LESSON 3

MASTERCLASS: SPECTRAL TRIPLES AS NC MANIFOLDS

↳ The states space and the GNS-construction

Def: A STATE on an unital C^*-algebra $\mathfrak{A}$ is a linear map

$$\omega : \mathfrak{A} \longrightarrow \mathbb{C} \quad \text{which is:}$$

(1) POSITIVE: i.e. $\omega(A) \geq 0 \quad \forall A \in \mathfrak{A}^+$

(2) NORMALIZED: $\omega(\mathbb{I}) = 1$

NOTATION: $S(\mathfrak{A})$ = space of all states on $\mathfrak{A}$ ≀ STATE SPACE

EXAMPLE: If we consider a C^*-algebra $\mathfrak{A} \subseteq B(\mathcal{H})$ for $\mathcal{H}$ a certain Hilbert sp. then every $\psi \in \mathcal{H}$ s.t. $\|\psi\| = 1$, defines a state. Indeed:

$$\omega(A) := \langle \psi, A\psi \rangle$$

⇒ positivity follows from the fact that:

$$\omega(\underbrace{B^*B}_{≀}) = \langle \psi, B^*B\psi \rangle = \langle B\psi, B\psi \rangle = \|B\psi\|^2 \geq 0$$

This is generic: indeed each positive element $A \in \mathfrak{A}^+$ can be written as:

$$A = B^*B, \quad B \in \mathfrak{A}$$

⟶ Normalization: $\omega(\mathbb{I}) = \langle \psi, \psi \rangle = 1$

THEOREM: $S(C(X)) = \{ \mu : X \longrightarrow [0,1],$ probability measure on $X \}$

≀

This theorem follows from Riesz theorem: each positive linear map $\omega : C(X) \to \mathbb{C}$ gives a regular positive measure μ_ω on X

Hence, the normalization ensures that:

$$\omega(1_X) = \mu_\omega(X) = 1$$

⇒ μ_ω is a probability measure.

NOTE: As a consequence of the positivity of ω:

$(A,B)_\omega := \omega(A^*B)$ ~ is a PRE-INNER PRODUCT on $\mathfrak{U}$

~~Indeed~~: Then, being an inner product, it satisfies the Cauchy-Schwarz inequality:

$$* \ |\omega(A^*B)|^2 \le \omega(A^*A)\,\omega(B^*B)$$

Moreover: $\omega(A^*) = \overline{\omega(A)}$ ~ indeed: $\omega(A^*) = \omega(A^*\mathbb{I})$

$$= (A,\mathbb{I})_\omega = \overline{(\mathbb{I},A)_\omega} = \overline{\omega(A)}$$

PROP: A linear map $\omega: \mathfrak{U} \longrightarrow \mathbb{C}$ on a unital C^*-algebra $\mathfrak{U}$ is POSITIVE $\Leftrightarrow$ is BOUNDED & $\|\omega\| = \omega(\mathbb{I})$

In particular: (1) A state on a unital C^*-algebra is bounded with norm 1

(2) An element $\omega \in \mathfrak{U}^*$ s.t. $\|\omega\| = \omega(\mathbb{I}) = 1$ is a state in $\mathfrak{U}$

NOW: We want to define states on a C^*-algebra without unit.

Def: A POSITIVE MAP $Q: \mathfrak{U} \longrightarrow \mathcal{B}$ with $\mathfrak{U}, \mathcal{B}$ = C^*-algebras is a linear map s.t. $A \ge 0$ in $\mathfrak{U}$ $\Rightarrow$ $Q(A) \ge 0$ in $\mathcal{B}$

PROP: A positive map between two C^*-algebras is bounded (continuous)

NOTE: * If we choose $\mathcal{B} = \mathbb{C}$, we see that the definition of a state on a unital algebra is a special case of a positive map between C^*-algebras

Def: A STATE on a C^*-algebra $\mathfrak{U}$ is a linear map $\omega: \mathfrak{U} \to \mathbb{C}$ s.t.

(1) ω is POSITIVE
(2) $\|\omega\| = 1$

PROP: A state ω on a C^*-algebra without unit has a unique extension to a state $\omega_\mathbb{I}$ on the unitization ~~$\mathfrak{U}_\mathbb{I}$~~ $\mathfrak{U}_\mathbb{I}$ given by:

$$\omega_\mathbb{I}(A + \lambda\mathbb{I}) := \omega(A) + \lambda \qquad \lambda \in \mathbb{C},\ A \in \mathfrak{U}$$

LEMMA: $\forall A \in \mathcal{U}$, $a \in \sigma(A) \Rightarrow \exists\, \omega_a$, state on $\mathcal{U}$ s.t. $\omega(A) = a$

Moreover:

if $A = A^* \Rightarrow \exists\, \omega : \mathcal{U} \to \mathbb{C}$ state s.t. $|\omega(A)| = \|A\|$

PROPERTY: $S(\mathcal{U})$ is a CONVEX SET $\rightsquigarrow$ it's a set $C \subseteq V$, with V = vector space, s.t.

$\lambda v + (1-\lambda) w \in C \quad \forall v, w \in C$, $\forall \lambda \in [0,1]$

Hence: $\forall v_i \in C$, $\sum_i p_i = 1$, $p_i \geq 0$

$\Rightarrow$ then: $\sum_i p_i v_i \in C$

$S(\mathcal{U})$, $\mathcal{U}$ = unital C^*-algebra

Then: * each $\omega \in S(\mathcal{U})$ is CONTINUOUS $\Rightarrow S(\mathcal{U}) \subseteq \mathcal{U}^*$

* w^*-limits preserve POSITIVITY & NORMALIZATION

$\hookrightarrow$ $S(\mathcal{U})$ is closed in $\mathcal{U}^*$ w.r.t the w^*-topology

* $S(\mathcal{U})$ = closed subset in $\mathcal{U}^*_1$ = unit ball

$\hookrightarrow$ By Banach-Alaoglu theorem, this ball is compact w.r.t the w^*-topology

Hence: $S(\mathcal{U})$ = compact in the relative w^* topology induced by $\mathcal{U}^*_1$

* $S(\mathcal{U})$ is COMPACT & CONVEX SET.

EXAMPLE: * $\mathcal{U} = \mathbb{C} \Rightarrow$ then: $S(\mathcal{U})$ is a point

* $\mathcal{U} = \mathbb{C}^2 \Rightarrow \omega(\lambda_1 + \mu\lambda_2) = c_1\lambda_1 + c_2\lambda_2$ $\rightsquigarrow$ positivity requires that $c_1 \geq 0$, $c_2 \geq 0$

normalization imposes that:

$c_1 + c_2 = 1$

Hence: $S(\mathbb{C} \oplus \mathbb{C}) = [0,1]$

* $\mathcal{U} = M_2(\mathbb{C})$

$S(\mathcal{U}) = \{$ 2x2-matrices s.t. they are positive & $Tr(\rho) = 1 \}$ $\rightsquigarrow$ density matrices in quantum mechanics

$\cong$

$B_1(0)$ = unit ball in $\mathbb{R}^3$