(\frac{t}{1-q^2}X^+), \tag{1}$$

where

$$\exp_q(x) := \sum_{n=1}^{\infty} \frac{x^n}{(n)_q!} = \exp(\sum_{n=1}^{\infty} \frac{(1-q)^{n-1}\, x^n}{n\,(n)_q}), \tag{2}$$

and $(n)_q := (1-q^n)/(1-q)$, $(\exp_q(x))^{-1} = \exp_{q^{-1}}(-x)$.

Since $\Delta(X^+) = X^+ \otimes 1 + K^{-2} \otimes X^+$the coboundary twist is

$$\mathcal{F}^{(cob)}(q) =$$
$$= \exp_{q^2}(\frac{t}{1-q^2}X^+) \otimes \exp_{q^2}(\frac{t}{1-q^2}X^+)$$
$$\exp_{q^{-2}}\left(-\frac{t}{1-q^2}(X^+ \otimes 1 + K^{-2} \otimes X^+)\right). \quad (3)$$

Using a functional equation for the q-exponential

$$\exp_q(x+y) = \exp_q(x)\exp_q(y); \quad \text{provided that} \quad yx = qxy.$$

and commutation relations $K^{-2}X^+ = q^{-2}X^+K^{-2}$ we can factorize the third q-exponential in (3). Then the expression for $\mathcal{F}^{(cob)}(q)$ is

$$\mathcal{F}^{(cob)}(q) = \exp_{q^2}\big(\frac{t}{1-q^2}1 \otimes X^+\big)\ \exp_{q^{-2}}\big(-\frac{t}{1-q^2}(K^{-2} \otimes X^+)\big).$$

Using the representation of the q-exponential as standard exponential of the q-dilogarithm (2), and realization $K^2 = q^h$ one can show that there are no singular terms in $\mathcal{F}^{(cob)}(q)$ in the limit $q \to 1$. The explicit expression is

$$\lim_{q\to 1}\mathcal{F}^{(cob)}(q) = \exp(\sum_{n=1}^{\infty} -\frac{1}{2}h \otimes \frac{(tX^+)^n}{n}) = \exp\big(\frac{1}{2}h \otimes \ln(1 - tX^+)\big),$$

which gives for $t = -2\xi$ the jordanian twist $\mathcal{F}^{(j)}$:

$$\mathcal{F}^{(j)} = \exp(\frac{1}{2}h \otimes \ln(1 + 2\xi X^+)),$$

where h, $X^\pm$ are the generators of the Lie algebra $sl(2)$.

R-matrix of XXZ spin-1chain

$$R(\lambda,\eta)=\left(\begin{array}{ccc|ccc|ccc}
a_1 & & & & & & & & \\
 & a_2 & & b_1 & & & & & \\
 & & a_3 & & b_2 & & b_3 & & \\
\hline
 & c_1 & & a_2 & & & & & \\
 & & c_2 & & a_4 & & b_2 & & \\
 & & & & & a_2 & & b_1 & \\
\hline
 & & c_3 & & c_2 & & a_3 & & \\
 & & & & & c_1 & & a_2 & \\
 & & & & & & & & a_1
\end{array}\right), \tag{4}$$

where the functions are

$$\begin{aligned}
a_1 &= \sinh(\lambda+\eta)\sinh(\lambda+2\eta), & b_2 &= e^{\lambda}\sinh\lambda\sinh 2\eta,\\
a_2 &= \sinh\lambda\sinh(\lambda+\eta), & b_3 &= e^{2\lambda}\sinh\eta\sinh 2\eta,\\
a_3 &= \sinh\lambda\sinh(\lambda-\eta), & c_1 &= e^{-\lambda}\sinh(\lambda+\eta)\sinh 2\eta,\\
a_4 &= \sinh\lambda\sinh(\lambda+\eta)+\sinh\eta\sinh 2\eta, & c_2 &= e^{-\lambda}\sinh\lambda\sinh 2\eta,\\
b_1 &= e^{\lambda}\sinh(\lambda+\eta)\sinh 2\eta, & c_3 &= e^{-2\lambda}\sinh\eta\sinh 2\eta.
\end{aligned}$$

The R-matrix satisfies the YB-eq in the space $\mathbb{C}^3\otimes\mathbb{C}^3\otimes\mathbb{C}^3$

$$R_{12}(\lambda)R_{13}(\lambda+\mu)R_{23}(\mu)=R_{23}(\mu)R_{13}(\lambda+\mu)R_{23}(\lambda), \tag{5}$$

where we use the standard notation of the QISM

Relation with the symmetric form $R^t_{12}(\lambda,\eta)=R_{12}(\lambda,\eta)$ by the similarity transformation

$$R_{12}(\lambda,\eta)\to \mathrm{Ad}\exp(\alpha\lambda(h_1-h_2))R_{12}(\lambda,\eta), \tag{6}$$

with $\alpha = \frac{1}{2}$ and $h = \text{diag}(1,0,-1)$. The transformed R-matrix obeys the YB-equation due to the $U(1)$ symmetry

$$[h_1 + h_2, R_{12}(\lambda,\eta)] = 0. \tag{7}$$

The R-matrix (4) has a few important properties: regularity, unitarity, PT- and crossing- symmetries.

$$R(0,\eta) = \sinh(\eta)\sinh(2\eta)\mathcal{P}, \tag{8}$$

where $\mathcal{P}$ is the permutation matrix of $\mathbb{C}^3 \otimes \mathbb{C}^3$. The unitarity relation

$$R_{12}(\lambda)R_{21}(-\lambda) = \rho(\lambda)I, \tag{9}$$

here $R_{21}(\lambda) = \mathcal{P}R_{12}(\lambda)\mathcal{P}$. The PT-symmetry

$$R_{12}^t(\lambda) = R_{21}(\lambda). \tag{10}$$

The crossing symmetry property

$$R(\lambda) = (Q \otimes I)\, R^{t_2}(-\lambda - \eta)\, (Q \otimes I)\,, \tag{11}$$

where the matrix Q is given by $$Q = \begin{pmatrix} 0 & 0 & -e^{-\eta} \\ 0 & 1 & 0 \\ -e^{\eta} & 0 & 0 \end{pmatrix}. \tag{12}$$

The R-matrix in the braid group form

$$\check{R}(\lambda,\eta) = \mathcal{P}R(\lambda,\eta), \tag{13}$$

admits the spectral decomposition

$$\check{R}(\lambda,\eta) = \rho_5(\lambda,\eta)P_5(\eta) + \rho_3(\lambda,\eta)P_3(\eta) + \rho_1(\lambda,\eta)P_1(\eta), \tag{14}$$

$$P_5(\eta) = I - P_3(\eta) - P_1(\eta), \tag{15}$$

$$P_3(\eta) = \frac{1}{e^{2\eta} + e^{-2\eta}} \left(\begin{array}{ccc|ccc|ccc} 0 & & & & & & & & \\ & e^{2\eta} & & -1 & & & & & \\ & & 1 & & \omega & & -1 & & \\ \hline & -1 & & e^{-2\eta} & & & & & \\ & & \omega & & \omega^2 & & -\omega & & \\ & & & & & e^{2\eta} & & -1 & \\ \hline & & -1 & & -\omega & & 1 & & \\ & & & & & -1 & & e^{-2\eta} & \\ & & & & & & & & 0 \end{array} \right), \tag{16}$$

here $\omega(e^{\eta}) = e^{\eta} - e^{-\eta}$ and

$$P_1(\eta) = \frac{1}{e^{2\eta} + 1 + e^{-2\eta}} \left(\begin{array}{ccc|ccc|ccc} 0 & & & & & & & & \\ & 0 & & & & & & & \\ & & e^{2\eta} & & -e^{\eta} & & 1 & & \\ \hline & & & 0 & & & & & \\ & & -e^{\eta} & & 1 & & -e^{-\eta} & & \\ & & & & & 0 & & & \\ \hline & & 1 & & -e^{-\eta} & & e^{-2\eta} & & \\ & & & & & & & 0 & \\ & & & & & & & & 0 \end{array} \right). \tag{17}$$

The R-matrix (4) has four degeneration points $\lambda = \pm\eta$, and $\lambda = \pm 2\eta$.

The R-matrix (13) can also be expressed in the following form

$$\check{R}(\lambda,\eta) = \frac{e^{\eta}}{4}\left(e^{2\lambda} - 1\right)\check{R}(\eta) + (\sinh\eta \sinh 2\eta)\, I + \frac{e^{-\eta}}{4}\left(e^{-2\lambda} - 1\right)\check{R}^{-1}(\eta). \tag{18}$$

The constant R-matrix

$$\check{R}^{\pm 1}(\eta) = \lim_{\lambda \to \pm\infty} \big(4 \exp(\mp(2\lambda + \eta))\check{R}(\lambda, \eta)\big) \tag{19}$$

being a solution of the YB-equation in the braid group form

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23}, \tag{20}$$

has the spectral decomposition ($q = e^{2\eta}$)

$$\check{R}(\eta) = qP_5(\eta) - \frac{1}{q}P_3(\eta) + \frac{1}{q^2}P_1(\eta). \tag{21}$$

Hence, $\check{R}(\eta)$ satisfies the cubic equation

$$\big(\check{R}(\eta) - qI\big)\left(\check{R}(\eta) + \frac{1}{q}I\right)\left(\check{R}(\eta) - \frac{1}{q^2}I\right) = 0. \tag{22}$$

Its matrix form is

$$\check{R}(\eta) = \left(\begin{array}{ccc|ccc|ccc}
e^{2\eta} & & & & & & & & \\
 & 0 & & 1 & & & & & \\
 & & 0 & & & & e^{-2\eta} & & \\
\hline
 & 1 & & \omega & & & & & \\
 & & & & 1 & & e^{-\eta}\omega & & \\
 & & & & & 0 & & 1 & \\
\hline
 & & e^{-2\eta} & & e^{-\eta}\omega & & (1-e^{-2\eta})\omega & & \\
 & & & & & 1 & & \omega & \\
 & & & & & & & & e^{2\eta}
\end{array}\right), \tag{23}$$

here $\omega(e^{2\eta}) = e^{2\eta} - e^{-2\eta}$.

For the purpose of establishing a relation with the Birman-Wenzl-Murakami algebra, the one dimensional projector $P_1(\eta)$ is related to the rank one matrix $\mathcal{E}(\eta) = \mu P_1(\eta)$ with $\mu = q+1+1/q$ and $q = e^{2\eta}$, which satisfies

$$\mathcal{E}^2(\eta) = \mu \mathcal{E}(\eta), \tag{24}$$

$$\check{R}(\eta)\mathcal{E}(\eta) = \mathcal{E}(\eta)\check{R}(\eta) = \frac{1}{q^2}\mathcal{E}(\eta), \tag{25}$$

and also

$$\check{R}(\eta) - \check{R}^{-1}(\eta) = \omega(q)\left(I - \mathcal{E}(\eta)\right), \tag{26}$$

where $\omega(q) = q - 1/q$. From these relations we conclude that $\check{R}$, $\check{R}^{-1}$ and $\mathcal{E}$ provide a realisation of the Birman-Wenzl-Murakami algebra $W_N(q, 1/q^2)$ in the space $\mathcal{H} = \otimes_1^N \mathbb{C}^3$.

The projector $P_5(\eta)$ on five dimentional subspace of $\mathbb{C}^3 \otimes \mathbb{C}^3$ corresponds to a symmetrizer on spin 2 irreducible representation of the quantum algebra $\mathcal{U}_q(o(3))$. It can be used to construct an R-matrix for higher spin $R^{(2,1)}(\lambda, \eta) \in \mathrm{End}(\mathbb{C}^5 \otimes \mathbb{C}^3)$ by the fusion procedure

$$R^{(2,1)}(\lambda, \eta) \simeq \check{R}_{12}(2\eta, \eta) R_{13}(\lambda + \eta, \eta) R_{23}(\lambda - \eta, \eta). \tag{27}$$

One can use higher symmetrizers of the BMW-algebra $W_s(q, 1/q^2)$ to get R-matrices $R^{(s,1)}(\lambda, \eta) \in \mathrm{End}(\mathbb{C}^{(2s+1)} \otimes \mathbb{C}^3)$.

Birman-Wenzl-Murakami algebra $W_N(q,\nu)$

The defining relations of the BMW algebra $W_N(q,\nu)$, for the generators 1, σ_i , σ_i^{-1} and e_i, $i=1,\ldots,N-1$, are recalled for convenience,

$$\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \sigma_i\sigma_j=\sigma_j\sigma_i, \text{ for } |i-j|>1, \tag{28}$$

$$e_i\sigma_i=\sigma_i e_i=\nu e_i, \tag{29}$$

$$e_i\sigma_{i-1}^{\pm 1}e_i=\nu^{\mp 1}e_i, \tag{30}$$

$$\sigma_i-\sigma_i^{-1}=\omega(q)(1-e_i). \tag{31}$$

It can be shown that the dimension of the BMW-algebra $W_N(q,\nu)$ is $\dim W_N(q,\nu)=(2N-1)!!$.

Many useful relations follow from the definition above

$$e_i{}^2=\mu e_i, \quad \text{with} \quad \mu=\frac{\omega-\nu+1/\nu}{\omega}=\frac{(q-\nu)(\nu+1/q)}{\nu\omega}. \tag{32}$$

Another important consequence of the relations (29,28) is

$$(\sigma_i-q)(\sigma_i+q^{-1})(\sigma_i-\nu)=0. \tag{33}$$

There is the natural inclusion of $W_M(q,\nu)\subset W_N(q,\nu)$, $M<N$.

The Yang-Baxterization procedure yields two spectral parameter dependent elements

$$\sigma_i^{(\pm)}(u)=\frac{1}{\omega}\left(u^{-1}\sigma_i-u\,\sigma_i^{-1}\right)+\frac{\nu\pm q^{\pm 1}}{u\nu\pm q^{\pm 1}u^{-1}}e_i. \tag{34}$$

These elements satisfy the YB-equation in the braid group form

$$\sigma_i^{(\pm)}(u)\sigma_{i+1}^{(\pm)}(uv)\sigma_i^{(\pm)}(v)=\sigma_{i+1}^{(\pm)}(v)\sigma_i^{(\pm)}(uv)\sigma_{i+1}^{(\pm)}(u). \tag{35}$$

Their unitarity relation is

$$\sigma_i^{(\pm)}(u)\sigma_i^{(\pm)}(u^{-1})=\left(1-\omega^{-2}(u-u^{-1})^2\right). \tag{36}$$

In order to see the connection with the previous formulas we set $\nu = 1/q^2$ and find that

$$\sigma_i^{(-)}(e^{-\lambda}) \simeq \check{R}_{i,i+1}(\lambda, \eta)$$

of (18) and

$$\sigma_i^{(+)}(e^{\lambda/2}) \simeq \check{R}_{i,i+1}(\lambda, \eta)$$

of $A_2^{(2)}$-case.

The irreducible representations of the BMW algebra $W_N(q,\nu)$ are more complicated than the irreps of the symmetric group $\mathfrak{S}_N$ or the Hecke algebra $\mathcal{H}_N(q)$, although they can be parameterized by the Young diagrams. The simplest, one-dimensional irreps of $W_N(q,\nu)$ are defined by the symmetrizer and antisymmetrizer, respectively. The symmetrizer of the $W_N(q,\nu)$ is given by

$$\mathcal{S}_N = \frac{1}{[N]_q!}\sigma_1^{(-)}(q^{-1})\sigma_2^{(-)}(q^{-2})\cdots\sigma_{N-1}^{(-)}(q^{-(N-1)})\mathcal{S}_{N-1}, \qquad (37)$$

$$\mathcal{S}_1 = 1, \qquad \mathcal{S}_2 = \frac{1}{[2]_q}\sigma_1^{(-)}(q^{-1}). \qquad (38)$$

We use the standard notation for the q-factorial $[n]_q! = [n]_q[n-1]_q\cdots[2]_q[1]_q$ and the q-numbers $[n]_q = (q^n - q^{-n})/(q - q^{-1})$. The elements $\mathcal{S}_n$, $n = 1,\ldots,N$ are idempotents, i.e. $\mathcal{S}_n^2 = \mathcal{S}_n$. In addition, the symmetrizer $\mathcal{S}_N$ is also central.

In the realisation on $\mathbb{C}^3 \otimes \mathbb{C}^3$ of the BMW algebra $W_2(q, q^{-2})$

$$\sigma_1 = \check{R}(\eta) = qP_5 - q^{-1}P_3 + \nu P_1, \quad \nu = \frac{1}{q^2}, \qquad (39)$$

and e_1 is proportional to the rank one projector P_1

$$e_1 = \mu P_1 = (q + 1 + q^{-1})P_1. \qquad (40)$$

Thus

$$\sigma_1^{(-)}(q^{-1}) = (q+q^{-1})P_5, \tag{41}$$

$$\sigma_1^{\pm 1}P_5 = q^{\pm 1}P_5, \tag{42}$$

$$e_1 P_5 = 0. \tag{43}$$

Similarly, the antisymmetrizer of the $W_N(q,\nu)$ is given by

$$\mathcal{A}_N = \frac{1}{[N]_q!}\sigma_1^{(+)}(q)\sigma_2^{(+)}(q^2)\cdots\sigma_{N-1}^{(+)}(q^{N-1})\mathcal{A}_{N-1}, \tag{44}$$

with

$$\mathcal{A}_1 = 1, \qquad \mathcal{A}_2 = \frac{1}{[2]_q}\sigma_1^{(+)}(q). \tag{45}$$

The elements $\mathcal{A}_n$, $n = 1,\ldots,N$ are idempotents and the antisymmetrizer $\mathcal{A}_N$ is also central in $W_N(q,\nu)$.

$$\sigma_1^{(+)}(q)\sigma_1^{(+)}(q) = [2]_q\sigma_1^{(+)}(q). \tag{46}$$

It is straightforward to see that

$$\mathcal{A}_3 \simeq \sigma_1^{(+)}(q)\sigma_2^{(+)}(q^2)\sigma_1^{(+)}(q) = \sigma_2^{(+)}(q)\sigma_1^{(+)}(q^2)\sigma_2^{(+)}(q). \tag{47}$$

In the realisation (39,13)

$$\sigma_1^{(+)}(q) = [2]_q P_3, \tag{48}$$

$$\sigma_1^{\pm 1}P_3 = -q^{\mp 1}P_3, \tag{49}$$

$$e_1 P_3 = 0. \tag{50}$$

In addition, in this realisation, the antisymmetrizer $\mathcal{A}_3$ has rank one. A straightforward calculation yields $\mathcal{A}_4 = 0$. Consequently all the higher antisymmetrizers vanish identically for $n > 4$.

In a general case of $W_N(q,\nu)$, it can be shown that the following identities are valid

$$\sigma_i^{(-)}(q)\mathcal{S}_n = \mathcal{S}_n\sigma_i^{(-)}(q) = 0, \tag{51}$$

$$\sigma_i^{(+)}(q^{-1})\mathcal{A}_n = \mathcal{A}_n\sigma_i^{(+)}(q^{-1}) = 0, \tag{52}$$

for $i = 1,\ldots,n-1$ and $1 < n \leqslant N$. The relations (51,25) can also be written in the following form

$$\sigma_i\mathcal{S}_n = \mathcal{S}_n\sigma_i = q\mathcal{S}_n, \tag{53}$$

$$e_i\mathcal{S}_n = \mathcal{S}_n e_i = 0, \tag{54}$$

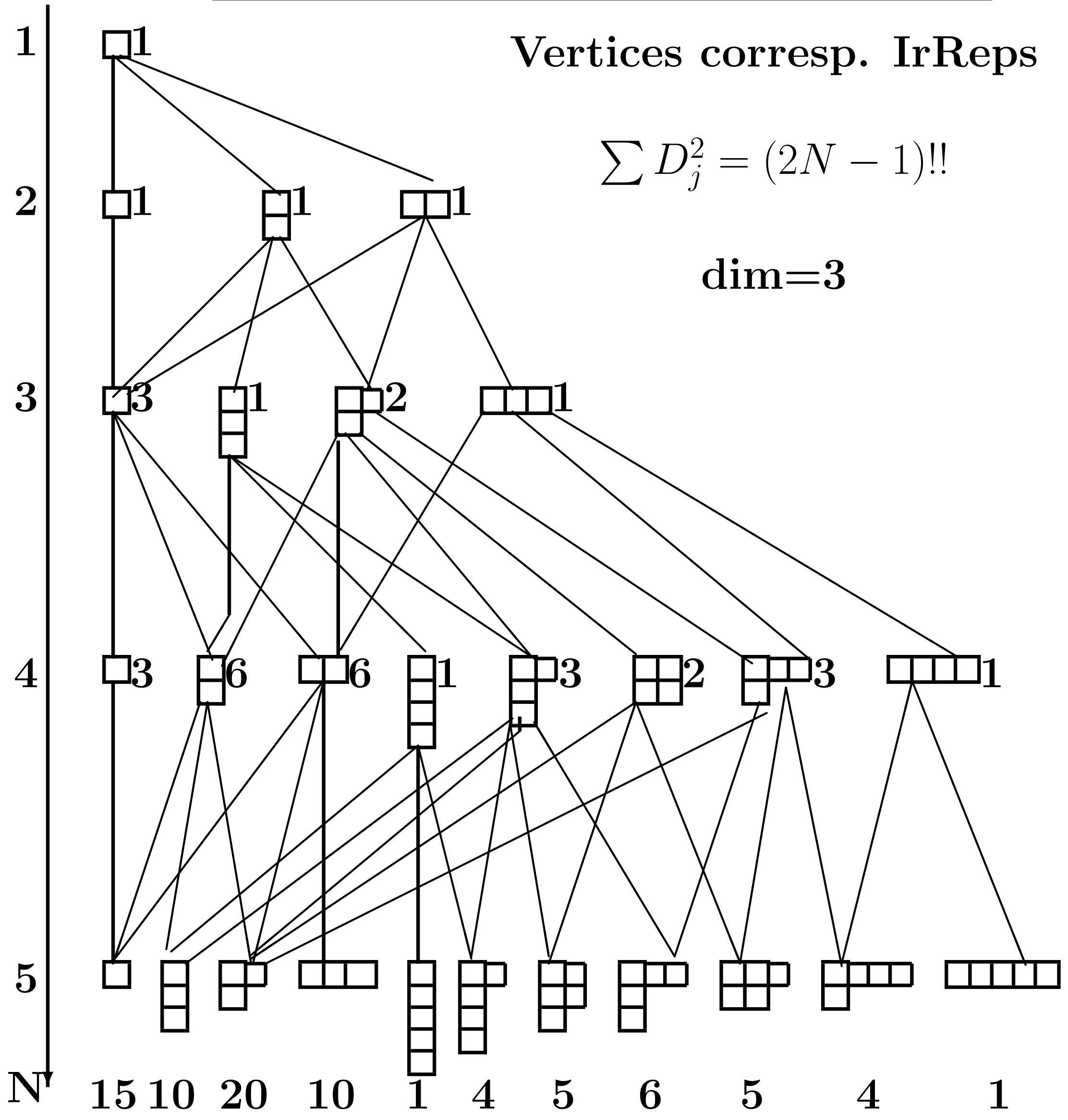
$$\sigma_i\mathcal{A}_n = \mathcal{A}_n\sigma_i = -\frac{1}{q}\mathcal{A}_n, \tag{55}$$

$$e_i\mathcal{A}_n = \mathcal{A}_n e_i = 0, \tag{56}$$

for $i = 1,\ldots,n-1$ and $1 < n \leqslant N$. From these identities it is evident that $\mathcal{S}_N$ and $\mathcal{A}_N$ are central in $W_N(q,\nu)$. Also, using the relations (53-29), it is straightforward to check that $\mathcal{S}_n$ and $\mathcal{A}_n$ are idempotents, i.e. $\mathcal{S}_n^2 = \mathcal{S}_n$ and $\mathcal{A}_n^2 = \mathcal{A}_n$, $n = 1,\ldots,N$.

The BMW algebra $W_N(q,q^{-2})$ can be used to describe the multiplet structure of the spectra of some open quantum spin chains.

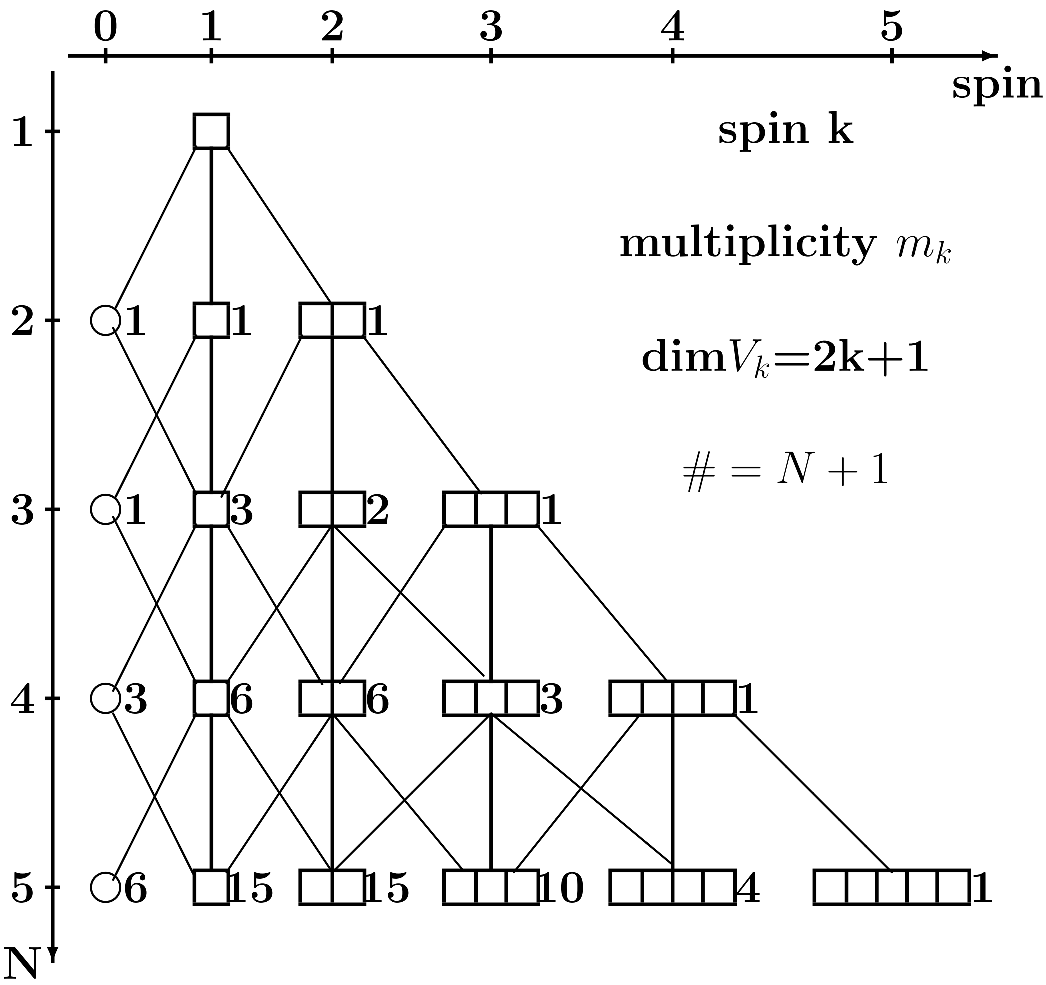
Bratteli diagram of BMW-algebra $W_N(q,\nu)$



Irreducible representations of $\mathcal{U}_q(so(3))$

$$\mathcal{H} = \bigotimes_1^N \mathbb{C}^3 = \sum_0^N V_k \otimes \mathbb{C}^{m_k}$$

$$V_k \otimes V_1 = V_{k-1} \oplus V_k \oplus V_{k+1}$$



Open Spin Chain

According to the QISM the R-matrix $R(u,q)$ can be used to construct an auxiliary L-operator

$$L_{0j}(u) = R_{0j}(u,q). \tag{57}$$

Notice that now we use the multiplicative spectral parameter, which in the case of the model XXZ_1 is given by $u = \exp(-\lambda)$. Then the monodromy matrix of a spin chain with N sites is the product of L-matrices in $\mathrm{End}(V_0)$ whose entries are in $\mathrm{End}(V_j)$

$$T(u) = L_{0\,N}(u)L_{0\,N-1}(u)\cdots L_{0\,1}(u), \tag{58}$$

while the entries of $T_{ab}(u)$ are operators on the whole space of states $\mathcal{H} = \otimes_{j=1}^{N} V_j$. As a consequence of the YB-eq one has

$$R_{00'}\left(\frac{u}{w}\right) L_{0j}(u)L_{0'j}(w) = L_{0'j}(w)L_{0j}(u)R_{00'}\left(\frac{u}{w}\right) \tag{59}$$

and

$$R_{12}\left(\frac{u}{w}\right) T_1(u)T_2(w) = T_2(w)T_1(u)R_{12}\left(\frac{u}{w}\right). \tag{60}$$

The transfer matrix

$$t(u) = \mathrm{tr}_0 T(u), \tag{61}$$

is the generating function of the integrals of motion with the periodic boundary condition.

For non-periodic boundary condition one has to use the Sklyanin formalism. The monodromy matrix $\mathcal{T}(u)$ consists of the two matrices $T(u)$ (58) and a reflection matrix $K^-(u) \in \mathrm{End}(V)$

$$\mathcal{T}(u) = T(u)K^-(u)T^{-1}(u^{-1}). \tag{62}$$

Using the unitarity relation (9) ($R_{12}^{-1}(u^{-1}) = R_{21}(u)$) one gets

$$T^{-1}(u^{-1}) = R_{1\,0}(u)R_{2\,0}(u)\cdots R_{N\,0}(u). \tag{63}$$

Taking into account $R_{12}(u,\eta) = \mathcal{P}_{12}R_{21}(u,\eta)\mathcal{P}_{12}$ one gets

$$\mathcal{T}(u) = \check{R}_{N\,0}(u)\check{R}_{N-1\,N}(u)\cdots\check{R}_{1\,2}(u)K_1^-(u)\check{R}_{1\,2}(u)\check{R}_{2\,3}(u)\cdots\check{R}_{N\,0}(u). \tag{64}$$

The generating function $\tau(u)$ of the integrals of motion is (with an extra reflection matrix $K^+(u)$)

$$\tau(u) = \mathrm{tr}_0\left(K_0^+(u)\mathcal{T}(u)\right). \tag{65}$$

The reflection matrices $K^{\pm}(u)$ are solutions to the reflection equation In particular, the Hamiltonian is given by $H = \frac{1}{2}\frac{d}{du}\ln\tau(u)|_{u=1}$,

$$H = \sum_{i=1}^{N-1}\check{R}'_{i,i+1}(1) + \frac{\mathrm{tr}_0K_0^+(1)\check{R}'_{N\,0}(1)}{\mathrm{tr}_0K_0^+(1)} + \\ + \frac{1}{2}\left(\frac{dK_1^-(1)}{du} + \frac{1}{\mathrm{tr}_0K_0^+(1)}\frac{d\,\mathrm{tr}_0K_0^+(1)}{du}\right). \tag{66}$$

The Hamiltonian density $h_{i,i+1} = \frac{d}{du}\check{R}_{i,i+1}(u)|_{u=1}$ is a function of the generators of $W_N(q,q^{-2})$ on the space $\mathcal{H} = \otimes_1^N\mathbb{C}^3$. In our case we can take the constant K-matrices $K^-(u) = 1$ and $K^+(u) = Q^tQ$.

Asymptotic expansion of $T(u)$ at $u \to 0$ (or at $u \to \infty$) results in some matrices

$$T(u) = u^{-N}L_{0\,N}^-L_{0,N-1}^-\cdots L_{0\,1}^- + \mathcal{O}(u^{-N+1}). \tag{67}$$

Here the constant L-matrices L_{0j}^- are upper triangular matrices which coincide with the asymptotic limit $\lambda \to +\infty$ (19) of the R-matrices (4),

$L^-_{0j} = R^-_{0j} = \mathcal{P}_{0j}\check{R}_{0j}$. Hence, the YB-equation for the constant R-matrix is

$$R^-_{i,i+1}L^-_{0,i+1}L^-_{0i} = L^-_{0i}L^-_{0,i+1}R^-_{i,i+1}. \tag{68}$$

With $R^-_{i,i+1} = \mathcal{P}_{i,i+1}\check{R}_{i,i+1}$ and multiplying the previous equation by the permutation operator $\mathcal{P}_{i,i+1}$ one gets

$$\left[\check{R}_{i,i+1}, L^-_{0,i+1}L^-_{0i}\right] = 0. \tag{69}$$

It is then obvious that $\rho_W(\sigma_i) = \check{R}_{i,i+1}$, $\rho_W(e_i) = \mu\left(P_1(\eta)\right)_{i,i+1}$. The representation ρ_W of the generators of the BMW algebra $W_N(q, q^{-2})$ in the space $\mathcal{H} = \otimes_1^N \mathbb{C}^3$, commute with the generators T^-_{ab} of the global (or diagonal) action of the quantum algebra $\mathcal{U}_q(o(3))$ on the space $\mathcal{H}$

$$\left[\check{R}_{i,i+1}, T^-\right] = 0, \quad T^- = L^-_{0\,N}L^-_{0,N-1}\cdots L^-_{0\,1}. \tag{70}$$

This product of L^-_{0j} can be represented as the image of a multiple co-product map $\Delta^N : \mathcal{U}_q(o(3)) \to (\mathcal{U}_q(o(3)))^{\otimes N}$

$$T^- = (\mathrm{id}\otimes\rho_W)(\mathrm{id}\otimes\Delta^N)\mathcal{L}^-_0. \tag{71}$$

Analogously, the asymptotic expansion of $T(u)$ at $u \to \infty$ yields the matrix $T^+ = L^+_{0\,N}L^+_{0\,N-1}\cdots L^+_{0\,1}$ (cf. (67)).

It is known that in the space $\mathcal{H}$ as a space of representation of $\mathcal{U}_q(o(3))$ and $W_N(q, q^{-2})$ these algebras are mutual centralizers.

According to the centralizer property this induces the decomposition of the representation space $\mathcal{H}$ into direct sum of irreps of both algebras, being a generalisation of the Schur-Weyl duality:

$$\mathcal{H} = \sum_{s=0}^{N} V_s \otimes U_s, \tag{72}$$

where V_s is the $(2s+1)$-dimensional irreducible representation of $\mathcal{U}_q(o(3))$ while U_s is corresponding irrep of $W_N(q,q^{-2})$. The dimension of an irrep of $W_N(q,q^{-2})$ is equal to the multiplicity m of the corresponding irrep of centralizer algebra $\mathcal{U}_q(o(3))$, and vice versa

$$m(V_s) = \dim U_s, \quad m(U_s) = \dim V_s. \tag{73}$$

The dimension of the irrep V_s of $\mathcal{U}_q(o(3))$ and the number n of the inequivalent irreps in the decomposition (72) are well known. It follows from the decomposition of the tensor product of the spin 1 representations of $o(3)$: $\dim V_s = 2s+1$,

$$n_N = N+1, \quad m_N(V_s) = \sum_{j=s,s\pm 1} m_{N-1}(V_j), \quad s \neq 0, N-1, N, \tag{74}$$

together with $m_N(V_0) = m_{N-1}(V_1)$, $m_N(V_{N-1}) = 1 + m_{N-1}(V_{N-2}) = N-1$ and $m_N(V_N) = 1$. However, the number and the dimensions of representations U_s of $W_N(q,q^{-2})$ can be obtained from its Bratteli diagram.

For $N = 2,3$ the number of existing irreducible representations of $W_N(q,q^{-2})$ and those entering into the decomposition of the space of states are the same 3, 4, respectively, while for $N \geqslant 4$ there are more irreps of W_N than of $\mathcal{U}_q(o(3))$, for example $n_4(W) = 8$ while $n_4(\mathcal{U}_q(o(3))) = 5$.

The decomposition (72) permits to determine the structure of the multiplets of the Hamiltonian, which is an element of the BMW algebra

$W_N(q, q^{-2})$

$$H = \sum_{i=1}^{N-1} h_{i,i+1}, \quad h_{i,i+1} = \frac{d}{d\lambda}\check{R}(\lambda,\eta)|_{\lambda=0} = f(\check{R}_i) \in W_N(q, q^{-2}). \quad (75)$$

According to the QISM, the R-matrices being regular at $\lambda = 0$, define the local Hamiltonian density for two sites of the chains. For the XXZ_1-model one gets

$$\begin{aligned} h_{XXZ} &= \frac{d}{d\lambda}\check{R}(\lambda,\eta)|_{\lambda=0} \simeq q\check{R}(\eta) - \check{R}^{-1}(\eta) \\ &= (q-1)\left((q+1+\frac{1}{q})(P_5 - P_1) + P_3\right). \end{aligned} \quad (76)$$

In the $A_2^{(2)}$-case

$$h_A = \frac{d}{d\lambda}\check{R}(\lambda,\eta)|_{\lambda=0} \simeq q\check{R}(\eta) + \frac{1}{q^2}\check{R}^{-1}(\eta) = (q^2 + \frac{1}{q^3})P_5 + (1+\frac{1}{q})(P_1 - P_3). \quad (77)$$

The Hamiltonian of the open spin chain with N-sites is then given by

$$H = \sum_{i=1}^{N-1} h_{i,i+1}. \quad (78)$$

As an example let us consider the case of $N = 3$ sites when the algebra $W_3(q, 1/q^2)$ is realised in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

$$H = h_{12} + h_{23}. \quad (79)$$

It follows

$$H_{XXZ}\mathcal{S}_3 = 2(q+1+\frac{1}{q})\mathcal{S}_3, \quad (80)$$

$$H_{XXZ}\mathcal{A}_3 = 2\mathcal{A}_3 \quad (81)$$

and similarly for the H_A (77)

$$H_A\mathcal{S}_3 = 2(q^2 + \frac{1}{q^3})\mathcal{S}_3, \tag{82}$$

$$H_A\mathcal{A}_3 = -2(1 + \frac{1}{q})\mathcal{A}_3. \tag{83}$$

In the case $N = 3$ there are four irreps of W_3: two one-dimensional irreps generated by $\mathcal{S}_3$ and $\mathcal{A}_3$, respectively, the three-dimensional irrep d_3 (corresponding to the one-box Young diagram) and the two-dimensional irrep d_2 (corresponding to the three-box Young diagram with two rows). Thus the Hamiltonian being restricted to invariant subspaces can have up to seven distinct eigenvalues. Their multiplicities are obtained from the correspondence between the irreps of W_3 and the irreps of $\mathcal{U}_q(o(3))$:

$$U(\mathcal{S}_3) \sim V_3, \quad U(\mathcal{A}_3) \sim V_0 \quad U(d_3) \sim V_1 \quad U(d_2) \sim V_2. \tag{84}$$

The degeneracies of energy values are ($j = 1, 2, 3;\ k = 1, 2$)

$$m(\epsilon(\mathcal{S}_3)) = 7,\ m(\epsilon(\mathcal{A}_3)) = 1,\ m(\epsilon_j(d_3)) = 3,\ m(\epsilon_k(d_2)) = 5. \tag{85}$$

The exact values of the energy are obtained by direct calculations.

For the XXZ-model of spin 1 the corresponding expressions are

$$\epsilon(\mathcal{S}_3) = 2(q + 1 + \frac{1}{q}), \quad \epsilon(\mathcal{A}_3) = 2,$$
$$\epsilon_1(d_3) = 1, \quad \epsilon_{2,3}(d_3) = \left(\frac{1}{2} \pm \sqrt{\frac{1}{2} + 2(q + 3 + \frac{1}{q})}\right),$$
$$\epsilon_1(d_2) = (q + 1 + \frac{1}{q}), \quad \epsilon_2(d_2) = (q + 3 + \frac{1}{q}).$$

In the $A_2^{(2)}$-case the corresponding expressions are

$$\epsilon(\mathcal{S}_3) = 2(q^2 + \frac{1}{q^3}), \quad \epsilon(\mathcal{A}_3) = -2(1 + \frac{1}{q}),$$
$$\epsilon_1(d_3) = (q^2 + \frac{1}{q^3}),$$
$$\epsilon_{2,3}(d_3) = \frac{1}{2}\left((q^2 + \frac{1}{q^3}) \pm \sqrt{q^4 + 8q^2 - 8q + \frac{34}{q} - \frac{8}{q^3} + \frac{8}{q^4} + \frac{1}{q^6}}\right),$$
$$\epsilon_1(d_2) = (1 + \frac{1}{q})(q^2 - 1 + \frac{1}{q^2}),$$
$$\epsilon_2(d_2) = (1 + \frac{1}{q})(q^2 - 2q + 1 - \frac{2}{q} + \frac{1}{q^2}).$$