Eynard-Orantin Recursion and Quantum Algebraic Curves

Quantum Curves for Geometric Enumeration

Problems

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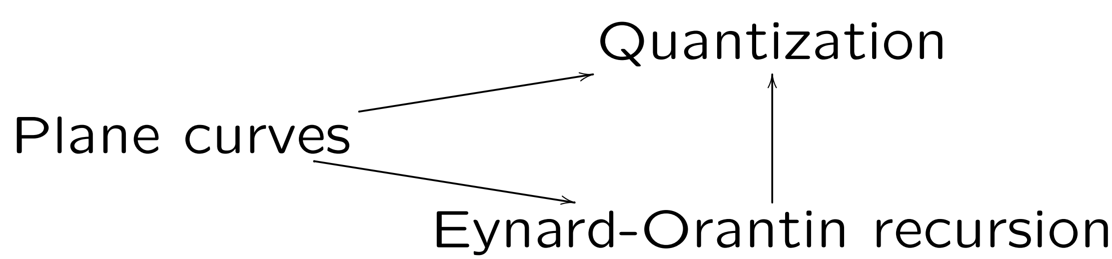
“A Tale of Two Theories”

(a) Knot/link invariants via Chern-Simons theory

(b) Gromov-Witten theory of toric Calabi-Yau 3-folds

They are “dual to each other”:

Both involve the following objects:



1. Quantization

1.1. Classical mechanics

In classical mechanics, a particle moving in a line is described by two variables: x: its coordinate; p: its momentum.

So the particle is moving in the phase space $\mathbb{R}^2$, along a trajectory $c(t) = (x(t), p(t))$.

Observable physical quantities are functions $f(x, p)$.

The dynamics of the particle is governed by a function called the Hamitonian function $H(x, p)$.

The Hamiltonian equation is:

$$\frac{d}{dt} f(x(t), p(t)) = -\{H(x, p), f(x, p)\},$$

where the $\{\cdot, \cdot\}$ is the Poisson bracket defined as follows:

$$\{f(x, p), g(x, p)\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}.$$

1.2. Canonical quantization

In quantum mechanics, physical observables are operators on certain Hilbert spaces of functions.

Canonical quantization is the following procedure:

Replace x by an operator $\widehat{x} := x\cdot$,

and replace p by an operator $\widehat{p} = -i\hbar\partial_x$.

I.e.,

$$\widehat{x}f(x) = x \cdot f(x),$$
$$\widehat{p}f(x) = -i\hbar\partial_x f(x).$$

These two operators do not commute with each other:

$$[\widehat{x}, \widehat{p}] = i\hbar.$$

This is called the Heisenberg commutation relation.

Normal ordering: replace a monomial $x^m p^n$ by the operator $\widehat{x}^m \widehat{p}^n$.

The polynomial ring $\mathbb{R}[x, p]$ is a Lie algebra under the Poisson bracket:

$$\{x^{m_1} p^{n_1}, x^{m_2} p^{n_2}\} = (m_1 n_2 - n_1 m_2) x^{m_1+m_2-1} p^{n_1+n_2-1}.$$

In particular,

$$\{x^{m+1} p, x^{n+1} p\} = (m - n) x^{m+n+1} p.$$

I.e. $\mathbb{R}[x]p$ is a Lie algebra under $\{\cdot, \cdot\}$.

After the quantization, a polynomial $\sum_{m,n} a_{m_1,n} x^m p^n$ becomes a differential operator

$$\sum_{m,n\geq 0} a_{m,n}\widehat{x}^m(-i\hbar\partial_x)^n.$$

The space of differential operators is noncommutative.

Dirac wished that after the canonical quantization, one has

$$\widehat{\{f,g\}} = \frac{1}{i\hbar}[\widehat{f},\widehat{g}].$$

This holds for $f, g \in \mathbb{R}[x]p$, but not in general.

1.3. Schrödinger equation

After the quantization, the Hamiltonian equation becomes the Schrödinger equation:

$$\frac{d}{dt}\psi = \widehat{H(x,p)}\psi.$$

Suppose that $\frac{d}{dt}\psi = 0$, then the Schrödinger equation becomes:

$$\widehat{H(x,p)}\psi = 0.$$

1.4 The quantum plane

Sometimes we will also be concerned with quantization of elements in $\mathbb{R}[e^x, e^p]$.

We will take

$$\sum_{m,n} \widehat{a_{m_1,n} e^{mx}} e^{np} = \sum_{m,n} a_{m_1,n} e^{m\widehat{x}} e^{n\widehat{p}} = \sum_{m,n \geq 0} a_{m,n} e^{m\widehat{x}} e^{n(-i\hbar\partial_x)}.$$

From the Heisenberg commutation relation one can see that

$$e^{\widehat{x}} e^{\widehat{p}} = e^{i\hbar} \cdot e^{\widehat{p}} e^{\widehat{x}}.$$

If one write $X = e^{\widehat{x}}$, $Y = e^{\widehat{p}}$ and $q = e^{i\hbar}$, then one gets:

$$XY = qYX.$$

This gives the defining relation of the coordinate ring of the quantum plane.

2. Plane Curves

2.1. Plane curves

They are defined by

$$\{(u,v) \in \mathbb{C}^2 : \;\; A(u,v) = 0\}.$$

We are also interested in the cases when A are polynomials in $x = e^u$ and $y = e^v$.

2.2 Examples

Airy curve:

$$\frac{1}{2}v^2 - u = 0.$$

Lambert curve:

$$y - xe^y = 0.$$

Local mirror curve for $\mathbb{C}^3$:

$$1 - y - xy^{-a} = 0.$$

Local mirror curve the resolved conifold:

$$y + xy^{-a} - 1 - e^{-t}xy^{-a-1} = 0.$$

A-polynomial for the figure-8 knot:

$$(1 - x^2 - 2x^4 - x^6 + x^8)y - x^4 - x^4y^2 = 0.$$

2.3 Why are these curves interesting?

They are examples of such curves which appear naturally in various mathematical contexts.

(1) The Airy curve arises in the study of intersection theory of Deligne-Mumford moduli spaces of algebraic curves $\overline{\mathcal{M}}_{g,n}$.

(2) The Lambert curve arises in the study of Hurwitz numbers.

(3) According to the recipe of Hori-Iqbal-Vafa, the mirror of a toric Calabi-Yau 3-fold is locally a curve in $(\mathbb{C}^*)^2$.

(4) For a knot one can associated a plane algebraic curve.

3. AJ Conjecture for Knots

3.1 A-polynomial of a knot in S^3.

Given a knot $K \subset S^3$, let U_K be a tubular neighborhood of K in S^3. Let $M_K := S^3 \backslash U_K$. We are interested in the representation variety

$$X(M_K) := \text{Hom}(\pi_1(M_K), SL(2, \mathbb{C}))/\sim,$$

where $\sim$ means modulo conjugations.

We have a homomorphism

$$\pi_1(\partial M_K) \to \pi_1(M_K),$$

and so an induced restriction map

$$r : X(M_K) \to \text{Hom}(\pi_1(\partial M_K), SL(2, \mathbb{C}))/\sim .$$

Because $\partial(M_K) \cong T^2$, we have $\pi_1(M_K) \cong \pi_1(T^2) \cong \mathbb{Z}^2$ is a free abelian group with two generators.

Hence we have

$$\mathrm{Hom}(\pi_1(\partial M_K), SL(2,\mathbb{C})/\sim \cong (\mathbb{C}^*)^2.$$

The union of one-dimensional images of components of $X(M_K)$ in $(\mathbb{C}^*)^2$ form an algebraic defined by a polynomial A_K in two variables.

It is called the A-polynomial of the knot K.

3.2 AJ Conjecture

Garoufalidis proposed a conjecture on q-defference equations for the colored Jones polynomials of knots:

$$\widehat{A}_K(l, m; q) J_n(K; q) = 0,$$

where the actions of the operators $\widehat{l}$, $\widehat{m}$ are defined by

$$\widehat{l} J_n(K; q) = J_{n+1}(K; q), \;\; \widehat{m} J_n(K; q) = q^{n/2} J_n(K; q).$$

This operator $\widehat{A}_K$ is conjectured to be obtained by suitable quantization of the A-polynomial A_K of K.

4. Duality between Link Invariants and Gromov-Witten Invariants

4.1 Toric Calabi-Yau 3-folds

Let S be a toric Fano surface. The total space of its canonical line bundle K_S is a noncompact toric Calabi-Yau 3-fold.

The toric geometry means K_S can be obtained by gluing pieces of $\mathbb{C}^3$ according to the toric data. (For example, $\kappa_{\mathbb{P}2}$ can be obtained by gluing three pieces of $\mathbb{C}^3$.)

This is analogous to the construction of a real hyperbolic 3-manifold by gluing tetrahedra: $\mathbb{C}^3 \Leftrightarrow$ tetrahedra.

This oversimplified picture suggests some deeper analogies between the two kinds of objects.

4.2 Local Gromov-Witten invariants by link invariants

Vafa and his collaborators discovered by physics argument a method to compute the local Gromov-Witten invariants of toric Calabi-Yau 3-folds by the colored HOMFLY polynomials of the Hopf link, called the topological vertex.

In a series of papers, a mathematical theory of the topological vertex has been established (Liu-Liu-Z., JDG 03; Liu-Liu-Z. JAMS 07; Li-Liu-Liu-Z., Geom. Top. 09).

Such results suggests that link invariants of real hyperbolic 3-manifolds and the Gromov-Witten invariants of toric Calabi-Yau 3-folds share many common features, and the method used to study one of them might be also useful to study the other.

4.3 Local mirror curves

We have seen plane curves associated to knots.

One can also associate plane algebraic curves to toric CY 3-folds: local mirror curves (Hori-Iqbal-Vafa recipe).

In knot/link theory, one has the issue of framing. Same in the CY case.

For framed $\mathbb{C}^3$, the framed local mirror curve is given by:

$$1 - y - xy^{-a} = 0.$$

For the resolved conifold, the framed local mirror curve is given by:

$$y + xy^{-a} - 1 - e^{-t}xy^{-a-1} = 0.$$

5. "A Tale of Two Theories": More details

Theory 1. Knot theory

K1. Knot K,

K2. Colored Jones polynomials $J_N(K)$,

K4. Plane curve defined by A-polynomial A_K,

K5. Quantization $\widehat{A}_K$ of A_K,

K6. AJ Conjecture $\widehat{A}_K J_N = 0$.

Theory 2. Gromov-Witten theory of toric Calabi-Yau 3-folds

CY1. Toric Calabi-Yau 3-fold X

CY3. Open Gromov-Witten invariants $F_{g,n}$ of X

CY4. Local mirror curve of X

CY5. Quantization of the local mirror curve

CY7. BKMP conjecture: Open GW invariants of X satisfy the Eynard-Orantin topological recursion determined by the local mirror curve.

What are missing from the above pictures:

K3, K7 on the knot theory side

CY2, CY6 on the CY side

K3=pertubative invariants.

Colored Jones polynomials are sums of such invariants.

CY2=partition function = exp of certain sum of $F_{g,n}$.

K6 = Dijkgraaf-Fuji-Manabe Conjecture: There is a correspondence between the perturbative invariants of $SL(2;\mathbb{C})$ Chern-Simons gauge theory and the free energies of the topological string defined by Eynard-Orantin on the algebraic curve defined by the A-polynomial.

CY5= Gukov-Sułkowski Conjecture: $\hat{A}Z = 0$.

6. Enyard-Orantin Recursion and BKMP Conjecture

6.1 Eynard-Orantin topological recursion

Eynard and Orantin studied the spectral curve in matrix model and discover a method to recursively define a sequence of differential forms $W_{g,n}(u_1, \ldots, u_n)$ on a plane algebraic curve

$$A(u, v) = 0.$$

In matrix model theory,

$$\left\langle \mathrm{tr}\left(\frac{1}{u_1 - M}\right) \cdots \mathrm{tr}\left(\frac{1}{u_k - M}\right)\right\rangle_{conn} = \sum_{g=0}^{\infty} \hbar^{2g-2+k} \frac{W_{g,k}(u_1, \ldots, u_k)}{du_1 \cdots du_k},$$

$$Z = \left\langle \mathrm{tr}\left(\frac{1}{u - M}\right)\right\rangle.$$

6.2 BKMP Conjecture

First Mariño then Bouchard-Klemm-Mariño-Pasqetti conjectured that: Starting with the local mirror curve of a toric CY 3-fold X, one can use the Eynard-Orantin recursion to determines its open Gromov-Witten invariants $F_{g,n}(u_1, \dots, u_n)$:

$$W_{g,n}(u_1, \dots, u_n) = \partial_{u_1} \cdots \partial_{u_n} F_{g,n}(x_1, \dots, x_n).$$

The first case of this conjecture (the $X = \mathbb{C}^3$ case) was proved independently by Chen and Z., based on idea of proof of the Hurwitz number case by Eynard-Mulase-Safnuk.

Eynard and Orantin have produced a proof in general.

7. Gukov-Sułkowski Conjecture

Fix one parametrization of the curve

$$A(u, v) = 0$$

given by

$$u = u(z), \qquad v = v(z).$$

Let

$$W_{g,n}(p_1, \dots, p_n) = \mathcal{W}_{g,n}(p_1, \dots, p_n) dp_1 \cdots dp_n.$$

be defined using the EO topological recursion.

Define

$$S_0 = \int^z v(z)du(z) = \int^z z^2 dz,$$

$$S_1 = -\frac{1}{2}\log\frac{du}{dz},$$

$$S_n = \sum_{2g-1+k=n} \frac{(-1)^k}{k!}\int^z dz'_1 \cdots \int^z dz'_n \mathcal{W}_{g,k}(z'_1, \ldots, z'_k)$$

$$= \sum_{2g-1+k=n} \frac{(-1)^k}{k!}\Xi_{g,k}(z, \ldots, z),$$

where

$$\Xi_{g,n}(z_1, \ldots, z_n) = \int^{z_1} \cdots \int^{z_n} \mathcal{W}_{g,n}(z_1, \ldots, z_n) dz_1 \cdots dz_n.$$

For example,

$$S_2 = -\Xi_{1,1}(z) - \frac{1}{3!}\Xi_{0,3}(z,z,z).$$

$$S_3 = \frac{1}{2!}\Xi_{1,2}(z,z) + \frac{1}{4!}\Xi_{0,4}(z,z,z,z).$$

Remark. (a) Need C to have genus 0.

(b) There is ambiguity in taking antiderivatives.

Define

$$Z = \exp \sum_{n=0}^{\infty} \hbar^{n-1} S_n.$$

This Z corresponds to the colored Jones polynomial J_N in the knot theory.

The enumerative meaning of S_n is

$$S_n = \sum_{2g-1+k=n} \frac{1}{k!} F_{g,k}(z, \dots, z),$$

where $F_{g,n}(x_1, \dots, x_n)$ is some suitably defined n-point function in genus g.

Gukov-Sułkowski Conjecture. There is a quantization $\widehat{A}(\widehat{u}, \widehat{v})$ of $A(u, v)$ such that

$$\widehat{A}(\widehat{u}, \widehat{v}) Z = 0.$$

This corresponds to the AJ Conjecture in the knot theory.

7.2 Our results

Theorem. Gukov-Sułkowski Conjecture holds for the following curves: Airy curve:

$$\frac{1}{2}v^2 - u = 0.$$

Lambert curve:

$$y - xe^y = 0.$$

Framed local mirror curve for $\mathbb{C}^3$:

$$1 - y - xy^{-a} = 0.$$

Framed local mirror curve the resolved conifold:

$$y + xy^{-a} - 1 - e^{-t}xy^{-a-1} = 0.$$

8. The Airy curve case of Gukov-Sułkowski Conjecture

8.1. Topological recursion for the Airy curve = DVV relations:

Theorem (Bennett et al., Z.) Eynard-Orantin recursion for the Airy curve is equivalent to the Dijikgraaf-Verlinde-Verlinde Virasoro constraints satisfied by the Witten-Kontsevich tau-function.

On $\overline{\mathcal{M}}_{g,n}$, the Deligne-Mumford moduli spaces of stable algebraic curves, there are line bundles $L_1, \dots, L_n$.

The Witten-Kontsevich tau-function is some generating function of intersection numbers introduced by Mumford:

$$\langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} c_1(L_i)^{a_i}.$$

They can be computed by the Dijkgraaf-Verlinde-Verlinde recursion relations are:

$$
\begin{aligned}
&\langle \tilde{\tau}_{a_0} \prod_{i=1}^{n} \tilde{\tau}_{a_i} \rangle_g \\
= \ & \sum_{i=1}^{n} (2a_i + 1) \langle \tilde{\tau}_{a_0 + a_i - 1} \prod_{j \in [n]_i} \tilde{\tau}_{a_j} \rangle_g \\
+ \ & \frac{1}{2} \sum_{b_1 + b_2 = a_0 - 2} \Big(\langle \tilde{\tau}_{b_1} \tilde{\tau}_{b_2} \prod_{i=1}^{n} \tilde{\tau}_{a_i} \rangle_{g-1} \\
+ \ & \sum_{A_1 \coprod A_2 = [n]} \sum_{g_1 + g_2 = g} \langle \tilde{\tau}_{b_1} \prod_{i \in A_1} \tilde{\tau}_{a_i} \rangle_{g_1} \cdot \langle \tilde{\tau}_{b_2} \prod_{i \in A_2} \tilde{\tau}_{a_i} \rangle_{g_2} \Big),
\end{aligned}
$$

where $\tilde{\tau}_a = (2a+1)!! \cdot \tau_a$ and $[n] = \{1, \ldots, n\}$, $[n]_i = [n] - \{i\}$.

These relations can be rewritten as differential equations:

$$L_m Z = 0, \quad m \geq -1,$$

where $\{L_m : \ m \geq -1\}$ satisfy

$$[L_m, L_n]Z = (m-n)L_{m+n}.$$

I derived the Gukov-Sułkowski Conjecture for the Airy curve case from the DVV relations.

In this case $\hat{A}Z = 0$ is the Airy function and the partition function Z can be identified with the Airy functions $Ai(x)$ or $Bi(x)$.

The Airy curve is just given by:

$$A(u,v) = \frac{1}{2}v^2 - u = 0.$$

We use the following parametrization:

$$u(p) = \frac{1}{2}p^2, \qquad v(p) = p.$$

Eynard-Orantin recursion has as initial values:

$$W_{0,1}(p) = 0, \quad W_{0,2}(p_1, p_2) = B(p_1, p_2) = \frac{dp_1 dp_2}{(p_1 - p_2)^2},$$

and in general:

$$\begin{aligned} & W_{g,n+1}(z_0, z_1, \ldots, z_n) \\ = \ & Res_{z=0}\Bigg[K(z, z_0) \cdot \Bigg(W_{g-1,n+2}(z, -z, z_{[n]}) \\ + & \sum_{\substack{g_1+g_2=g \\ A_1 \coprod A_2=[n]}} W_{g_1,|A_1|+1}(z, z_{A_1}) \cdot W_{g_2,|A_2|+1}(-z, z_{A_2})\Bigg)\Bigg]. \end{aligned}$$

Here we have used the following notations: First of all, $[n]$ denotes the set of indices $\{1, 2, \dots, n\}$; secondly, for $A \subset [n]$, when $A = \emptyset$, z_A is empty; otherwise, if $A = \{i_1, \dots, i_k\}$, then $z_A = z_{i_1}, \dots, z_{i_k}$.

Write $W_{g,n}(z_1, \dots, z_n) = \mathcal{W}_{g,n}(z_1, \dots, z_n) dz_1 \cdots dz_n$, one then has

$$\begin{aligned} & \mathcal{W}_{g,n+1}(z_0, z_1, \dots, z_n) \\ = \; & \frac{1}{2} Res_{z=0} \Bigg[\frac{1}{z(z_0^2 - z^2)} \cdot \Bigg(\mathcal{W}_{g-1,n+2}(z, -z, z_{[n]}) \\ + & \sum_{\substack{g_1+g_2=g \\ A_1 \coprod A_2 = [n]}} \mathcal{W}_{g_1,|A_1|+1}(z, z_{A_1}) \cdot \mathcal{W}_{g_2,|A_2|+1}(-z, z_{A_2}) \Bigg) \Bigg]. \end{aligned}$$

Our theorem states:

$$W_{g,n}(z_1,\ldots,z_n)=\sum_{a_1,\ldots,a_n\geq 0}\langle\tau_{a_1}\cdots\tau_{a_n}\rangle_g\prod_{i=1}^{n}\frac{(2a_i+1)!!}{z_i^{2a_i+2}}.$$

We show this directly by showing the topological recursion relations IS the DVV relations.

8.2. Combinatorial identity predicted by the quantum Airy curve

Choose $z = u^{1/2}$ or $z = -u^{1/2}$, one then expresses S_n in the u-coordinate. For example,

$$S_0 = \pm\frac{1}{3}(2u)^{3/2},$$

$$S_1 = -\frac{1}{4}\log(2u) + constant,$$

$$S_2 = \pm\frac{5}{24(2u)^{3/2}},$$

$$S_3 = \frac{5}{16}\frac{1}{(2u)^3}.$$

The equation $\hat{A}Z = 0$ is equivalent to the following sequence of equations:

$$\frac{1}{2}(\partial_u S_0)^2 = u,$$
$$\frac{1}{2}\partial_u^2 S_0 + \partial_u S_0 \cdot \partial_u S_1 = 0,$$
$$\frac{1}{2}\partial_u^2 S_1 + \partial_u S_0 \cdot \partial_u S_2 + \frac{1}{2}\partial_u S_1 \cdot \partial_u S_1 = 0,$$
$$\frac{1}{2}\partial_u^2 S_{n-1} + \partial_u S_0 \cdot \partial_u S_n + \partial_u S_1 \cdot \partial_u S_{n-1} + \frac{1}{2}\sum_{\substack{i+j=n \\ i,j\geq 2}} \partial_u S_i \cdot \partial_u S_j = 0,$$

where $n > 2$. One can directly check the first three equations, and one can rewrite the last equation as follows:

$$\partial_u S_n = \pm\frac{1}{2(2u)^{1/2}}\left(-\partial_u^2 S_{n-1} + \frac{1}{2u}\partial_u S_{n-1} - \sum_{\substack{i+j=n \\ i,j\geq 2}} \partial_u S_i \cdot \partial_u S_j\right).$$

8.3 Change of coordinates

To prove the Gukov-Sułkowski Conjecture, we change to new coordinates

$$w_i = \frac{1}{z_i^2}.$$

We reformulate the equation to be proved as:

$$w^{5/2}\partial_w S_n = \pm\left((w^{5/2}\partial_w)^2 S_{n-1} + \sum_{\substack{i+j=n\\ i,j\geq 2}} w^{5/2}\partial_w S_i \cdot w^{5/2}\partial_w S_j\right).$$

This can be derived from the Eynard-Orantin type relations reformulated in the w-coordinates.

One gets polynomial expressions for $2g-2+n>0$:

$$\begin{aligned}\omega_{g,n}(w_1,\ldots,w_n) &= \mathcal{W}_{g,n}(z_1,\ldots,z_n)\\ &= \sum_{a_1,\ldots,a_n\geq 0}\langle\tau_{a_1}\cdots\tau_{a_n}\rangle_g\prod_{i=1}^{n}(2a_i+1)!!w_i^{a_i+1}.\end{aligned}$$

We have the following recursion relations:

$$\begin{aligned}&\omega_{g,n+1}(w_0,w_1,\ldots,w_n)\\ =\;& \frac{1}{2}w_0\omega_{g-1,n+2}(w_0,w_0,w_{[n]})\\ +\;& \frac{1}{2}w_0\sum_{\substack{g_1+g_2=g\\ A_1\coprod A_2=[n]}}^{s}\omega_{g_1,|A_1|+1}(w_0,w_{A_1})\cdot\omega_{g_2,|A_2|+1}(w_0,w_{A_2})\\ +\;& \sum_{i=1}^{n}D_{w_0,w_i}\omega_{g,n}(x,w_{[n]_i}),\end{aligned}$$

where for $m \geq 0$, $D_{u,v} : \mathbb{Z}[x] \to \mathbb{Z}[u,v]$ is a linear operator defined by:

$$D_{u,v}x^m = uv(u^m + 3u^{m-1}v + 5u^{m-2}v^2 + \cdots + (2m+1)v^m).$$

Consider the antiderivatives of $\omega_{g,n}$ defined by:

$$\Omega_{g,n} = \sum_{a_1,\ldots,a_n \geq 0} \langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g \prod_{i=1}^{n} (2a_i - 1)!! w_i^{a_i+1/2}.$$

Here we use the following convention: $(-1)!! = 1$.

The functions $\omega_{g,n}$ and $\Omega_{g,n}$ are related by:

$$\omega_{g,n} = 2^n \prod_{j=1}^{n} w_j^{3/2} \cdot \partial_{w_1} \cdots \partial_{w_n} \Omega_{g,n}.$$

The following recursion relations hold:

$$
\begin{aligned}
& \partial_{w_0}\Omega_{g,n+1}(w_0, w_1, \ldots, w_n) \\
= \;& w_0^{5/2}\partial_x\partial_y\Omega_{g-1,n+2}(x, y, w_{[n]})|_{x=y=w_0} \\
+ \;& w_0^{5/2} \sum_{\substack{g_1+g_2=g \\ A_1 \coprod A_2=[n]}}^{s} \partial_{w_0}\Omega_{g_1,|A_1|+1}(w_0, w_{A_1}) \cdot \partial_{w_0}\Omega_{g_2,|A_2|+1}(w_0, w_{A_2}) \\
+ \;& w_0^{-3/2} \sum_{i=1}^{n} \mathcal{D}_{w_0,w_i}\partial_x\Omega_{g,n}(x, w_{[n]_i}).
\end{aligned}
$$

where $\mathcal{D}_{u,v} : \mathbb{C}[x]x^{-1/2} \to \mathbb{C}[u, v]uv^{1/2}$ is a linear operator defined by:

$$\mathcal{D}_{u,v}x^{a-1/2} = uv^{1/2}(u^{a+1} + u^a v + \cdots + v^{a+1}).$$

From this one can derive the identity needed to complete the proof.

9. Other three cases of Gukov-Sułkowski Conjecture

9.1. Our strategy

Step 0. Start with some enumeration problem.

Step 1. Show that suitable generating functions satisfy the topological recursion.

Step 2. Use alternative computations to EXPLICITLY write down the partition functions.

Step 3. Find the difference equations predicted by Gukov and Sułkowski, i.e., find the suitable quantization of the local mirror curve.

9.2 Step 1.

For the Lambert curve, Borot, Eynard, Mulase and Safnuk prove the Hurwitz numbers satisfy the TR.

For the framed local mirror curve of $\mathbb{C}^3$, Chen and Z. proved the one-legged topological vertex satisfies the TR.

For the framed local mirror curve of the resolved conifold, use the Eynard-Orantin proof of the BJMP conjecture.

9.3. Step 2.

Lambert curve case: Burnside formula for Hurwitz numbers.

Framed local mirror curve of $\mathbb{C}^3$ case: Mariño-Vafa formula for one-partition Hodge integrals

Framed local mirror curve of the resolved conifold: Generalization of the Mariño-Vafa formula for two-partition Hodge integral (Z. LLZ), and proof of the full Mariño-Vafa formula (Z.).

These formula establishes the correspondence of the relevant CY 3-folds to the colored HOMFLY polynomials of the unknot.

To give a flavor of what is involved, let me just mention that in the Lambert curve case, it boils down to first show that

$$Z = 1 + \sum_{|\mu|>0} \sum_{|\nu|=|\mu|} e^{\kappa_\nu \lambda/2} \frac{\dim R_\nu}{|\nu|!} \frac{\chi_\nu(\mu)}{z_\mu} \cdot (x/\lambda)^{|\mu|},$$

where

μ: a partition of a positive integer,

R_ν: irreducible representation of the symmetric group indexed by μ,

$\chi_\mu(R_\nu)$: character value of irreducible representation R_ν on the conjugacy class indexed by μ.

Step 3.

Using the representation theory of the symmetric group, one can show that

$$Z = \sum_{n=0}^{\infty} e^{n(n-1)\lambda/2} \frac{x^n}{n!\lambda^n}.$$

It is then straightforward to see that

$$(\widehat{y} - \widehat{x} e^{\widehat{y}}) Z = 0,$$

where

$$\widehat{x} = x\cdot, \quad \widehat{y} = \lambda x \frac{\partial}{\partial x}.$$

The curve

$$x = ye^{-y}$$

is the Lambert curve, by Lagrange inversion formula:

$$y = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} x^n$$

is the series that counts the rooted trees.

The series

$$W(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^{n-1}}{n!} x^n$$

is called the Lambert W-function.

In the framed $\mathbb{C}^3$ case:

$$Z = \sum_{n=0}^{\infty} \frac{q^{-an(n-1)/2+n/2}}{\prod_{j=1}^{n}(1-q^j)} x^n.$$

When $a = 0$, one gets the quantum dilogarithm:

$$Z = \sum_{n=0}^{\infty} \frac{q^{n/2}}{\prod_{j=1}^{n}(1-q^j)} x^n = \prod_{n=1}^{\infty} \frac{1}{1 - xq^{n-1/2}}.$$

This is also the partition for the hyperbolic tetrahedra in Chern-Simons theory.

The following equation is satisfied:

$$(1 - \widehat{y} - e^{\lambda/2}\widehat{x}\widehat{y}^{-a})Z = 0,$$

where

$$\widehat{x} = x\cdot, \quad \widehat{y} = e^{\lambda x \frac{\partial}{\partial x}}.$$

Quantum anomaly:

I have shown by computing the disc invariants that the mirror curve can be written as:

$$x - y^a + y^{a+1} = 0.$$

Our result shows that one needs to change the above equation to

$$A(x, y) = 1 - y - xy^{-a} = 0$$

before taking the quantization. Furthermore, higher order quantum corrections introduce an extra factor of $e^{\lambda/2}$ for $\widehat{x}$, i.e., one should take $\widehat{x} = e^{\lambda/2} \cdot x$.

In the framed resolved conifold case:

$$Z = \sum_{n=0}^{\infty} \prod_{j=1}^{n} \frac{1 - e^{-t} q^{-(j-1)}}{1 - q^{-j}} q^{an(n-1)/2 - n/2} x^n.$$

The following equation is satisfied:

$$(1 - \widehat{y} + q^{1/2} \widehat{x} \widehat{y}^{a+1} - q^{1/2} e^{-t} \widehat{x} \widehat{y}^{a}) Z = 0,$$

where

$$\widehat{x} = x\cdot, \quad \widehat{y} = e^{-\sqrt{-1}\lambda x \frac{\partial}{\partial x}}.$$

Quantum anomaly:

I have show that by counting the disc invariants one can get the following equation of the framed mirror curve of the resolved conifold with an outer brane and framing a:

$$y + xy^{-a} - 1 - e^{-t}xy^{-a-1} = 0.$$

By changing x to $-x$ and a to $-a-1$, one gets the following equation:

$$A(x,y) = 1 - y + x^{a+1} - e^{-t}xy^{a} = 0.$$

Our result indicates that when taking the quantization higher order quantum corrections introduce an extra factor of $q^{1/2}$ for $\widehat{x}$, i.e., one should take $\widehat{x} = q^{1/2} \cdot x$.

Question: How to explain the quantum anomaly?

Question: Is it always possible that $\widehat{x} = q^{1/2}x\cdot$ when the spectral curve lies in $\mathbb{C}^* \times \mathbb{C}^*$, after suitable rewriting of the defining equation of the curve?

10. Some references

Some references

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Thank you very much for your attentions!