QGM Master Class “Cluster algebras”
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Lectures 11-15

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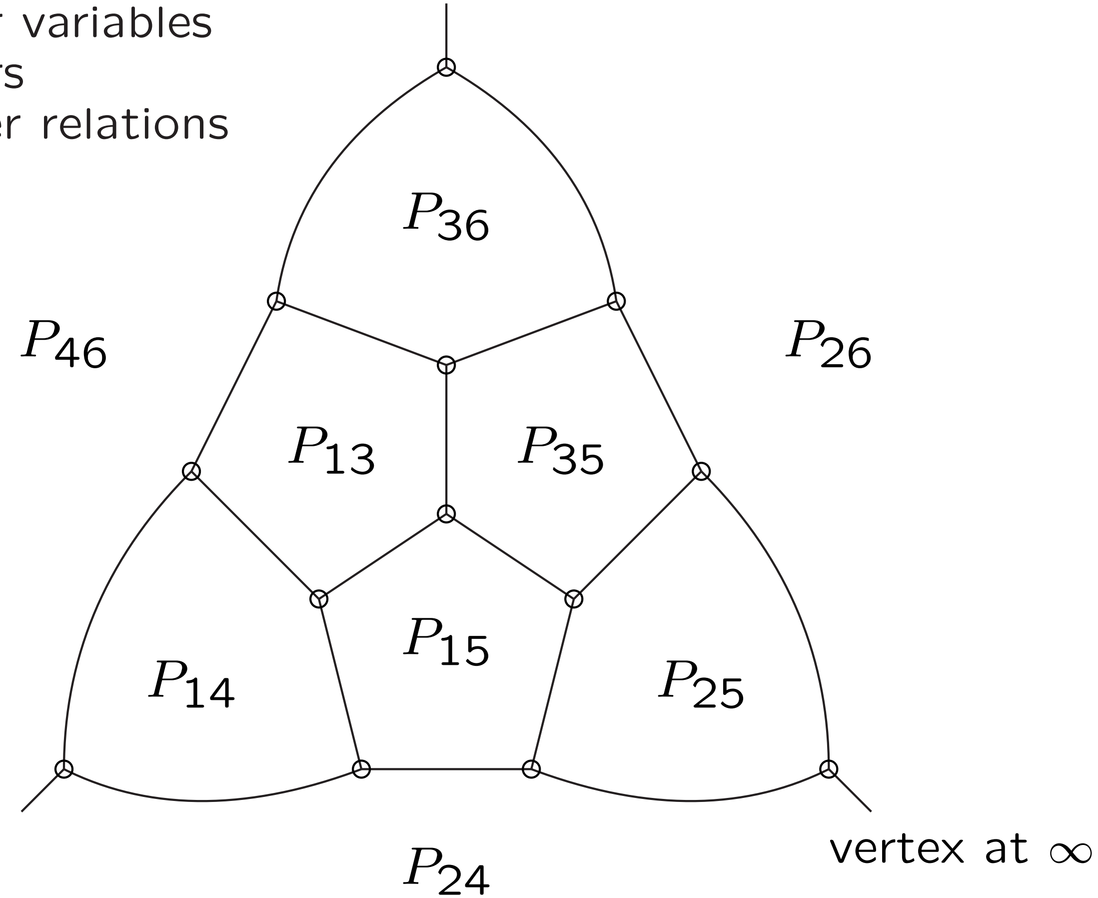
Example 1: Clusters and exchange relations in $\mathbb{C}[\mathrm{Gr}_{2,6}]$

faces	$\longleftrightarrow$	cluster variables
vertices	$\longleftrightarrow$	clusters
edges	$\longleftrightarrow$	Plücker relations

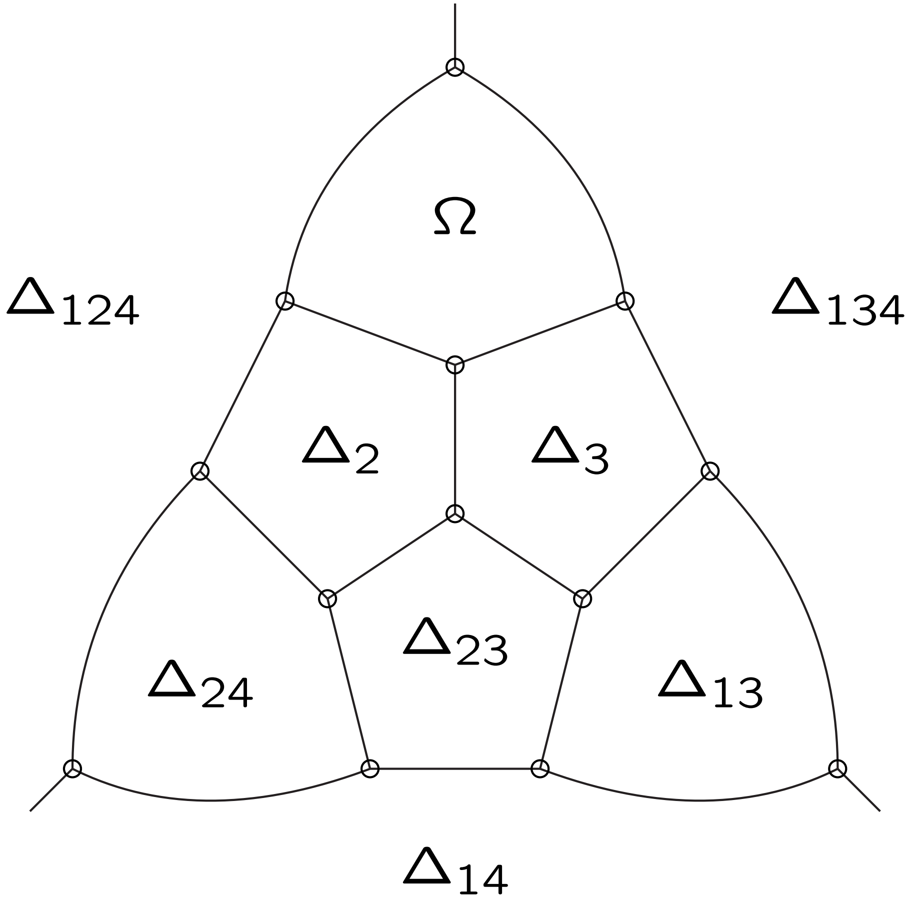


$$P_{13}\,P_{25} = P_{12}\,P_{35} + P_{23}\,P_{15}$$

$$P_{35}\,P_{14} = P_{34}\,P_{15} + P_{45}\,P_{13}$$

$$P_{15}\,P_{36} = P_{56}\,P_{13} + P_{16}\,P_{35}$$

Example 2: Clusters and exchange relations in $\mathbb{C}[\mathsf{SL}_4]^N$



$$\Delta_2 \Delta_{13} = \Delta_{12} \Delta_3 + \Delta_1 \Delta_{23}$$

$$\Delta_3 \Delta_{24} = \Delta_4 \Delta_{23} + \Delta_{34} \Delta_2$$

$$\Delta_{23}\, \Omega = \Delta_{123} \Delta_{34} \Delta_2 + \Delta_{12} \Delta_{234} \Delta_3$$

Comparing matrices $\tilde{B}$

Example 1

	P_{13}	P_{35}	P_{15}
P_{13}	0	−1	1
P_{35}	1	0	−1
P_{15}	−1	1	0
P_{12}	1	0	0
P_{23}	−1	0	0
P_{34}	0	1	0
P_{45}	0	−1	0
P_{56}	0	0	1
P_{16}	0	0	−1

Example 2

	Δ_2	Δ_3	Δ_{23}
Δ_2	0	−1	1
Δ_3	1	0	−1
Δ_{23}	−1	1	0
Δ_1	−1	0	0
Δ_{12}	1	0	−1
Δ_{123}	0	0	1
Δ_4	0	1	0
Δ_{34}	0	−1	1
Δ_{234}	0	0	−1

The exchange matrices B in these two examples are the same; the bottom parts of $\tilde{B}$ (hence the coefficients) are different.

We say that these two cluster algebras are of the same *type*.

Exchange graph and cluster complex

Theorem 1 [M. Gekhtman, M. Shapiro, and A. Vainshtein, *Math. Res. Lett.* **15** (2008)] *In any exchange pattern,*

- *every seed is uniquely determined by its cluster;*
- *two seeds are related by a mutation if and only of their clusters share all elements but one.*

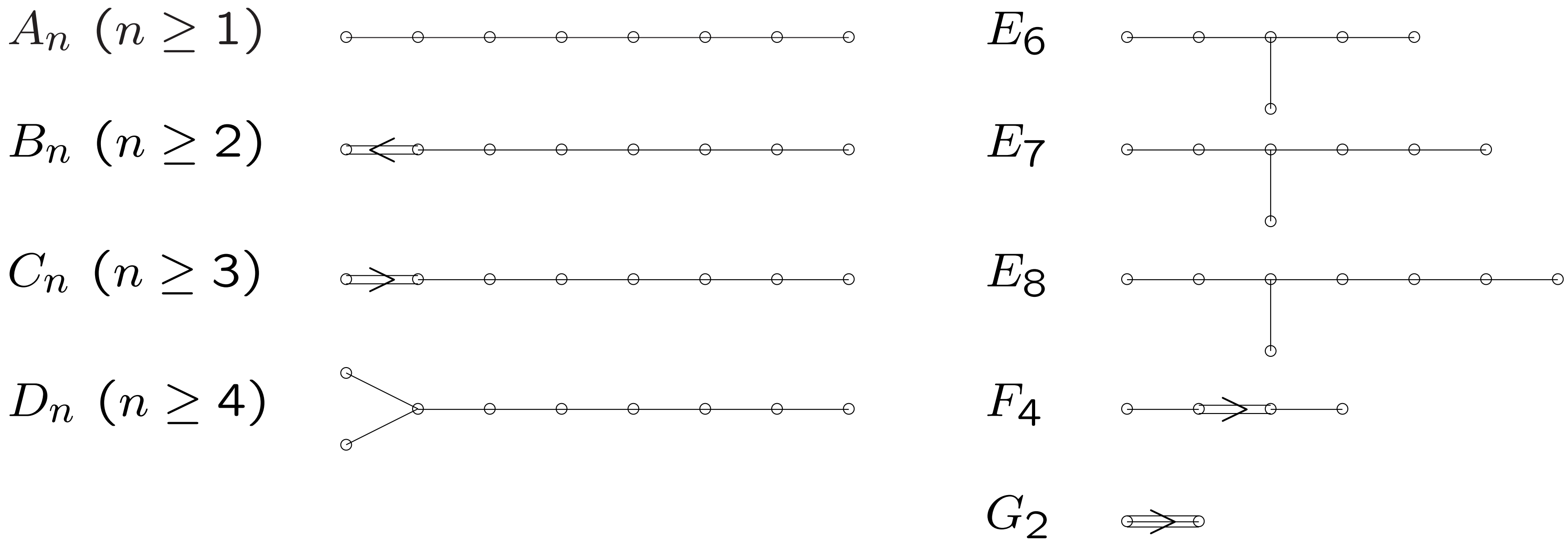
The *cluster complex* $\Delta(\mathcal{A})$ is the simplicial complex whose vertices are the cluster variables in $\mathcal{A}$ and whose maximal simplices are the clusters. By Theorem 1, the cluster complex is an $(n-1)$-dimensional *pseudomanifold*. Its dual graph is the *exchange graph* of $\mathcal{A}$, the connected, n-regular graph whose vertices are the seeds/clusters and whose edges correspond to mutations.

Conjecture 2 *The cluster complex and the exchange graph depend only on the type of $\mathcal{A}$.*

Cluster algebras of finite type

A cluster algebra $\mathcal{A}(\mathcal{S})$ is of *finite type* if the mutation class $\mathcal{S}$ is finite (equivalently, there are finitely many cluster variables).

The classification of cluster algebras of finite type turns out to be completely parallel to the classical Cartan-Killing classification of semisimple Lie algebras and finite root systems.

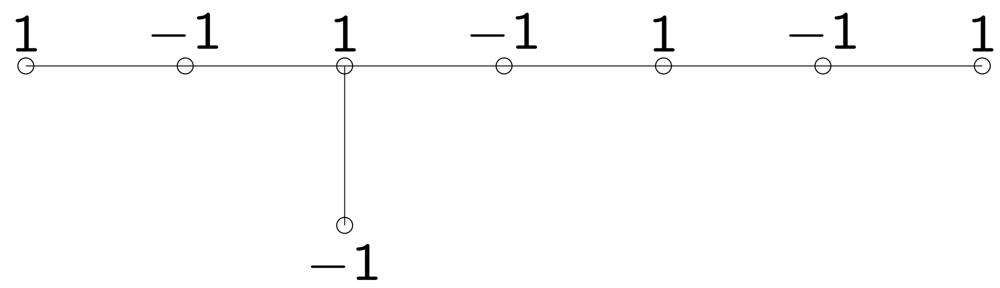


Bi-partition of a Dynkin diagram

Let $A = (a_{ij})$ be an $n \times n$ Cartan matrix of finite type. Let

$$\varepsilon : [1, n] \to \{1, -1\}$$

be a sign function such that $a_{ij} < 0 \implies \varepsilon(i) = -\varepsilon(j)$.



Let $B(A) = (b_{ij})$ be the skew-symmetrizable matrix defined by

$$b_{ij} = \begin{cases} 0 & \text{if } i = j; \\ \varepsilon(i)\, a_{ij} & \text{if } i \neq j. \end{cases}$$

Finite type classification

Theorem 3 *A cluster algebra is of finite type if and only if the exchange matrix at some seed is of the form* $B(A)$*, where* A *is a Cartan matrix of finite type.*

The type of the Cartan matrix A in the Cartan-Killing nomenclature is uniquely determined by the cluster algebra, and is called its *cluster type*.

Cluster types of some coordinate rings

The symmetry exhibited by the cluster type of a cluster algebra is usually not apparent at all from its geometric realization.

$\mathbb{C}[\mathrm{Gr}_{2,n+3}]$	A_n	(Example 1)
$\mathbb{C}[\mathrm{Gr}_{3,6}]$	D_4	
$\mathbb{C}[\mathrm{Gr}_{3,7}]$	E_6	
$\mathbb{C}[\mathrm{Gr}_{3,8}]$	E_8	
$\mathbb{C}[\mathrm{SL}_3]^N$	A_1	
$\mathbb{C}[\mathrm{SL}_4]^N$	A_3	(Example 2)
$\mathbb{C}[\mathrm{SL}_5]^N$	D_6	
$\mathbb{C}[\mathrm{Sp}_4]^N$	B_2	

(beyond this table—infinite types)

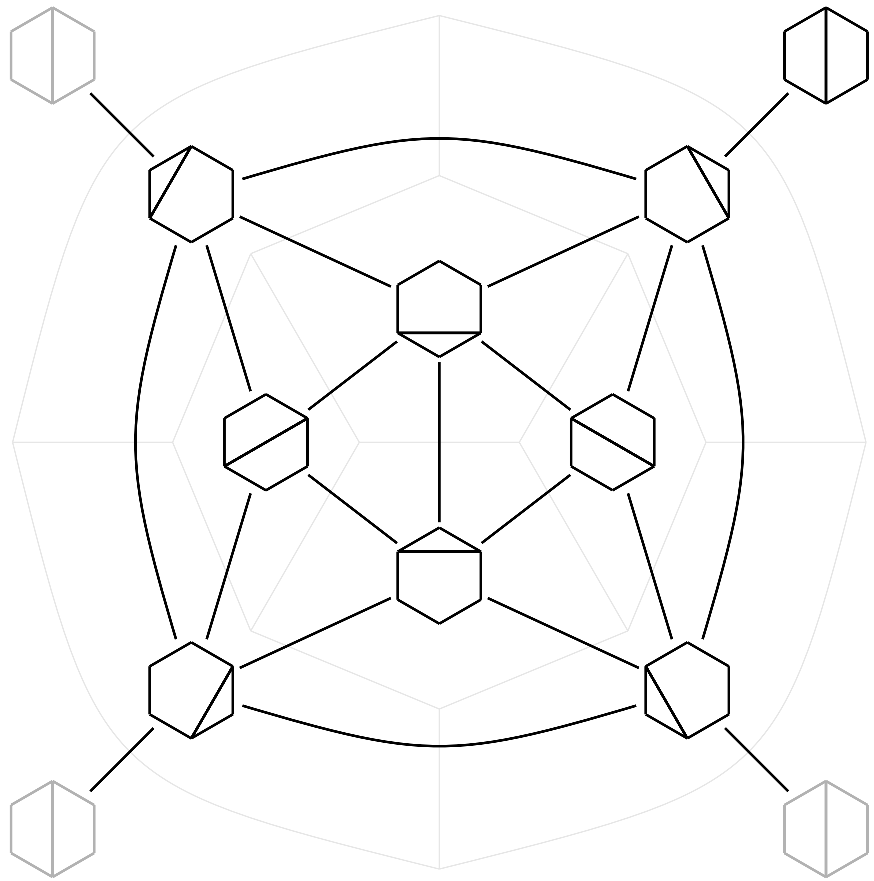
Cluster complexes in finite type

Theorem 4 [F.Chapoton, S.F., and A.Zelevinsky, *Canad. Math. Bull.* **45** (2002)] *The cluster complex of a cluster algebra of finite type is the dual simplicial complex of a simple convex polytope.*

This polytope is the *generalized associahedron* of the appropriate Cartan-Killing type. In types A_n and B_n, we recover, respectively, Stasheff's *associahedron* and Bott-Taubes' *cyclohedron*.

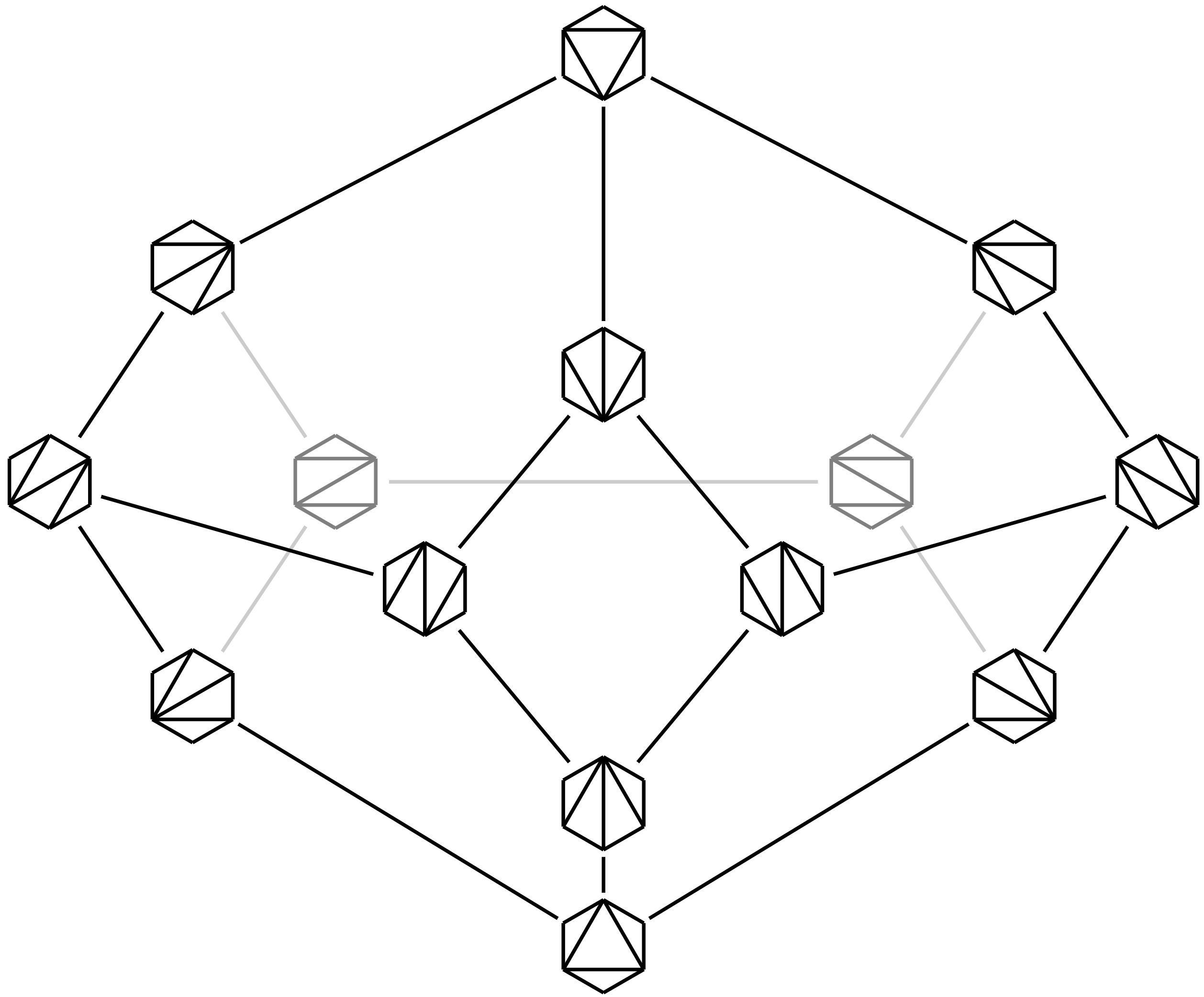
Cluster complex of type A_n

The simplices of the cluster complex $\Delta(\mathcal{A})$ associated with a cluster algebra $\mathcal{A}$ of type A_n are naturally identified with collections of non-crossing diagonals in a convex $(n+3)$-gon.

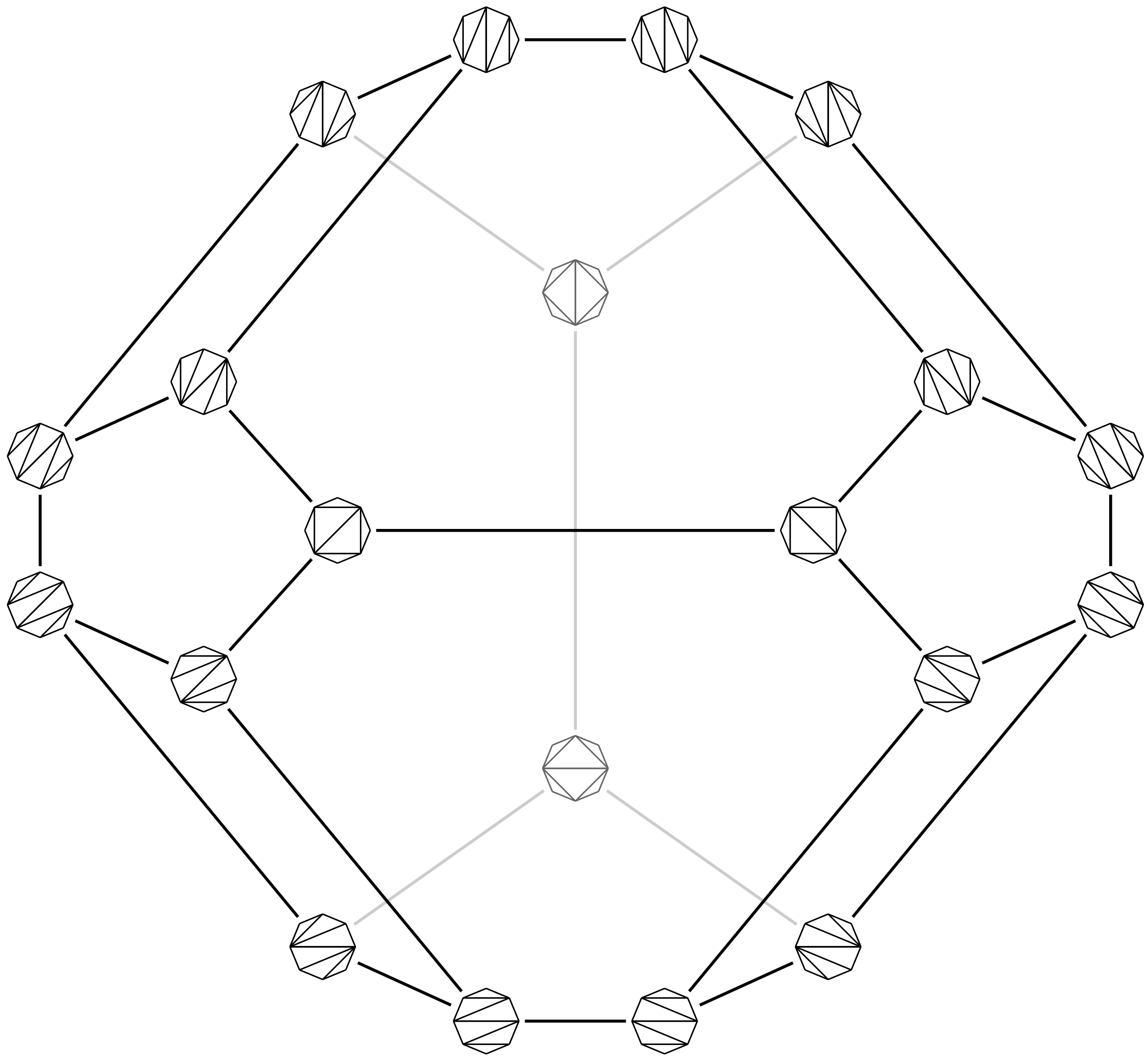


This is the dual complex of Stasheff's *associahedron*.

Associahedron of type A_3



Generalized associahedron of type B_3 (cyclohedron)



Polyhedral realization of the cluster complex

Let $\mathcal{A}$ be a cluster algebra $\mathcal{A}$ of finite type defined by a Cartan matrix A. Let Φ be the associated crystallographic root system.

Theorem 5 *The number of cluster variables in $\mathcal{A}$ is equal to the number of roots in Φ that are either positive or negative simple.*

Let $\Phi_{\geq -1}$ denote the set of these "almost positive" roots. The cluster variables in $\mathcal{A}$ are naturally labeled by the roots in $\Phi_{\geq -1}$. The labeling is determined by the denominators of the Laurent expansions with respect to the distinguished cluster.

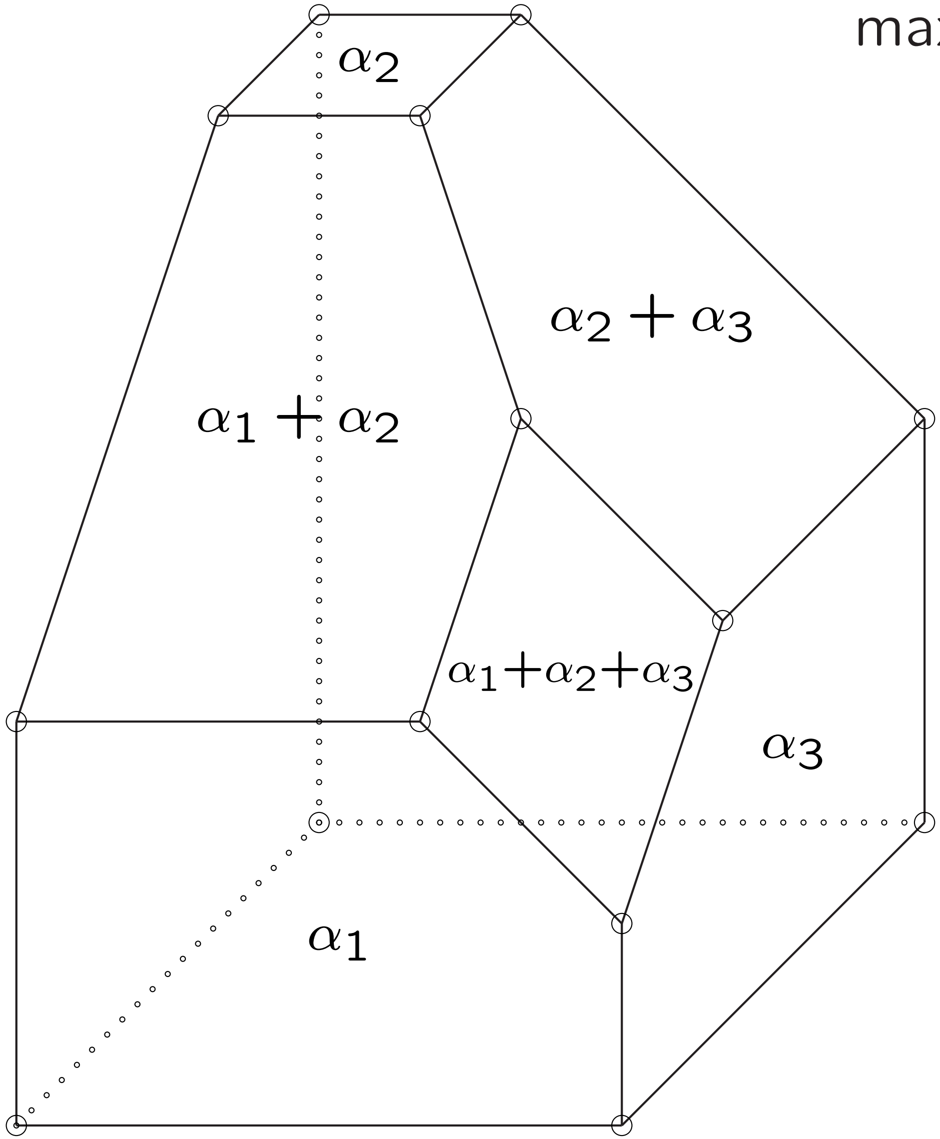
The cluster fan

The cluster complex $\Delta(\mathcal{A})$ can be built on the ground set $\Phi_{\geq -1}$. Its combinatorics, and the geometry of the associated simplicial fan (the normal fan of the generalized associahedron $P(\Phi)$) can be explicitly described in root-theoretic terms.

Polyhedral realization of the associahedron of type A_3

$$\max(-z_1, -z_3, z_1, z_3, z_1+z_2, z_2+z_3) \leq 3/2$$

$$\max(-z_2, z_2, z_1+z_2+z_3) \leq 2$$



Enumerative results

Theorem 6 *The number of clusters in a cluster algebra of finite type is equal to*

$$N(\Phi)=\prod_{i=1}^{n}\frac{e_i+h+1}{e_i+1},$$

where $e_1,\dots,e_n$ are the exponents, and h is the Coxeter number.

$N(\Phi)$ is the *Catalan number* associated with the root system Φ.

A_n	B_n, C_n	D_n	E_6	E_7	E_8	F_4	G_2
$\frac{1}{n+2}\binom{2n+2}{n+1}$	$\binom{2n}{n}$	$\frac{3n-2}{n}\binom{2n-2}{n-1}$	833	4160	25080	105	8

Catalan combinatorics of arbitrary type

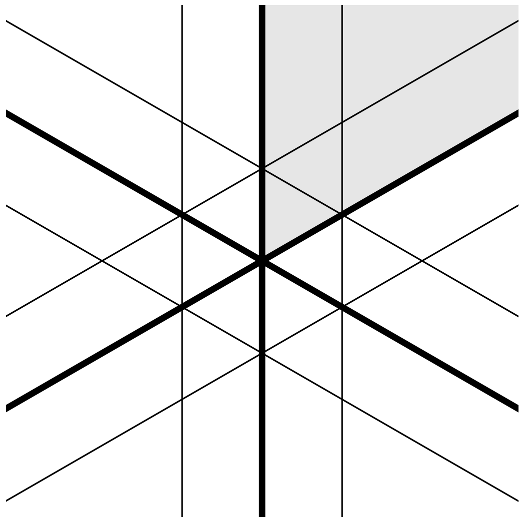
The numbers $N(\Phi)$ can be viewed as generalizations of the *Catalan numbers* to arbitrary Cartan-Killing type. Besides clusters, they are known to enumerate a variety of combinatorial objects related to the root system Φ:

- *ad-nilpotent ideals* in a Borel subalgebra of a semisimple Lie algebra;
- antichains in the *root poset*;
- regions of the *Catalan arrangement* contained in the fundamental chamber;
- orbits of the Weyl group action on the quotient $Q/(h+1)Q$ of the root lattice;
- conjugacy classes of elements x of a semisimple Lie group which satisfy $x^{h+1} = 1$;
- *non-crossing partitions* of the appropriate type.

$N(\Phi)$ **and hyperplane arrangements**
[Jian-Yi Shi]

The *Catalan arrangement* associated with a root system Φ is the arrangement of affine hyperplanes defined by the equations

$$\begin{array}{l}\langle \beta, x\rangle = 0 \\ \langle \beta, x\rangle = 1\end{array} \quad \text{for all } \beta \in \Phi.$$



Theorem 7 *The number of regions of the Catalan arrangement contained in the fundamental chamber is equal to* $N(\Phi)$.

References

V. V. Fock and A. B. Goncharov, Moduli spaces of local systems and higher Teichmüller theory, *Publ. Math. IHES* **103** (2006).

M. Gekhtman, M. Shapiro, and A. Vainshtein, Cluster algebras and Weil-Petersson forms, *Duke Math. J.* **127** (2005).

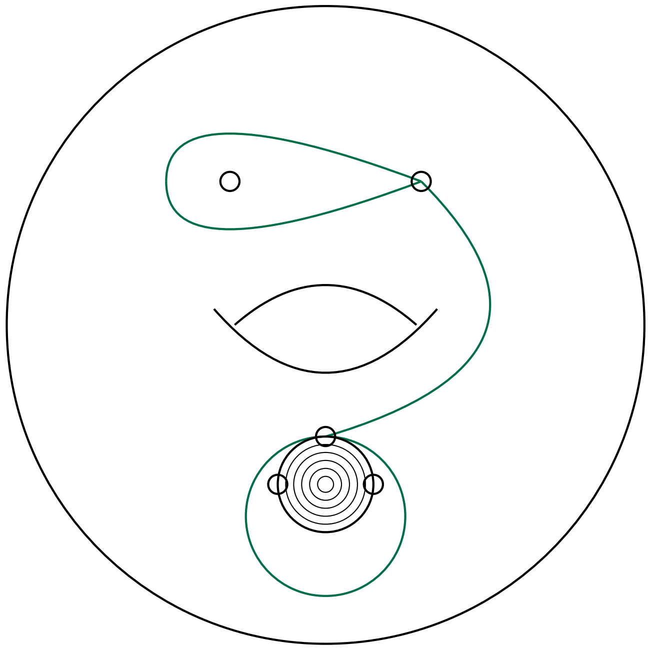
S. F., M. Shapiro, and D. Thurston, Cluster algebras and triangulated surfaces I: Cluster complexes, *Acta Math.* **201** (2008).

S. F. and D. Thurston, Cluster algebras and triangulated surfaces II: Lambda lengths.

R. Penner, Lambda lengths, *CTQM master class*, Aarhus, 2006, `http://www.ctqm.au.dk/events/2006/August/`.

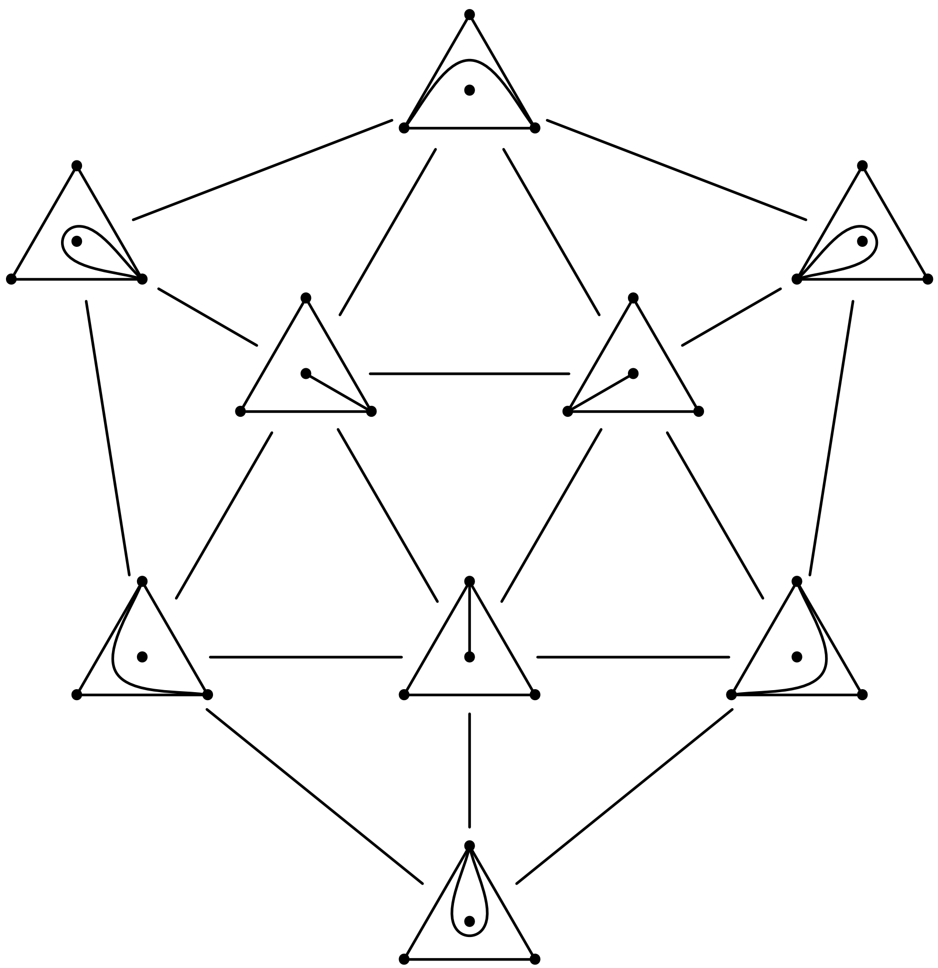
Arcs on a surface

Let $\mathbf{S}$ be a connected oriented surface with boundary. (Several small degenerate cases must be excluded.) Fix a finite non-empty set $\mathbf{M}$ of *marked points* in the closure of $\mathbf{S}$. An *arc* in $(\mathbf{S}, \mathbf{M})$ is a non-selfintersecting curve in $\mathbf{S}$, considered up to isotopy, which connects two points in $\mathbf{M}$, does not pass through $\mathbf{M}$, and does not cut out an unpunctured monogon or digon.



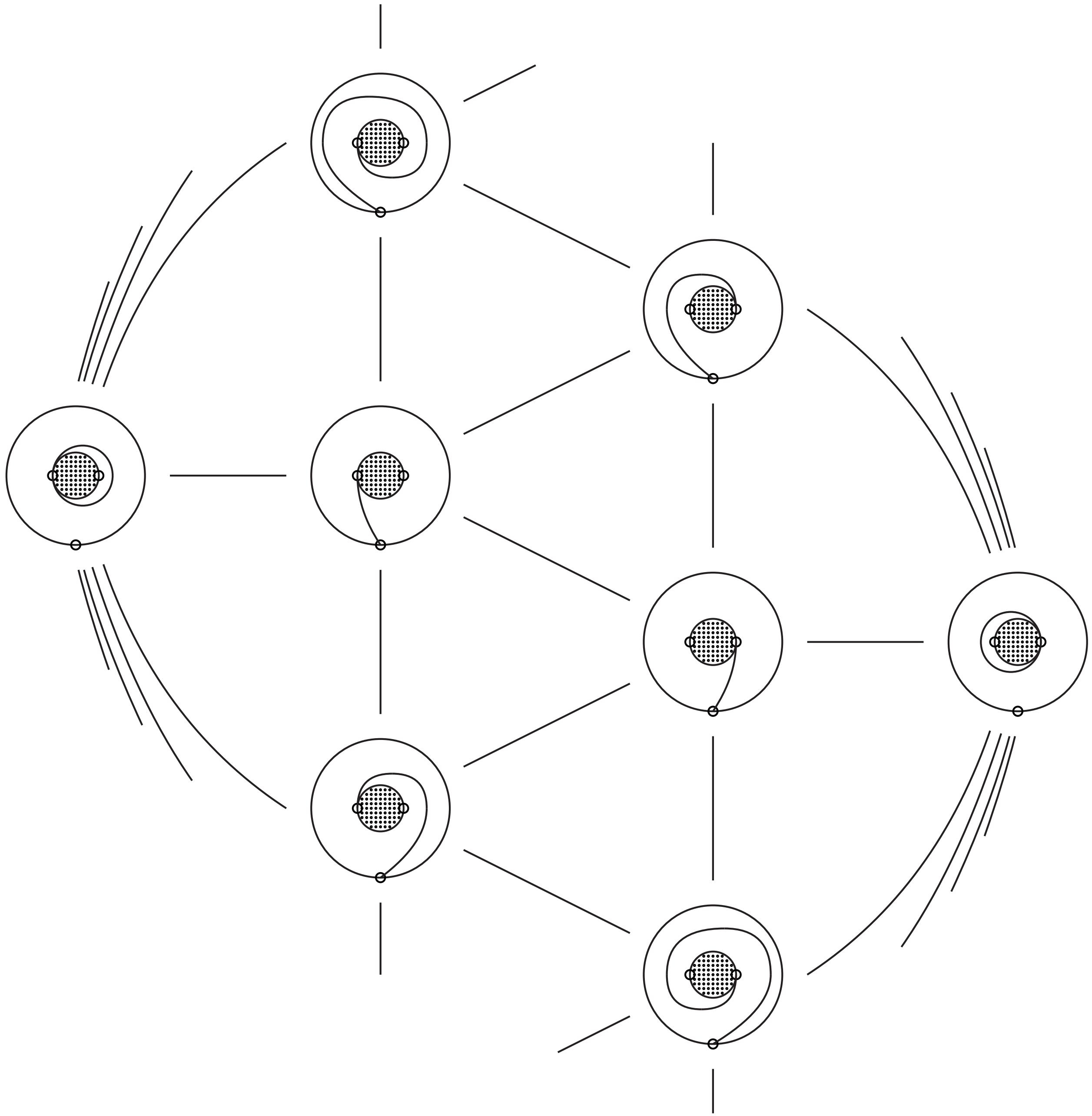
Arc complex

Arcs are *compatible* if they have non-intersecting realizations. Collections of pairwise compatible arcs are the simplices of the *arc complex*. Its facets correspond to *ideal triangulations*.



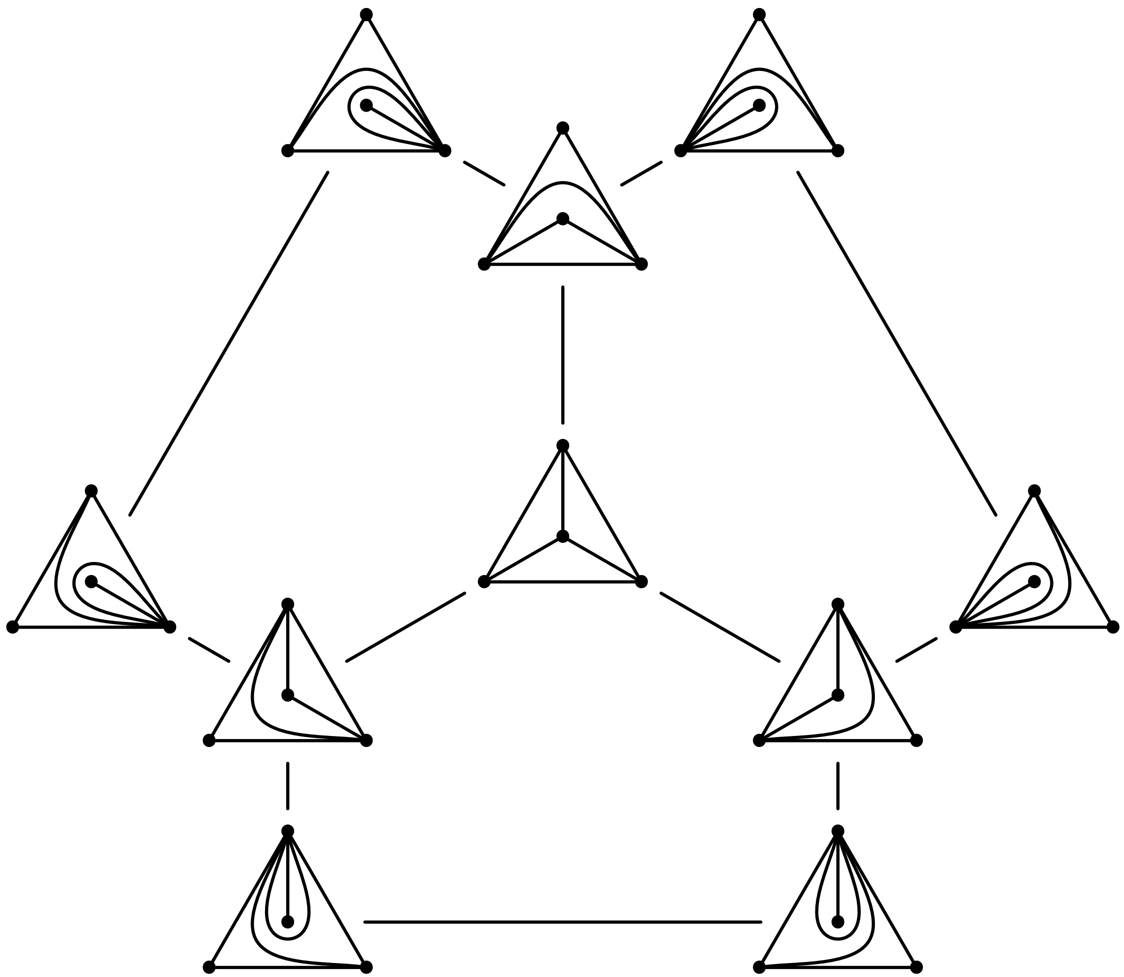
The arc complex is a *pseudomanifold with boundary*.

Arc complex for an annulus of type $\tilde{A}(2,1)$



Flips on a surface

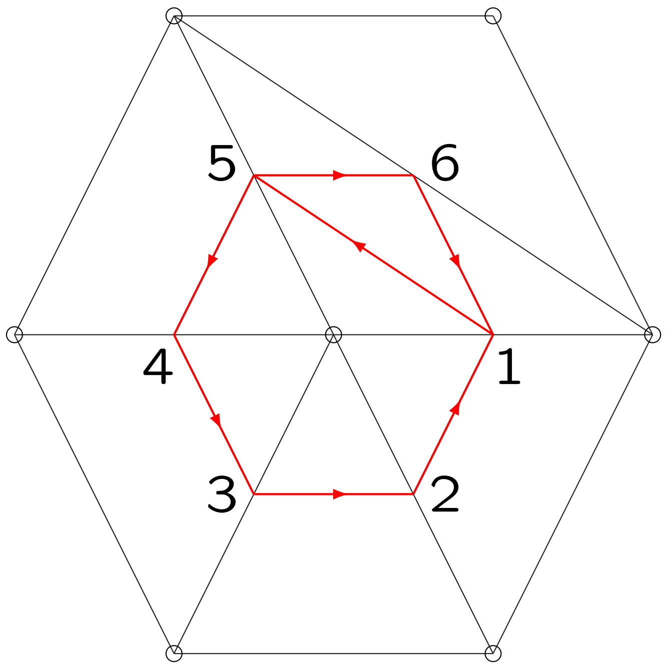
The edges of the dual graph of the arc complex correspond to *flips*.



An edge inside a *self-folded triangle* cannot be flipped.

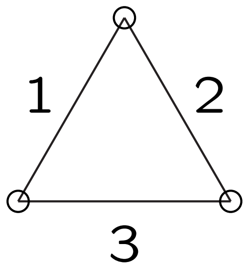
Signed adjacency matrices

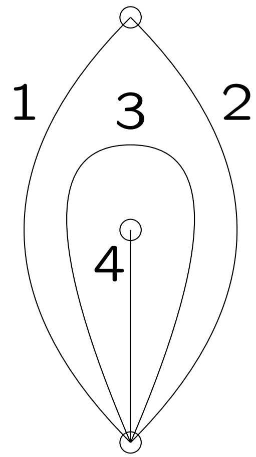
To a triangulation T we associate its *signed adjacency matrix* $B(T)$.

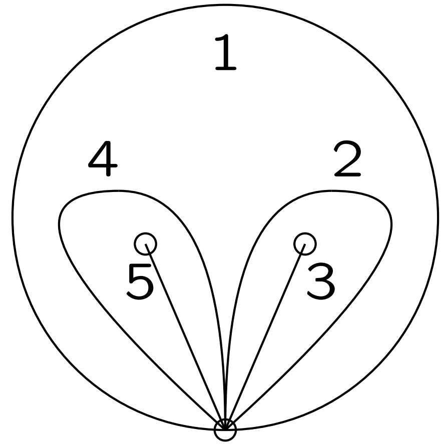


$$B(T) = \begin{bmatrix} 0 & -1 & 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

Signed adjacency matrices, continued







$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 1 & -1 & -1 \\ -1 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & -1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 \end{bmatrix}$$

Lemma 8 *Flips in ideal triangulations translate into mutations of the associated signed adjacency matrices.*

That is, if triangulations T' and T are related by a flip of the arc labeled k, then $B(T') = \mu_k(B(T))$.

From the arc complex to the cluster complex

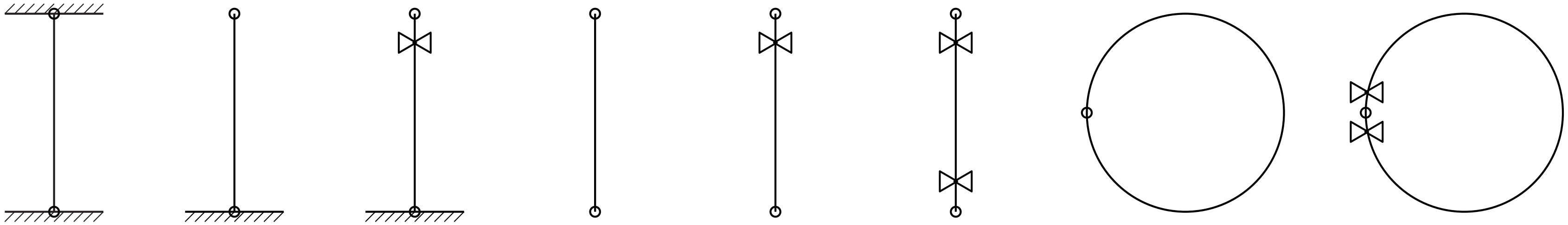
If a cluster algebra $\mathcal{A}$ has an exchange matrix that can be interpreted as a signed adjacency matrix of a triangulation of some marked surface $(\mathbf{S}, \mathbf{M})$, then one might hope that the underlying combinatorics of $\mathcal{A}$ can be modeled similarly to Example 1:

- cluster variables correspond to arcs;
- clusters correspond to triangulations;
- exchanges correspond to flips.

If $\mathbf{M}$ contains interior points, then we have a problem: flips in some directions are not allowed. This problem can be resolved by introducing *tagged arcs*.

Tagged arcs

A *tagged arc* is obtained by taking an arc that does not cut out a once-punctured monogon, and "tagging" each of its ends in one of two ways, *plain* or *notched*, obeying certain rules:

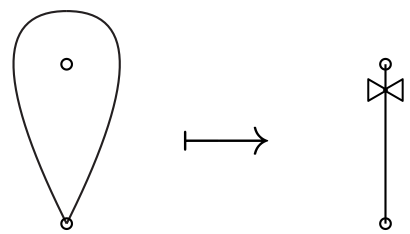


Compatibility of tagged arcs

Tagged arcs α and β are *compatible* if and only if

- their untagged versions α° and β° are compatible;
- if α and β share an endpoint, then the ends of α and β connecting to it must be tagged in the same way—unless $\alpha^\circ = \beta^\circ$, in which case at least one end of α must be tagged in the same way as the corresponding end of β.

Ordinary arcs can be viewed as a special case of tagged arcs:

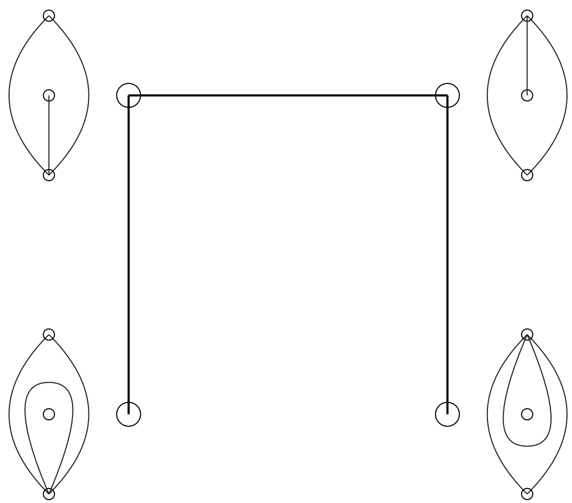


Under this identification, the notion of compatibility of tagged arcs extends the corresponding notion for ordinary arcs.

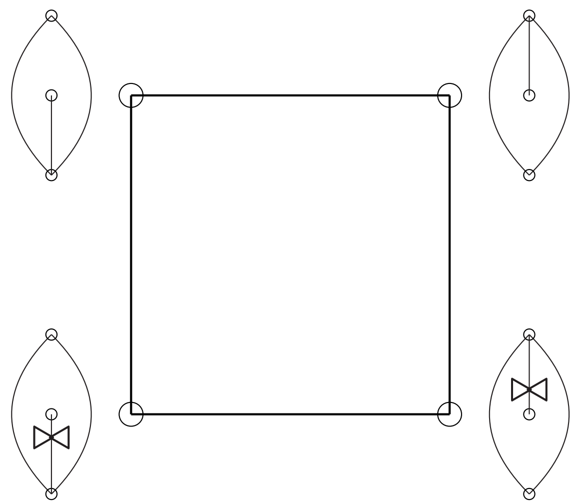
Tagged arc complex

The *tagged arc complex* is the simplicial complex whose vertices are tagged arcs and whose simplices are collections of pairwise compatible tagged arcs. The maximal simplices are called *tagged triangulations*.

The arc complex is a *subcomplex* of the tagged arc complex.

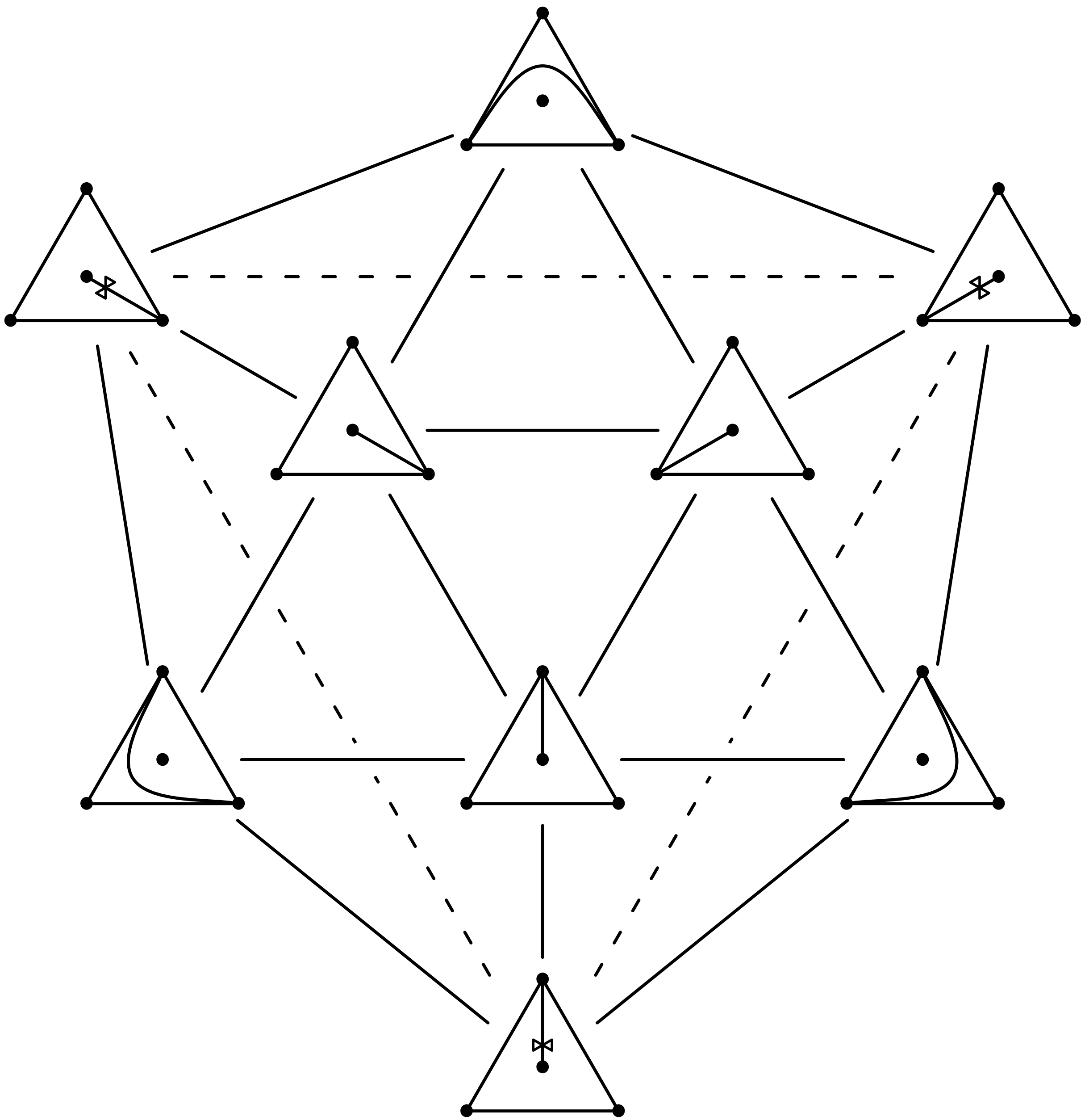


arc complex



tagged arc complex

Tagged arc complex of a once-punctured triangle



Properties of the tagged arc complex

Theorem 9 *The tagged arc complex is a pseudomanifold.*

That is: (a) all tagged triangulations have the same cardinality; (b) in each of them, any tagged arc can be flipped in a unique way.

Theorem 10 *The tagged arc complex is connected unless* $(\mathbf{S}, \mathbf{M})$ *is a closed surface with a single puncture, in which case it consists of two isomorphic connected components.*

Cluster complex associated with a marked surface

The definition of matrices $B(T)$ can be extended to tagged triangulations T so that tagged flips correspond to matrix mutations.

Theorem 11 *Let $\mathcal{A}$ be a cluster algebra whose exchange matrix at a particular seed can be interpreted as the signed adjacency matrix $B(T_\circ)$ of a triangulation $T_\circ$ of some marked surface $(\mathbf{S},\mathbf{M})$. Then the cluster complex $\Delta(\mathcal{A})$ is canonically isomorphic to a connected component of the tagged arc complex of $(\mathbf{S},\mathbf{M})$. Under this isomorphism,*

- *cluster variables $\longleftrightarrow$ tagged arcs,*
- *clusters/seeds $\longleftrightarrow$ tagged triangulations,*
- *mutations $\longleftrightarrow$ flips,*
- *exchange matrix at a seed associated with any T is $B(T)$.*

Topological properties of cluster complexes

Theorem 12 *The cluster complex is either contractible or homotopy equivalent to a sphere. Specifically:*

- *For a polygon with at most one puncture, the cluster complex is homeomorphic to an* $(n-1)$*-dimensional sphere* S^{n-1}.
- *For a closed surface with* $p \geq 2$ *punctures, the cluster complex is homotopy equivalent to* S^{p-1}.
- *In all other cases, the cluster complex is contractible.*

Growth rate of the cluster complex

A cluster complex has *polynomial growth* if the number of distinct seeds which can be obtained from a fixed initial seed by at most n mutations is bounded from above by a polynomial in n.

A cluster complex has *exponential growth* if the number of such seeds is bounded from below by an exponentially growing function of n.

Theorem 13 *The cluster complex has polynomial growth for a disk with* ≤ 2 *punctures, an annulus with* ≤ 1 *puncture, or an unpunctured pair of pants. Otherwise, it has exponential growth.*

Which cluster algebras come from surfaces?

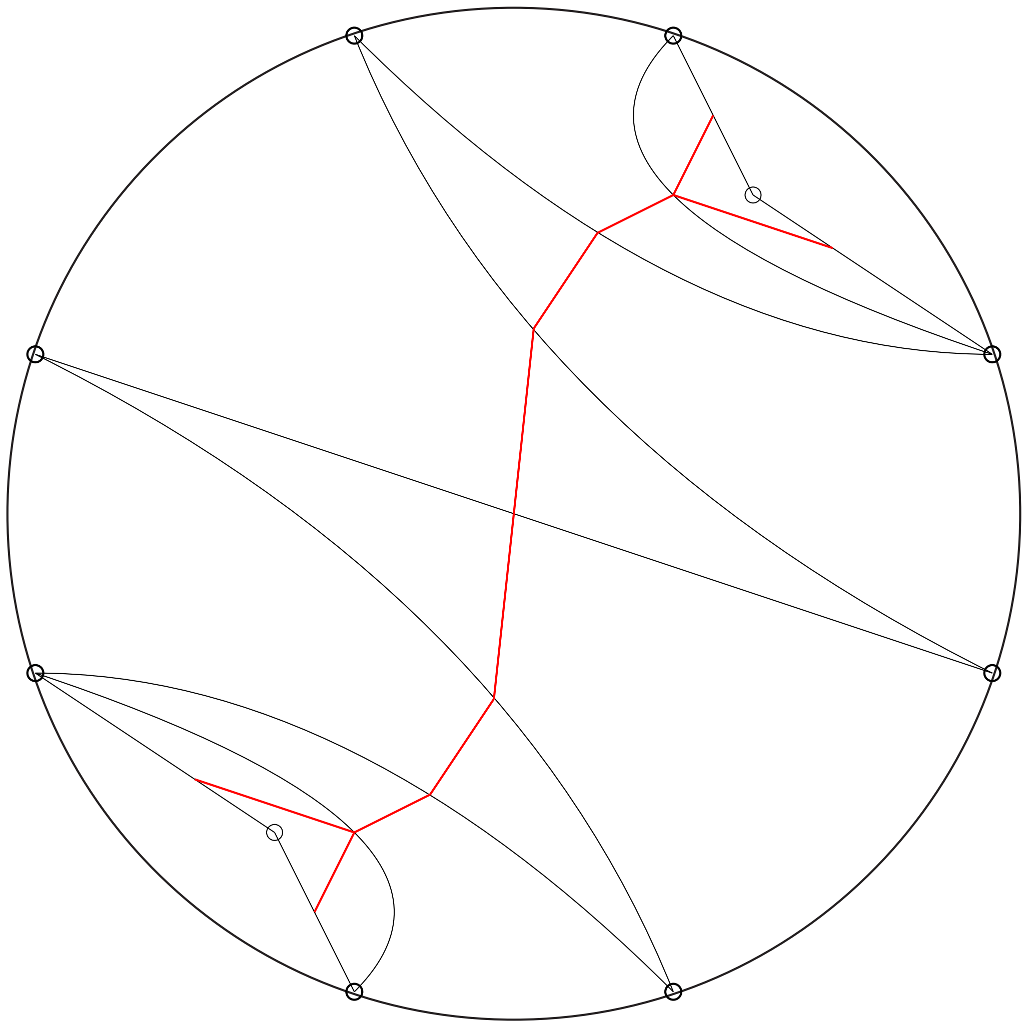
Examples of cluster algebras whose exchange matrices can be interpreted as signed adjacency matrices of triangulated surfaces include the following cluster types:

- finite type A_n (Example 1: unpunctured $(n+3)$-gon);
- finite type D_n (once-punctured n-gon);
- affine type $\widetilde{A}(n_1, n_2)$ (unpunctured annulus);
- affine type $\widetilde{D}_{n-1}$ (twice-punctured $(n-3)$-gon).

Exceptional finite cluster types E_6, E_7, E_8 cannot be modeled by triangulated surfaces.

Example: Cluster type $\widetilde{D}_{n-1}$

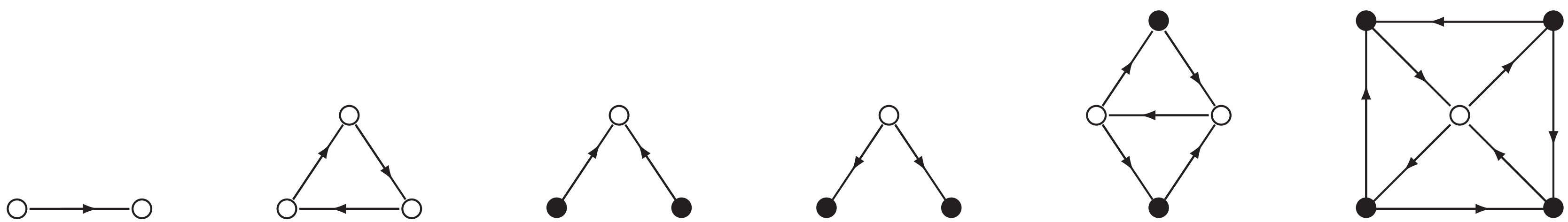
A topological model for cluster algebras of type $\widetilde{D}_{n-1}$ can be given using tagged arcs in a twice-punctured $(n-3)$-gon.



Quivers. Block decompositions

A cluster algebra associated with a marked bordered surface must have *skew-symmetric* exchange matrices. Such matrices are represented by *quivers*.

Theorem 14 *A quiver describes signed adjacencies of arcs in some triangulated marked bordered surface if and only if it can be obtained from a collection of blocks of the form shown below by gluing together some pairs of white vertices.*



Theorem 15 [A. Felikson, P. Tumarkin, M. Shapiro, 2008] *Apart from a finite number of exceptions, a quiver is of finite mutation type if and only if it comes from a triangulated surface.*

Beyond the cluster complex

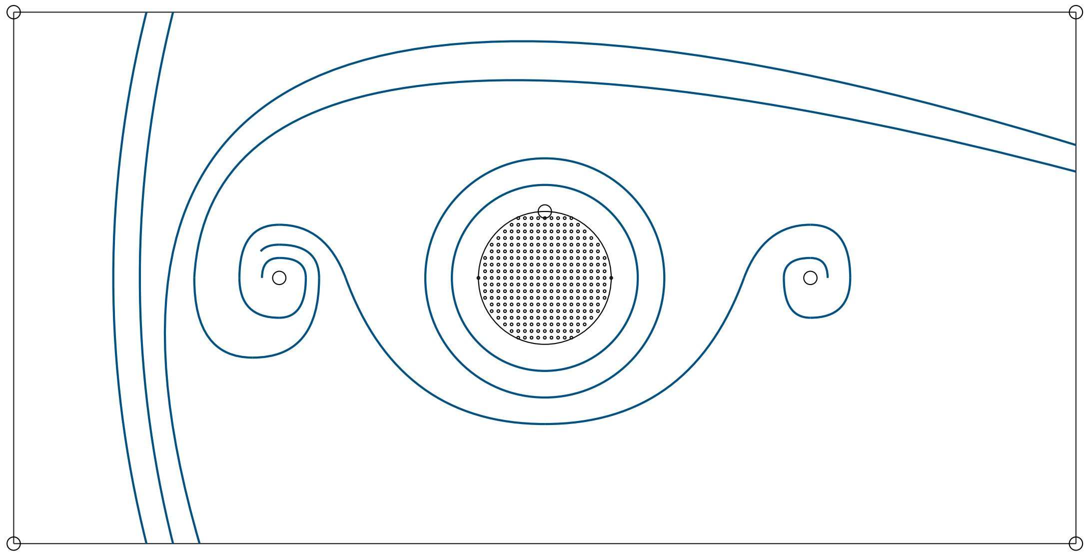
Even when the combinatorics of the cluster complex $\Delta(\mathcal{A})$ is well understood (e.g., in finite type, or in the case of surfaces), more needs to be done to understand the cluster algebra $\mathcal{A}$, since $\Delta(\mathcal{A})$ does not capture all relevant algebraic data.

Most importantly, we would like to have a direct (rather than recursive) description of the coefficients appearing in the exchange relations. A case in point is Example 2.

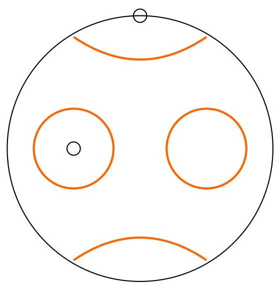
For cluster algebras associated with marked bordered surfaces, solving this problem requires new combinatorial machinery that employs W. Thurston's theory of *laminations*.

Integral laminations

An *integral (unbounded measured) lamination* on $(\mathbf{S}, \mathbf{M})$ is a finite collection of non-selfintersecting and pairwise non-intersecting curves in $\mathbf{S}$, modulo isotopy:

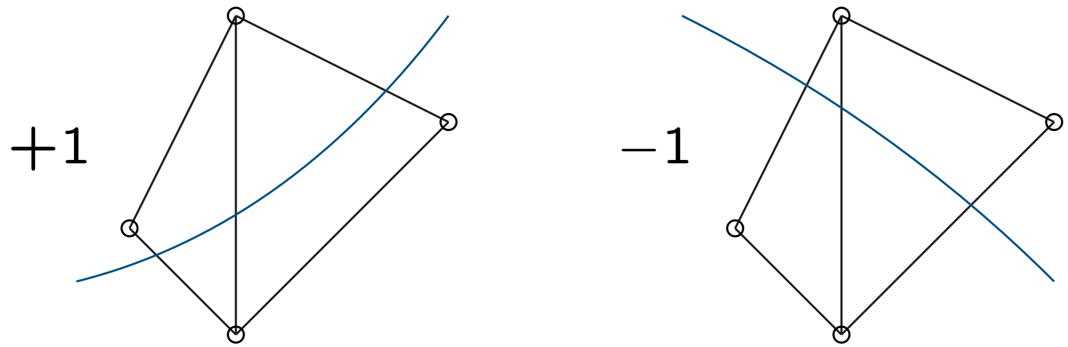


Curves that are not allowed in a lamination:



Shear coordinates

Let L be an integral lamination, and T a triangulation without self-folded triangles. For each arc γ in T, the *shear coordinate* $b_\gamma(T, L)$ is defined as a sum of contributions from all curves in L. Each such curve contributes $+1$ (resp., -1) to $b_\gamma(T, L)$ if it cuts through the quadrilateral surrounding γ as shown:

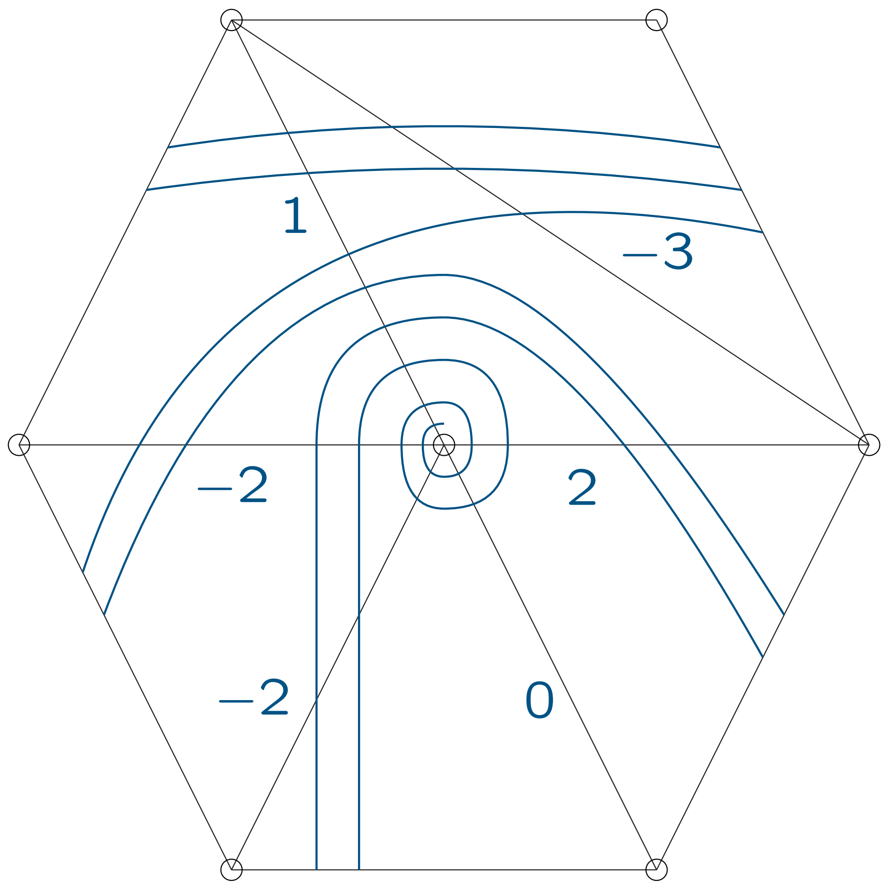


Thurston's coordinatization theorem

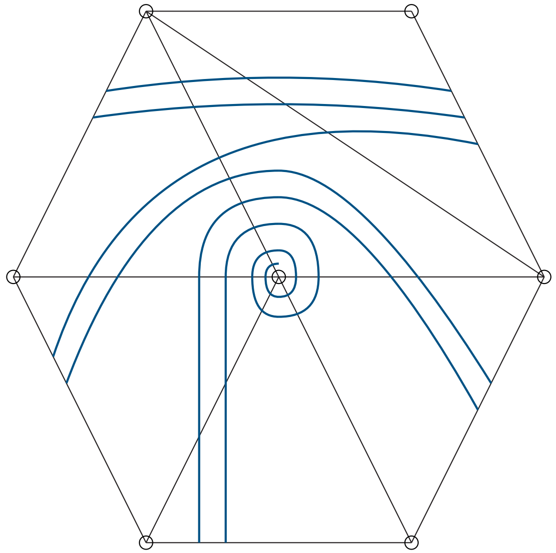
Theorem 16 [W.Thurston] *For a fixed triangulation T, the map*

$$L \mapsto (b_\gamma(T,L))_{\gamma\in T}$$

is a bijection between integral laminations and $\mathbb{Z}^n$.



Matrix $\tilde{B} = \tilde{B}(T, \mathbf{L})$ associated with a triangulation T and a multi-lamination $\mathbf{L}$



$$\tilde{B} = \begin{bmatrix} 0 & -1 & 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ \hline 2 & 0 & -2 & -2 & 1 & -3 \end{bmatrix}$$

The top part of $\tilde{B}$ is $B(T)$. For each lamination in $\mathbf{L}$, we add a row consisting of its shear coordinates with respect to T.

All of this can be extended to tagged triangulations T.

Theorem 17 *Under (tagged) flips, the matrices* $\tilde{B}(T, \mathbf{L})$ *change according to the matrix mutation rules.*

Cluster algebra associated with a multi-lamination

Theorem 18 *For any multi-lamination* $\mathbf{L}$ *on a bordered surface* $(\mathbf{S}, \mathbf{M})$*, there is a unique cluster algebra* $\mathcal{A}(\mathbf{S}, \mathbf{M}, \mathbf{L})$ *in which*

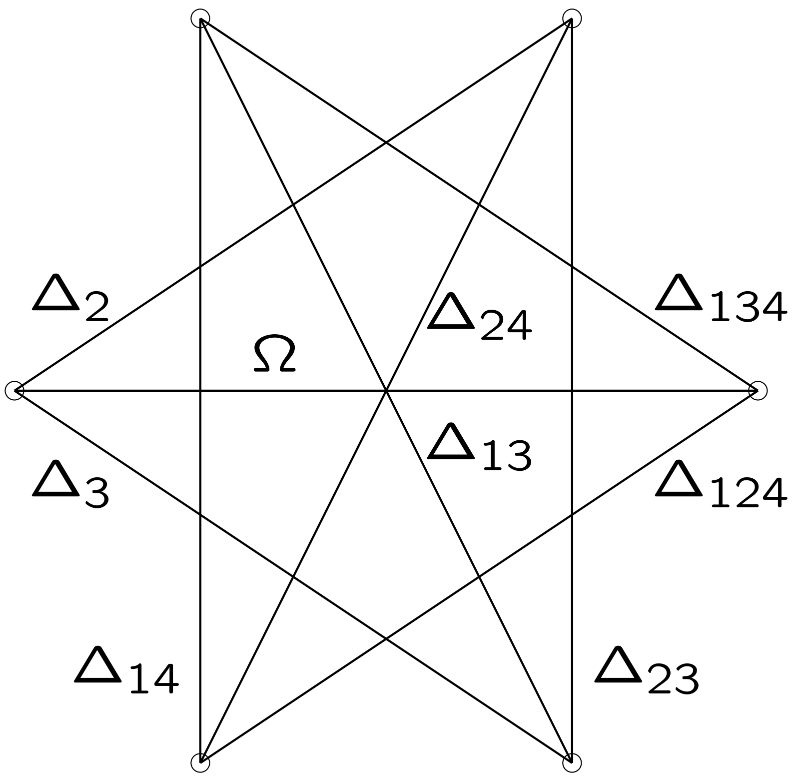
- *cluster variables are labeled by the tagged arcs in* $(\mathbf{S}, \mathbf{M})$*;*
- *coefficient variables are labeled by the laminations in* $\mathbf{L}$*;*
- *clusters correspond to the tagged triangulations in* $(\mathbf{S}, \mathbf{M})$*;*
- *exchange relations correspond to the tagged flips, and are encoded by the matrices* $\tilde{B}(T, \mathbf{L})$*.*

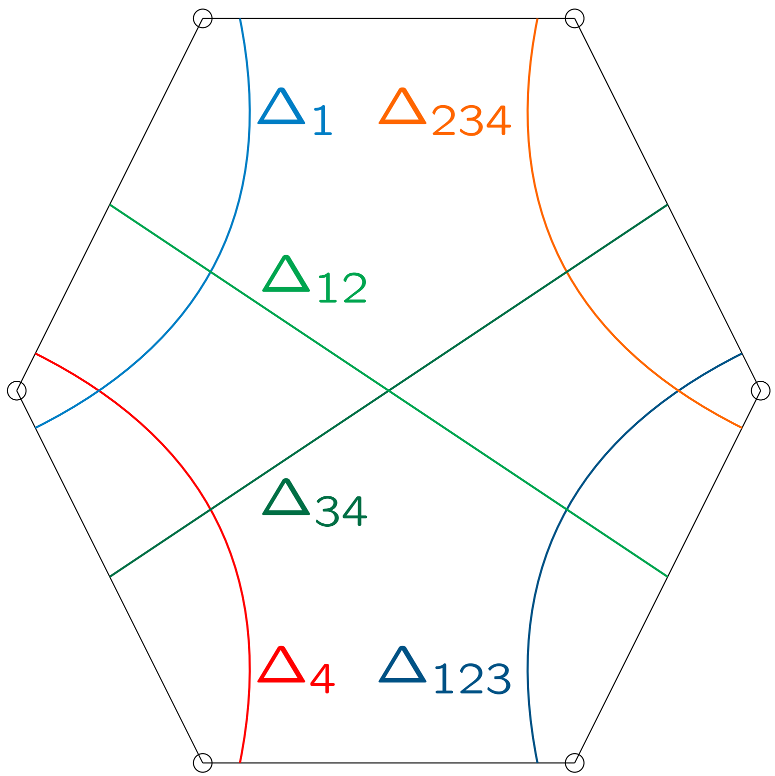
In view of Thurston's theorem, *any* cluster algebra (of geometric type) with exchange matrices $B(T)$ has a topological realization of the above form, for some choice of multi-lamination $\mathbf{L}$.

Example 2, revisited

The cluster algebra $\mathbb{C}[SL_4]^N$ can be described by the multi-lamination shown below. Its generators correspond to

- the 9 diagonals of the hexagon (cluster variables) and
- the 6 laminations shown below (coefficient variables).





Example 3: 3×3 matrices

The ring of polynomials in the 9 matrix entries of a 3×3 matrix

$$z = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix}$$

carries a natural cluster algebra structure of cluster type D_4. It has 21 distinguished generators: the 19 minors of z plus two additional polynomials:

$$\Omega_1 = z_{12}z_{21}z_{33} - z_{12}z_{23}z_{31} - z_{13}z_{21}z_{32} + z_{13}z_{22}z_{31},$$
$$\Omega_2 = z_{11}z_{23}z_{32} - z_{12}z_{23}z_{31} - z_{13}z_{21}z_{32} + z_{13}z_{22}z_{31}.$$

These 21 generators split into 5 coefficient variables

$$z_{13},\ z_{31},\ \Delta_{12,23} = \left\| \begin{matrix} z_{12} & z_{13} \\ z_{22} & z_{23} \end{matrix} \right\|,\ \Delta_{23,12} = \left\| \begin{matrix} z_{21} & z_{22} \\ z_{31} & z_{32} \end{matrix} \right\|,\ \det(z),$$

and 16 cluster variables (the remaining ones).

Example 3: The cluster structure

The cluster structure on $\mathbb{C}[\mathrm{Mat}_{3,3}]$ is determined by the following initial data. As an initial cluster, we take

$$\mathbf{x} = (z_{12}, z_{32}, \Delta_{13,12}, \Delta_{13,23}).$$

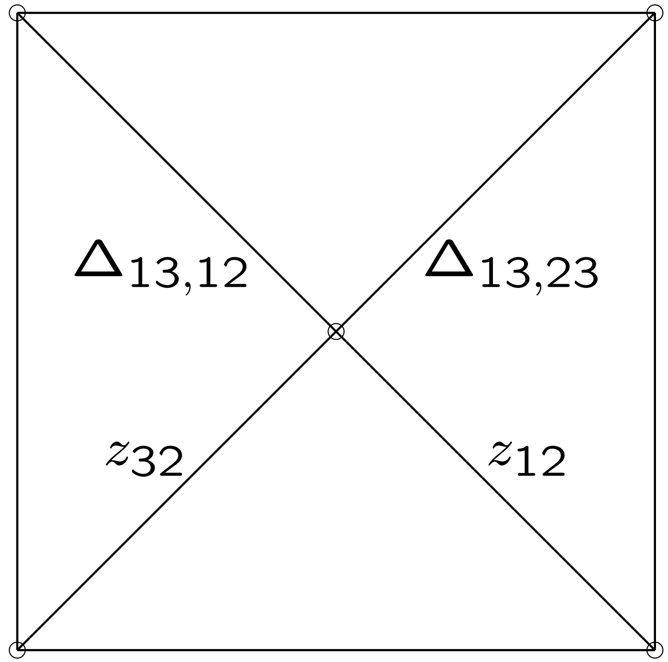
The exchange relations from $\mathbf{x}$ are:

$$\begin{aligned}
z_{12}\, z_{33} &= z_{13}\, z_{32} + \Delta_{13,23} \\
z_{32}\, z_{11} &= \Delta_{13,12} + z_{31}\, z_{12} \\
\Delta_{13,12}\, \Delta_{23,23} &= \Delta_{23,12}\, \Delta_{13,23} + \det(z)\, z_{32} \\
\Delta_{13,23}\, \Delta_{12,12} &= \det(z)\, z_{12} + \Delta_{12,23}\, \Delta_{13,12}
\end{aligned}$$

Example 3: Building a triangulated surface

These exchange relations are encoded in the matrix $\tilde{B}$ below:

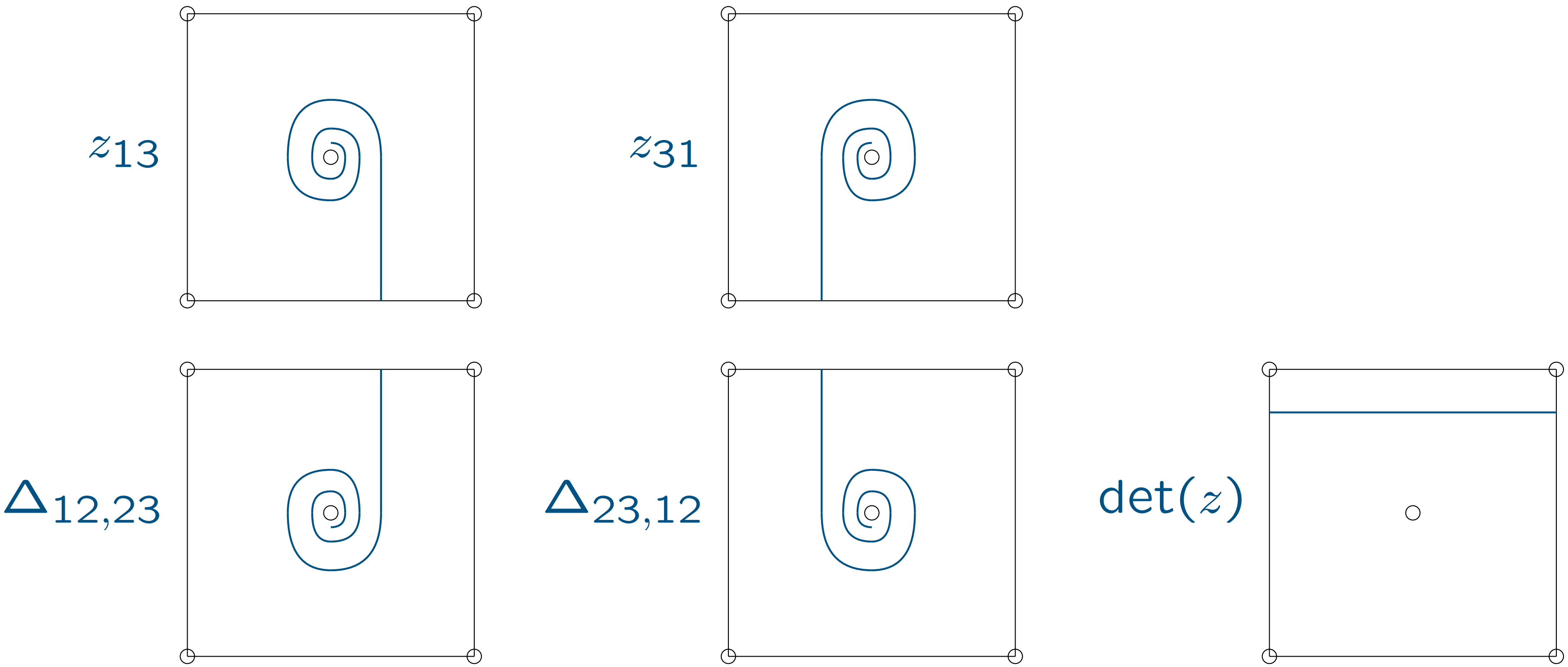
z_{12}	0	−1	0	1
z_{32}	1	0	−1	0
$\Delta_{13,12}$	0	1	0	−1
$\Delta_{13,23}$	−1	0	1	0
z_{13}	1	0	0	0
z_{31}	0	−1	0	0
$\Delta_{12,23}$	0	0	0	−1
$\Delta_{23,12}$	0	0	1	0
$\det(z)$	0	0	−1	1



The initial exchange matrix is a signed adjacency matrix for a (non-unique) triangulation of a once-punctured quadrilateral.

Example 3: Constructing the laminations

Interpreting the rows in the bottom part of $\tilde{B}$ as vectors of shear coordinates, we reconstruct the 5 laminations corresponding to the coefficient variables:



Example 3: The final picture

We can now give a complete description of the cluster algebra $\mathcal{A} = \mathbb{C}[\text{Mat}_{3,3}]$.

The cluster variables in $\mathcal{A}$ are labeled by the 16 tagged arcs in the quadrilateral. They form 50 clusters of size 4, one for each of the 50 tagged triangulations, or each of the 50 vertices of the type D_4 associahedron.

The exchange relations in $\mathcal{A}$ are encoded by the matrices $\tilde{B}(T, \mathbf{L})$ associated to a tagged triangulation T and the multi-lamination $\mathbf{L}$ that we constructed.

Exercise. *Determine which cluster variable corresponds to which tagged arc.*

Total positivity

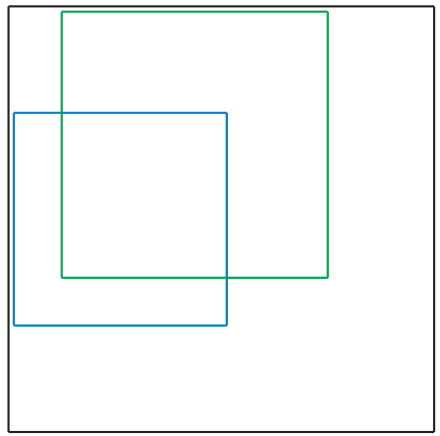
Let $\mathcal{A} = \mathbb{C}[Z]$ be a cluster algebra realized as a ring of regular functions on an algebraic variety Z. A point $z \in Z$ is *totally positive* if all generators of $\mathcal{A}$ take positive values at z. Such points form the *totally positive variety* $Z_{>0}$ associated with $\mathcal{A}$. Since cluster transformations are *subtraction-free*, it is enough to check the positivity at z of the elements of any extended cluster.

This notion can be related to the classical concept of total positivity for square matrices, leading to a family of efficient total positivity criteria.

Total positivity criteria

There is a natural cluster algebra structure in the ring $\mathbb{C}[\mathrm{Mat}_{n,n}]$ of polynomials in the matrix entries of an $n \times n$ matrix, generalizing Example 3. The set of coefficient and cluster variables includes all minors of the matrix.

Each extended cluster in $\mathbb{C}[\mathrm{Mat}_{n,n}]$ has cardinality $n^2 = \dim(\mathrm{Mat}_{n,n})$. One such extended cluster consists of all *solid initial minors*:



We then recover the following total positivity criterion.

Theorem 19 [M. Gasca and J. M. Peña, *Linear Algebra Appl.* **165** (1992)] *An $n \times n$ matrix is totally positive if and only if all its n^2 solid initial minors are positive.*

Cluster variables as lambda lengths

The cluster variables in a cluster algebra associated with a multi-lamination on a bordered surface can be intrinsically interpreted as suitably renormalized *lambda lengths* (=Penner coordinates) on an appropriate *decorated Teichmüller space*.

In this geometric realization, the Teichmüller space plays the role of the corresponding totally positive variety.

Decorated Teichmüller space

(unpunctured case, after R. Penner)

Assume that all marked points in $\mathbf{M}$ lie on the boundary of $\mathbf{S}$.

The (cusped) *Teichmüller space* $\mathcal{T}(\mathbf{S},\mathbf{M})$ consists of all complete finite-area hyperbolic structures with constant curvature -1 on $\mathbf{S}\setminus\mathbf{M}$, with geodesic boundary at $\partial\mathbf{S}\setminus\mathbf{M}$, considered up to diffeomorphisms of $\mathbf{S}$ fixing $\mathbf{M}$ that are homotopic to the identity. (Thus there is a cusp at each point of $\mathbf{M}$.)

A point in the *decorated Teichmüller space* $\widetilde{\mathcal{T}}(\mathbf{S},\mathbf{M})$ is a hyperbolic structure as above together with a collection of *horocycles*, one around each cusp at a marked point $p\in\mathbf{M}$.

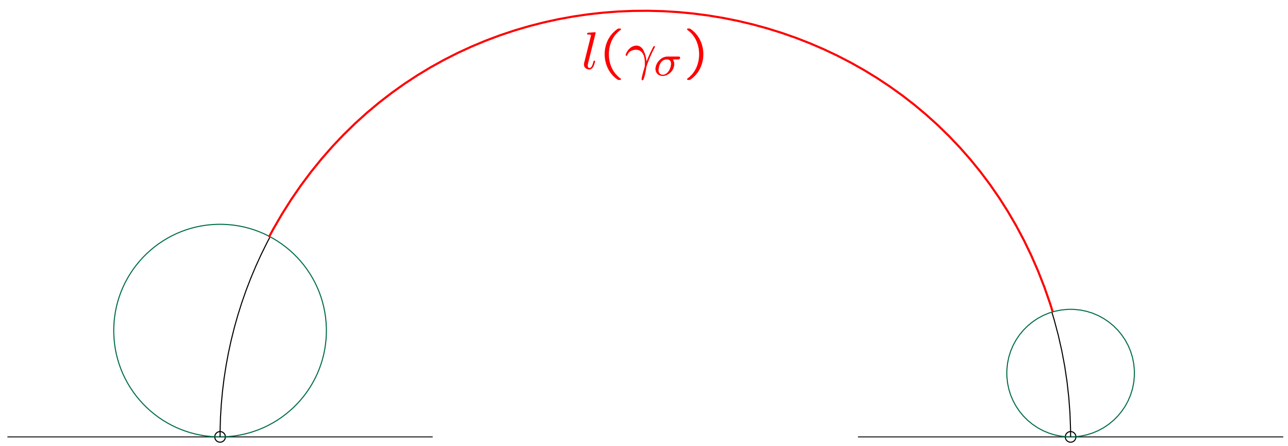
Such an horocycle can be viewed as a set of points "at an equal distance" from the cusp. This enables us to compare "distances" to the cusp from two different points.

Lambda lengths

Let γ be an arc in $(\mathbf{S}, \mathbf{M})$, or a boundary segment between two adjacent marked points. For a decorated hyperbolic structure in $\widetilde{\mathcal{T}}(\mathbf{S}, \mathbf{M})$, the *lambda length* $\lambda(\gamma) = \lambda_\sigma(\gamma)$ is defined as follows. Take the unique geodesic γ_σ representing γ, and let

$$\lambda(\gamma) = \exp(l(\gamma_\sigma)/2),$$

where $l_\sigma(\gamma)$ denotes the signed distance along γ_σ between the horocycles at either end of γ.



Penner's coordinatization

For a given γ, one can view the lambda length

$$\lambda(\gamma) : \sigma \mapsto \lambda_\sigma(\gamma)$$

as a function on the decorated Teichmüller space $\widetilde{\mathcal{T}}(\mathbf{S},\mathbf{M})$. These functions can be used to coordinatize $\widetilde{\mathcal{T}}(\mathbf{S},\mathbf{M})$:

Theorem 20 [R. Penner] *Let T be a triangulation of $(\mathbf{S},\mathbf{M})$ without self-folded triangles. Then the map*

$$\prod_\gamma \lambda(\gamma) : \widetilde{\mathcal{T}}(\mathbf{S},\mathbf{M}) \to \mathbb{R}^m_{>0}$$

is a homeomorphism. Here γ runs over the arcs in T and the boundary segments of $(\mathbf{S},\mathbf{M})$.

Laminated lambda lengths

Fix a multi-lamination $\mathbf{L} = (L_i)_{i\in I}$. For an arc or boundary segment γ, the *laminated lambda length* of γ is defined by

$$\lambda_{\sigma,\mathbf{L}}(\gamma) = \lambda_\sigma(\gamma) \prod_{i\in I} q_i^{l_{L_i}(\gamma)/2},$$

where $q = (q_i)_{i\in I}$ is a collection of positive real parameters, and $l_{L_i}(\gamma)$ denotes the *transversal measure* of γ with respect to L_i.

A point (σ, q) in the *laminated Teichmüller space* $\overline{\mathcal{T}}(\mathbf{S},\mathbf{M},\mathbf{L})$ is a decorated hyperbolic structure $\sigma \in \widetilde{\mathcal{T}}(\mathbf{S},\mathbf{M})$ together with a vector $q \in \mathbb{R}^I_{>0}$ satisfying the boundary conditions $\lambda_{\sigma,\mathbf{L}}(\gamma) = 1$ for all boundary segments in $(\mathbf{S},\mathbf{M})$. Penner's theorem implies that this space is coordinatized by the lambda lengths of arcs in a fixed triangulation together with the parameters q_i:

$$\overline{\mathcal{T}}(\mathbf{S},\mathbf{M},\mathbf{L}) \cong \mathbb{R}^{n+|I|}_{>0}.$$

Geometric model, unpunctured case

Theorem 21 *The laminated Teichmüller space* $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M}, \mathbf{L})$ *is canonically isomorphic to the totally positive variety associated with the cluster algebra* $\mathcal{A}(\mathbf{S}, \mathbf{M}, \mathbf{L})$*. Under this isomorphism, cluster variables are represented by the laminated lambda lengths of arcs, while the coefficient variables are represented by the parameters* q_i *associated with the laminations in* $\mathbf{L}$*.*

Thus, the lambda lengths $\lambda_{\sigma,\mathbf{L}}(\gamma)$ satisfy the exchange relations encoded by the extended signed adjacency matrices $\tilde{B}(T, \mathbf{L})$.

Geometric model, general case

When we allow marked points in the interior of $\mathbf{S}$, the construction becomes substantially more complicated:

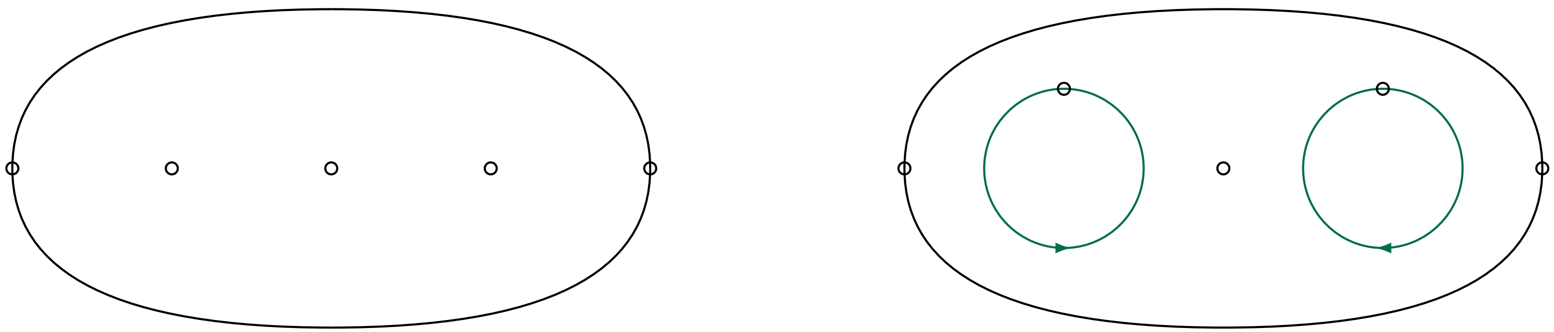
- we need to lift laminations and arcs to the covering space to handle intersection numbers and lambda lengths in the presence of spiralling;

- we need to set up the proper generalization of a decorated Teichmüller space;

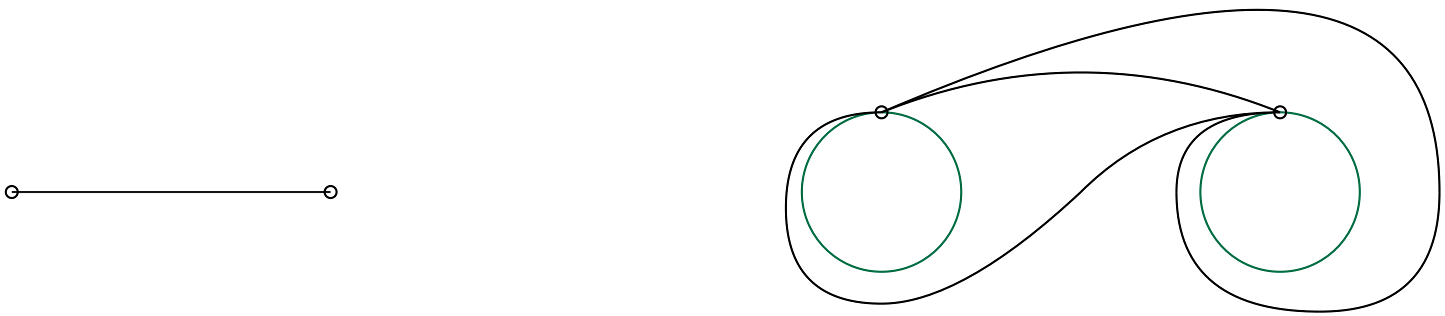
- we need to realize cluster variables as well-defined functions on this space, independent of the choices of lifts.

Opened surface

We open each puncture in $\mathbf{M}$ to a circular boundary component; orient it in one of three ways: clockwise, counterclockwise, or degenerate (no opening); and pick a marked point on it.

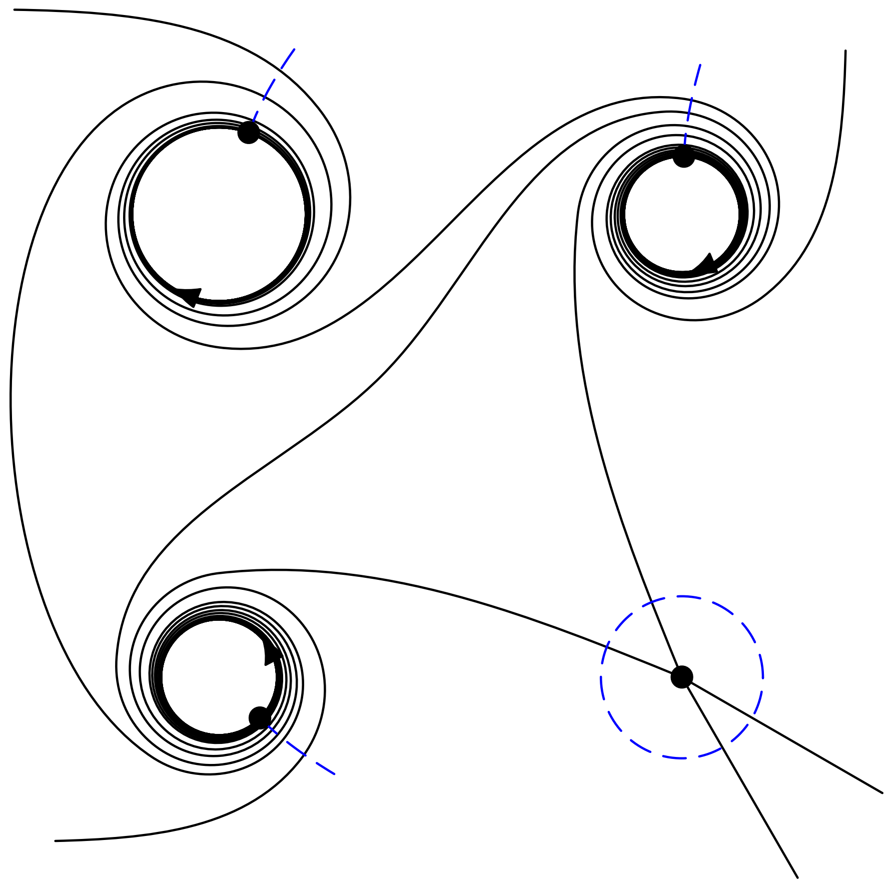


Each arc on $(\mathbf{S}, \mathbf{M})$ incident to a puncture can be *lifted* to infinitely many different arcs on the opened surface.



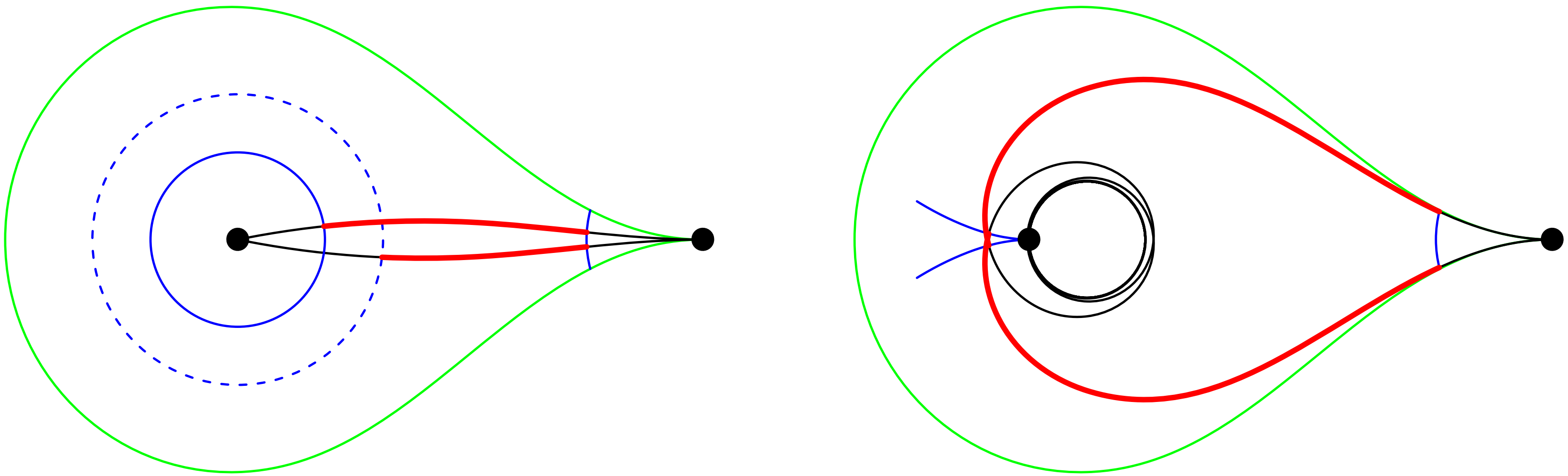
Teichmüller space $\overline{\mathcal{T}}(\mathrm{S}, \mathrm{M})$ associated with opened surfaces

A point in $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M})$ includes: an orientation of each nontrivial opening C; a decorated hyperbolic structure with geodesic boundary along C; and an horocycle at the marked point chosen on C. This horocycle should be perpendicular to C and to all geodesics that spiral into C in the chosen direction.



Lambda lengths of lifted tagged arcs

To define lambda lengths of tagged arcs on the opened surface, we use *conjugate* horocycles:



These lambda lengths can be used to coordinatize the space $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M})$.

Laminated lambda lengths, general case

Fix a lift $\overline{L}$ of each lamination $L \in \mathbf{L}$ to the opened surface.

Choose a lift $\overline{\gamma}$ of each tagged arc γ to the opened surface.

Both the lambda length of $\overline{\gamma}$ and its transverse measure with respect to $\overline{L}$ will depend on the choice of the lift $\overline{\gamma}$.

But: if the parameters q_i and the lengths of the boundary segments and openings satisfy appropriate boundary conditions, then the same formula as before defines a "laminated lambda length" $\lambda_{\sigma,\mathbf{L}}(\overline{\gamma})$ that depends only on γ but not on the choice of its lift $\overline{\gamma}$.

These functions satisfy the requisite exchange relations, providing a geometric realization of the corresponding cluster algebra, and identifying the appropriate modification of the decorated Teichmüller space with the associated totally positive variety.