

Rogers dilogarithm and periodicity of cluster Y -systems

F. Chapoton

July 30, 2010

- 1 Some dilogarithm identities can be written in a special manner.
- 2 They are related to Y -systems and cluster algebras.
- 3 The cluster category lurks in the background.

First, the classical dilogarithm (Leonhard Euler)

$$\operatorname{Li}_2(x) = \sum_{n \geq 1} \frac{x^n}{n^2} = - \int_0^x \log(1-y) \frac{dy}{y}.$$

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Then, the Rogers dilogarithm (Leonard James Rogers)

$$L(x) = \operatorname{Li}_2(x) + \frac{1}{2} \log(x) \log(1-x).$$

see reference book by Leonard Lewin.

Dilogarithm values and identities

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Famous **Spence-Abel identity**
with hidden cyclic symmetry of order 5.

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First two equations in a family indexed by $n \in \mathbb{N}$:

$$\sum_{f \in \mathbb{Y}_n} L(1/(1+f)) = n L(1),$$

where \mathbb{Y}_n is a set of fractions in n variables.

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where

$$\mathbb{Y}_1 = \{x, 1/x\}$$

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In general, \mathbb{Y}_n has $n + \binom{n+1}{2}$ elements.

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One important development is the **cluster category** due to Buan-Marsh-Reineke-Reiten-Todorov for the general acyclic case and to Caldero-Ch. in a special case, with further developments by Claire Amiot.

Y-systems are sets of algebraic relations on variables $y_{i,n}$ that arose in the physics of conformal field theory (**Thermodynamic Bethe Ansatz**) in the works of Al. B. Zamolodchikov and many others.

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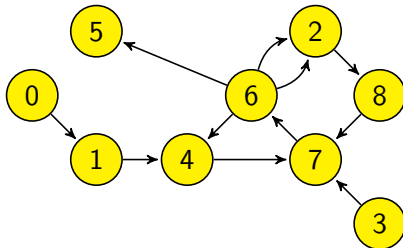
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As shown by Fomin and Zelevinsky, Y-systems are closely related to cluster algebras.

Cluster algebras from quivers



Initial data for a cluster algebra: quiver $Q = (Q_0, Q_1)$

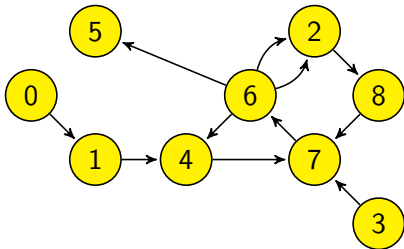
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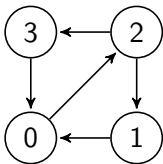
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One considers all quivers obtained from the initial one by **mutation of quivers**.

Mutation of quivers

One can mutate a quiver Q at any vertex $i \in Q_0$.

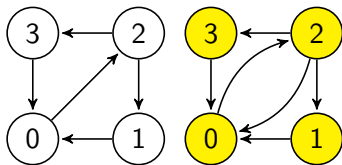


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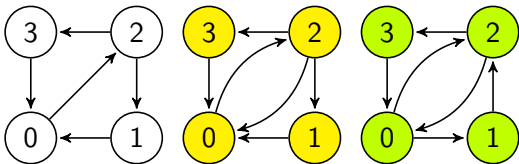
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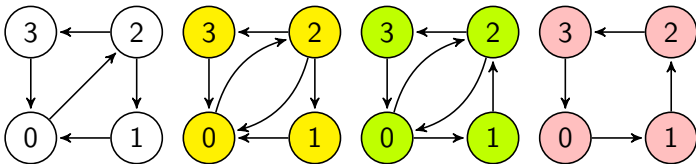
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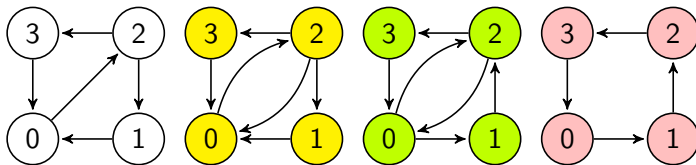
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This defines a new quiver $\mu_i(Q)$ and $\mu_i(\mu_i(Q)) = Q$.



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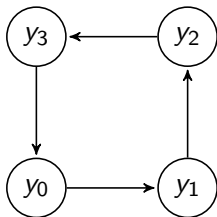
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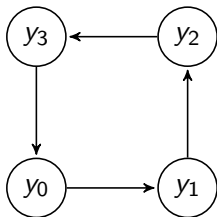
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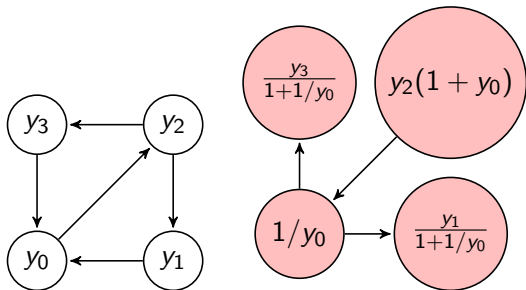
$$y'_j = \begin{cases} 1/y_i & \text{if } j = i, \\ y_j \prod_{i \rightarrow j} (1 + y_i) / \prod_{j \rightarrow i} (1 + 1/y_i) & \text{if } j \neq i. \end{cases}$$

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Q and $\mu_0(Q)$

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For example, for the quiver $\circ \leftarrow \circ$ of type \mathbb{A}_2 , with initial seed (y_1, y_2) one gets the following set of Y -variables:

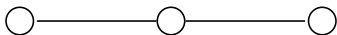
$$y_1, y_2, y_1(1 + y_2), (1 + y_1 + y_1y_2)/y_2, \frac{y_1 + 1}{y_1y_2}$$

and their inverses

$$1/y_1, 1/y_2, \frac{1}{y_1(1 + y_2)}, \frac{y_2}{1 + y_1 + y_1y_2}, \frac{y_1y_2}{y_1 + 1}.$$

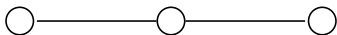
Precise setting for dilogarithm identities

Fix a Dynkin diagram of type $\mathbb{A}, \mathbb{D}, \mathbb{E}$. For example \mathbb{A}_3 .

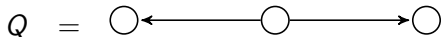


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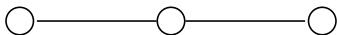


Let Q be an alternating orientation of this Dynkin diagram. For example,

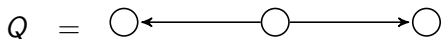


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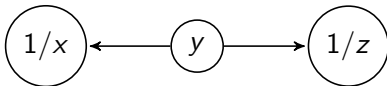
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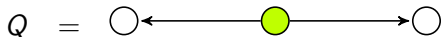


Choose the following initial Y -seed:

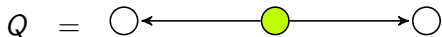


(variables y_i for sources, inverses $1/y_i$ for sinks)

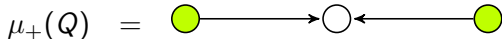
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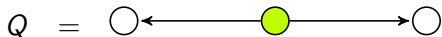
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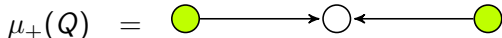
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Let μ_- be the composition of all mutations at the sources of $\mu_+(Q)$. One has $\mu_-(\mu_+(Q)) = Q$.

Theorem (Fomin & Zelevinsky)

The composition $\mu_- \circ \mu_+$ acting on Y -seeds is periodic. Let \mathbb{Y} be the subset of the Y -variables that correspond to sources in the Y -seeds obtained by iterating μ_+ and μ_- . The set \mathbb{Y} is made of Laurent polynomials in the variables y_i . The non-trivial denominators are in natural bijection with positive roots for the Dynkin diagram.

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For example, one gets in type A_3

$$x, y, z, \frac{x+z+xz+1}{y}, \frac{x+2y+z+y^2+xy+xz+yz+1}{xyz},$$
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Identity associated with a Dynkin diagram

Theorem (Ch.)

Let X_n be a simply-laced Dynkin diagram of rank n . Let $\mathbb{Y}(X_n)$ be the subset of the Y -system described before. Then

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Type \mathbb{D}_n gives another family of identities. Three other identities for \mathbb{E}_6 , \mathbb{E}_7 and \mathbb{E}_8 .

About the proof

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To compute the value of the sum, one takes the limit where all variables go to 0. Among elements of \mathbb{Y} , exactly n have limit 0 (hence add $L(1)$ to the sum) and the others have limit $+\infty$ (hence do not contribute to the sum). For example in type \mathbb{A}_3 :

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They use similar methods, namely the relation to cluster algebras and cluster categories.

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vector spaces M_i corresponding to vertices $i \in Q_0$ of Q ,

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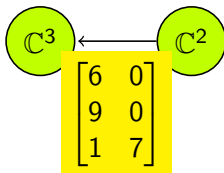
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For example, a representation of the quiver of type \mathbb{A}_2 is



The **dimension vector** of M is $\dim M = (\dim M_i)_{i \in Q_0}$.

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Introduce now the following polynomials in variables $(x_i)_i$:

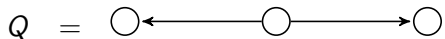
$$F_M = \sum_{e \leq m} \chi(\text{Gr}_e(M)) \prod_i x_i^{e_i},$$

where χ is the Euler characteristic.

Call them the **F -polynomials**.

Example of F -polynomial

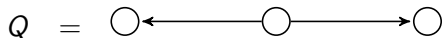
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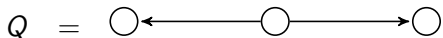


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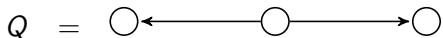
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Therefore the F -polynomial for this representation is

$$1 + x_1 + x_3 + x_1x_3 + x_1x_2x_3.$$

Fact

The polynomials F_M are sufficient to recover Y -variables.

In fact, the Y -variables are products and quotients of F -polynomials and monomials (There are explicit formulas. . .).

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One can also use the same F -polynomials to compute **cluster variables** (not defined in this talk, but even more important).

Cluster category

Cluster variables and Y -variables are closely related to F -polynomials, which are defined using the category $\text{mod } Q$ of representations of the quiver Q .

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The cluster category \mathcal{C}_Q is defined as a kind of quotient of the derived category $D \text{ mod } Q$ of the category $\text{mod } Q$ in which the shift functor $[1]$ becomes isomorphic with the Auslander-Reiten functor τ . It remains a triangulated category.

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This definition works for acyclic quivers. More and more general versions are available, for quivers with potentials, in works of Bernhard Keller, Yann Palu, Claire Amiot, Pierre-Guy Plamondon and others.

Periodicity in cluster category

In the category $D \bmod Q$, the Auslander-Reiten functor satisfies

$$\tau^h = [-2], \quad (1)$$

where h is the Coxeter number.

Therefore in the cluster category \mathcal{C}_Q , where $\tau = [1]$, one has

$$[h + 2] = \text{Id}. \quad (2)$$

This means that the shift functor is periodic of period $h + 2$.

For example, this explains the period 5 for the quiver \mathbb{A}_2 , for which $h = 3$.

Clusters and geometry of surfaces

There exists a combinatorial/geometric description of the cluster category in some cases related to triangulations of surfaces and Teichmüller space. (e.g. works of Ralf Schiffler)

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Quantum clusters

There is an obvious way to define quantum F -polynomials: replace the Euler characteristic by the number of points over finite field \mathbb{F}_q . They can probably be used to define quantum Y -systems.

Question: can one use these in quantum dilogarithm identities?

Advertisement

(Nothing to do with the talk)

arXiv:0909.1694

a note on q -Zeta operators and Bernoulli-Carlitz fractions