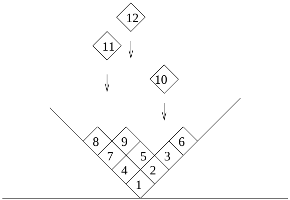
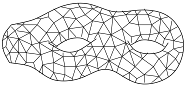
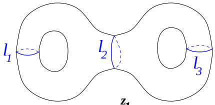
# Applications of Topological Recursion in combinatorics and enumerative geometry.

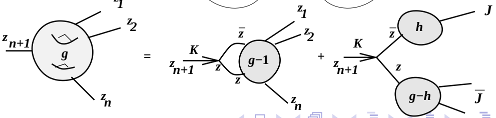

Bertrand Eynard
Institut de Physique Théorique CEA-SACLAY


Aarhus, Jan 2013





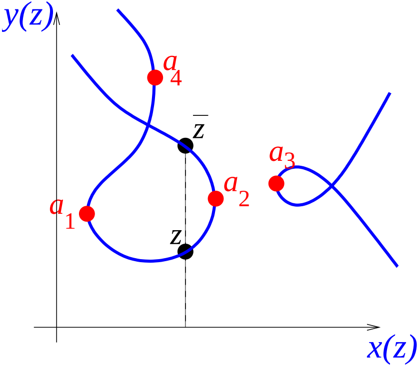



## Outline

- Algebraic definition:
  spectral curves and their invariants $W_n^{(g)}$
- Some applications
  - Topological expansion of Matrix models
  - Discrete surfaces
  - Intersection numbers and double scaling limits of large discrete surfaces, Kontsevich integral
  - Volumes of moduli spaces
  - Partitions, crystal growth and TASEP
  - Gromov-Witten invariants
  - Knot invariants
- Conclusion

# Spectral curve

- Spectral curve: $\mathcal{S} = (\mathcal{C}, x(z), y(z), B)$ $\qquad (x, y : \mathcal{C} \to \mathbb{C})$




- Branchpoints $a_i =$ points with vertical tangent $dx(a_i) = 0$

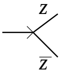
Assumption: regular spectral curve ( $\leftrightarrow \forall i,\ a_i =$ simple).

- Galois conjugate near $a_i$, $\exists!$ point $\bar{z}$ such that $x(\bar{z}) = x(z)$.
- $\mathcal{C}$ =Riemann surf., genus $\mathfrak{g} \to \theta$-function, Alg. Geometry...

# Diagrammatic rules

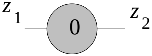
$$z_1 \longrightarrow z \;=\; K(z_1,z) = \frac{1}{2(y(z)-y(\bar z))\,dx(z)}\; d_{z_1} \ln \frac{\theta_*(z_1-\bar z)}{\theta_*(z_1-z)}$$

$$z_1 \;\text{———}\; z_2 \;=\; B(z_1,z_2) = d_{z_1} d_{z_2} \ln \theta_*(z_1-z_2) = \text{2nd kind form}$$

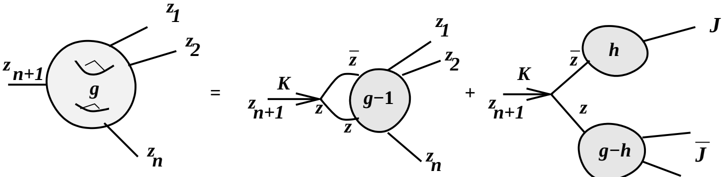



$$= \sum_i \mathrm{Res}_{z\to a_i}$$

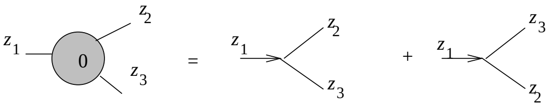
$(g,n)$ invariant: $W_n^{(g)}(z_1,\dots,z_n) = (1,\dots,1)$ form on $\mathcal{C}^n$:




$$= W_2^{(0)}(z_1,z_2) = B(z_1,z_2)$$



# Example $(0,3)$ invariant

$W_3^{(0)}(z_1,z_2,z_3) =$ 


$$= \sum_i \mathrm{Res}_{z\to a_i} K(z_1,z)(B(z,z_2)B(\bar{z},z_3) + B(z,z_3)B(\bar{z},z_2))$$

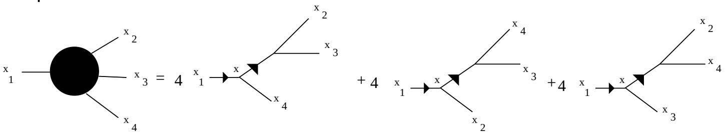
Example: rational curve $z_i \in \mathcal{C} = \mathbb{P}^1$:

$$B(z_1,z_2) = \frac{dz_1 dz_2}{(z_1-z_2)^2}$$

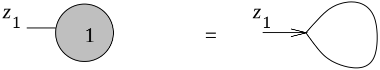
$$W_3^{(0)}(z_1,z_2,z_3) = \sum_i \frac{dz_1 dz_2 dz_3}{x''(a_i)y'(a_i)\ (z_1-a_i)^2(z_2-a_i)^2(z_3-a_i)^2}$$

# More examples

$$W_4^{(0)} =$$

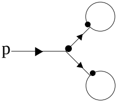



$$W_1^{(1)} =$$

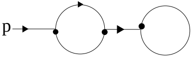



$$= \sum_i \mathrm{Res}_{z \to a_i} K(z_1, z) B(z, \bar{z})$$

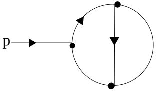
$$W_1^{(2)} =$$




$+ \ 2$




$+ \ 2$



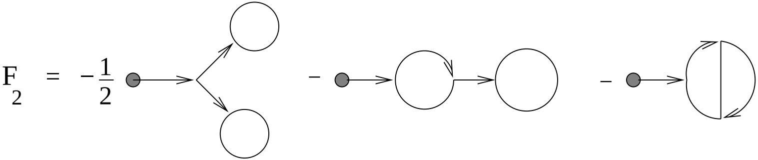

$\ldots$, etc

Definition of the Symplectic Invariant $F_g$, $g \geq 2$:

$$F_g = W_0^{(g)} = \frac{1}{2-2g} \sum_i \mathrm{Res}_{z \to a_i} W_1^{(g)}(z)\,\Phi(z)$$

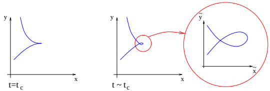
where  $= \Phi(z) = \int_*^z ydx$

Example $F_2$:



Similar definitions (but more involved) for $F_0$ and $F_1$.

- Symmetry: $W_n^{(g)}(z_1, z_2, \ldots, z_n)$ is symmetric in $n$ variables.
- Homogeneity: $y \to \lambda y \Rightarrow F_g \to \lambda^{2-2g} F_g$.
- Symplectic invariance: $dx \wedge dy = d\tilde{x} \wedge d\tilde{y} \Rightarrow F_g(\mathcal{S}) = F_g(\tilde{\mathcal{S}})$.
- Singular limits, universality: If $\mathcal{S}(t)$ is singular at $t = t_c$, then:




$F_g(\mathcal{S}(t)) \sim (t - t_c)^{(2-2g)\mu}\ F_g(\text{Blow up}_{t_c}(\mathcal{S}))$

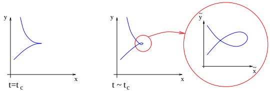
- Integrability: Tau-function $\tau = \exp\left(\sum_g N^{2-2g} F_g(\mathcal{S})\right)\Theta$.
- Holomorphic anomaly equations = modular transformations, backround independence.
- Derivatives Variation of the curve $y(x) \to y(x) + \delta y(x) \Rightarrow$

$$\delta W_n^{(g)}(z_1, \ldots, z_n) = \int_{(\delta y\, dx)^*} W_{n+1}^{(g)}(z_1, \ldots, z_n, z)$$

- And many more ...

# Main properties

- Symmetry: $W_n^{(g)}(z_1, z_2, \ldots, z_n)$ is symmetric in $n$ variables.
- Homogeneity: $y \to \lambda y \Rightarrow F_g \to \lambda^{2-2g} F_g$.
- Symplectic invariance: $dx \wedge dy = d\tilde{x} \wedge d\tilde{y} \Rightarrow F_g(\mathcal{S}) = F_g(\tilde{\mathcal{S}})$.
- Singular limits, universality: If $\mathcal{S}(t)$ is singular at $t = t_c$, then:




$F_g(\mathcal{S}(t)) \sim (t - t_c)^{(2-2g)\mu}\ F_g(\text{Blow up}_{t_c}(\mathcal{S}))$

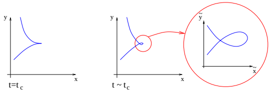
- Integrability: Tau-function $\tau = \exp\left(\sum_g N^{2-2g} F_g(\mathcal{S})\right) \Theta$.
- Holomorphic anomaly equations = modular transformations, backround independence.
- Derivatives Variation of the curve $y(x) \to y(x) + \delta y(x) \Rightarrow$

$$\delta W_n^{(g)}(z_1, \ldots, z_n) = \int_{(\delta y\, dx)^*} W_{n+1}^{(g)}(z_1, \ldots, z_n, z)$$

- And many more ...

# Main properties

- Symmetry: $W_n^{(g)}(z_1, z_2, \dots, z_n)$ is symmetric in $n$ variables.
- Homogeneity: $y \to \lambda y \Rightarrow F_g \to \lambda^{2-2g} F_g$.
- Symplectic invariance: $dx \wedge dy = d\tilde{x} \wedge d\tilde{y} \Rightarrow F_g(\mathcal{S}) = F_g(\tilde{\mathcal{S}})$.
- Singular limits, universality: If $\mathcal{S}(t)$ is singular at $t = t_c$, then:




$$F_g(\mathcal{S}(t)) \sim (t - t_c)^{(2-2g)\mu} \, F_g(\text{Blow up}_{t_c}(\mathcal{S}))$$

- Integrability: Tau-function $\tau = \exp\left(\sum_g N^{2-2g} F_g(\mathcal{S})\right) \Theta$.
- Holomorphic anomaly equations = modular transformations, backround independence.
- Derivatives Variation of the curve $y(x) \to y(x) + \delta y(x) \Rightarrow$

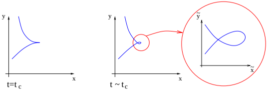
$$\delta W_n^{(g)}(z_1, \dots, z_n) = \int_{(\delta y\, dx)^*} W_{n+1}^{(g)}(z_1, \dots, z_n, z)$$

- And many more ...

# Main properties

- Symmetry: $W_n^{(g)}(z_1, z_2, \ldots, z_n)$ is symmetric in $n$ variables.
- Homogeneity: $y \to \lambda y \Rightarrow F_g \to \lambda^{2-2g} F_g$.
- Symplectic invariance: $dx \wedge dy = d\tilde{x} \wedge d\tilde{y} \Rightarrow$ $F_g(\mathcal{S}) = F_g(\tilde{\mathcal{S}})$.
- Singular limits, universality: If $\mathcal{S}(t)$ is singular at $t = t_c$, then:

$$F_g(\mathcal{S}(t)) \sim (t - t_c)^{(2-2g)\mu} \, F_g(\mathrm{Blow\,up}_{t_c}(\mathcal{S}))$$




$F_g(\mathcal{S}(t)) \sim (t - t_c)^{(2-2g)\mu} \, F_g(\mathrm{Blow\,up}_{t_c}(\mathcal{S}))$

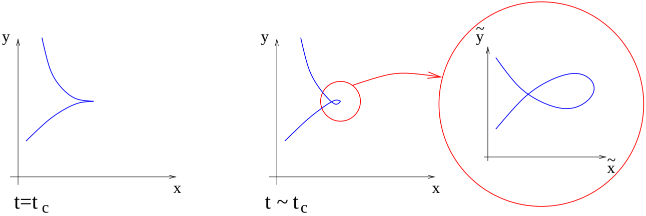
- Integrability: Tau-function $\tau = \exp\left(\sum_g N^{2-2g} F_g(\mathcal{S})\right) \Theta$.
- Holomorphic anomaly equations = modular transformations, backround independence.
- Derivatives Variation of the curve $y(x) \to y(x) + \delta y(x) \Rightarrow$

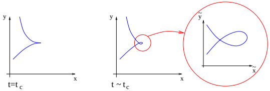
$$\delta W_n^{(g)}(z_1, \ldots, z_n) = \int W_{n+1}^{(g)}(z_1, \ldots, z_n, z)$$

# Main properties

- Symmetry: $W_n^{(g)}(z_1, z_2, \ldots, z_n)$ is symmetric in $n$ variables.
- Homogeneity: $y \to \lambda y \Rightarrow F_g \to \lambda^{2-2g} F_g$.
- Symplectic invariance: $dx \wedge dy = d\tilde{x} \wedge d\tilde{y} \Rightarrow$ $F_g(\mathcal{S}) = F_g(\tilde{\mathcal{S}})$.
- Singular limits, universality: If $\mathcal{S}(t)$ is singular at $t = t_c$, then:

$$F_g(\mathcal{S}(t)) \sim (t - t_c)^{(2-2g)\mu} \, F_g(\mathrm{Blow\,up}_{t_c}(\mathcal{S}))$$

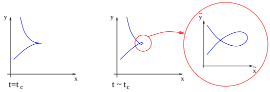



$F_g(\mathcal{S}(t)) \sim (t - t_c)^{(2-2g)\mu} \, F_g(\mathrm{Blow\,up}_{t_c}(\mathcal{S}))$

# Main properties

- Symmetry: $W_n^{(g)}(z_1, z_2, \ldots, z_n)$ is symmetric in $n$ variables.
- Homogeneity: $y \to \lambda y \Rightarrow F_g \to \lambda^{2-2g} F_g$.
- Symplectic invariance: $dx \wedge dy = d\tilde{x} \wedge d\tilde{y} \Rightarrow$ $F_g(\mathcal{S}) = F_g(\tilde{\mathcal{S}})$.
- Singular limits, universality: If $\mathcal{S}(t)$ is singular at $t = t_c$,

  then: 

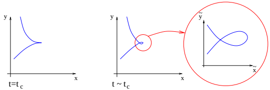

  $F_g(\mathcal{S}(t)) \sim (t - t_c)^{(2-2g)\mu}\ F_g(\text{Blow up}_{t_c}(\mathcal{S}))$
- Integrability: Tau-function $\tau = \exp\left(\sum_g N^{2-2g} F_g(\mathcal{S})\right) \Theta$.
- Holomorphic anomaly equations = modular transformations, backround independence.
- Derivatives Variation of the curve $y(x) \to y(x) + \delta y(x) \Rightarrow$

$$\delta W_n^{(g)}(z_1, \ldots, z_n) = \int_{(\delta y\, dx)^*} W_{n+1}^{(g)}(z_1, \ldots, z_n, z)$$

- And many more ...

# Main properties

- Symmetry: $W_n^{(g)}(z_1, z_2, \ldots, z_n)$ is symmetric in $n$ variables.
- Homogeneity: $y \to \lambda y \Rightarrow F_g \to \lambda^{2-2g} F_g$.
- Symplectic invariance: $dx \wedge dy = d\tilde{x} \wedge d\tilde{y} \Rightarrow$ $F_g(\mathcal{S}) = F_g(\tilde{\mathcal{S}})$.
- Singular limits, universality: If $\mathcal{S}(t)$ is singular at $t = t_c$,

  then: 

  $F_g(\mathcal{S}(t)) \sim (t - t_c)^{(2-2g)\mu} \; F_g(\text{Blow up}_{t_c}(\mathcal{S}))$
- Integrability: Tau-function $\tau = \exp\left(\sum_g N^{2-2g} F_g(\mathcal{S})\right) \Theta$.

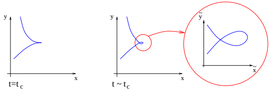
  Hirota equations, Determinantal formulae

  $\sum_g N^{2-2g-n} W_n^{(g)}(z_1, \ldots, z_n) = \det'(\hat{k}(z_i, z_j))$

  (proved in many cases, conjectured for the most general case.)
- Holomorphic anomaly equations = modular transformations, backround independence.
- Derivatives Variation of the curve $y(x) \to y(x) + \delta y(x) \Rightarrow$

# Main properties

- Symmetry: $W_n^{(g)}(z_1, z_2, \ldots, z_n)$ is symmetric in $n$ variables.
- Homogeneity: $y \to \lambda y \Rightarrow F_g \to \lambda^{2-2g} F_g$.
- Symplectic invariance: $dx \wedge dy = d\tilde{x} \wedge d\tilde{y} \Rightarrow$ $F_g(\mathcal{S}) = F_g(\tilde{\mathcal{S}})$.
- Singular limits, universality: If $\mathcal{S}(t)$ is singular at $t = t_c$,

  then: 

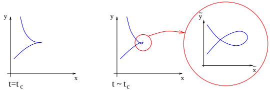

  $F_g(\mathcal{S}(t)) \sim (t - t_c)^{(2-2g)\mu} \; F_g(\text{Blow up}_{t_c}(\mathcal{S}))$
- Integrability: Tau-function $\tau = \exp\left(\sum_g N^{2-2g} F_g(\mathcal{S})\right) \Theta$.
- Holomorphic anomaly equations = modular transformations, backround independence.
- Derivatives Variation of the curve $y(x) \to y(x) + \delta y(x) \Rightarrow$

$$\delta W_n^{(g)}(z_1, \ldots, z_n) = \int_{(\delta y\, dx)^*} W_{n+1}^{(g)}(z_1, \ldots, z_n, z)$$

- And many more ...

# Main properties

- Symmetry: $W_n^{(g)}(z_1, z_2, \ldots, z_n)$ is symmetric in $n$ variables.
- Homogeneity: $y \to \lambda y \Rightarrow F_g \to \lambda^{2-2g} F_g$.
- Symplectic invariance: $dx \wedge dy = d\tilde{x} \wedge d\tilde{y} \Rightarrow$ $F_g(\mathcal{S}) = F_g(\tilde{\mathcal{S}})$.
- Singular limits, universality: If $\mathcal{S}(t)$ is singular at $t = t_c$,

then: 


$F_g(\mathcal{S}(t)) \sim (t - t_c)^{(2-2g)\mu} \; F_g(\text{Blow up}_{t_c}(\mathcal{S}))$

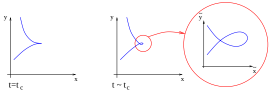
- Integrability: Tau-function $\tau = \exp\left(\sum_g N^{2-2g} F_g(\mathcal{S})\right) \Theta$.
- Holomorphic anomaly equations = modular transformations, backround independence.
- Derivatives Variation of the curve $y(x) \to y(x) + \delta y(x) \Rightarrow$

$$\delta W_n^{(g)}(z_1, \ldots, z_n) = \int_{(\delta y\, dx)^*} W_{n+1}^{(g)}(z_1, \ldots, z_n, z)$$

- And many more ...

# Main properties

- Symmetry: $W_n^{(g)}(z_1, z_2, \ldots, z_n)$ is symmetric in $n$ variables.
- Homogeneity: $y \to \lambda y \Rightarrow F_g \to \lambda^{2-2g} F_g$.
- Symplectic invariance: $dx \wedge dy = d\tilde{x} \wedge d\tilde{y} \Rightarrow$ $F_g(\mathcal{S}) = F_g(\tilde{\mathcal{S}})$.
- Singular limits, universality: If $\mathcal{S}(t)$ is singular at $t = t_c$,

then: 

$F_g(\mathcal{S}(t)) \sim (t - t_c)^{(2-2g)\mu}\ F_g(\text{Blow up}_{t_c}(\mathcal{S}))$

- Integrability: Tau-function $\tau = \exp\left(\sum_g N^{2-2g} F_g(\mathcal{S})\right) \Theta$.
- Holomorphic anomaly equations = modular transformations, backround independence.
- Derivatives Variation of the curve $y(x) \to y(x) + \delta y(x) \Rightarrow$

$$\delta W_n^{(g)}(z_1, \ldots, z_n) = \int_{(\delta y\, dx)^*} W_{n+1}^{(g)}(z_1, \ldots, z_n, z)$$

- And many more ...

# Application: Matrix models

Consider a $N \times N$ matrix integral with a topological expansion:

$$Z = \int_{H_N(\gamma)} dM \, \mathrm{e}^{-N \,\mathrm{Tr}\, V(M)} \sim \exp\left(\frac{1}{N^2}\sum_{g=0}^{\infty} N^{-2g} \mathcal{F}_{g_{\mathrm{M.M}}}\right)$$

Theorem 1: Schwinger-Dyson (=Loop equations) imply

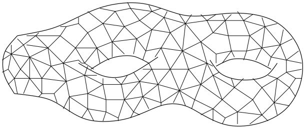
$$\mathcal{F}_{g_{\mathrm{M.M}}} = F_g(\mathcal{S}):$$

$$\text{Sp. curve: } \mathcal{S} = \begin{cases} y = \frac{1}{2}\left(V'(x) - \sqrt{V'(x)^2 - 4P(x)}\right) \\ = \text{'' } \lim_{N\to\infty} \frac{1}{N} < \mathrm{Tr}\, \frac{1}{x-M} > \text{ ''} \\ \\ \frac{1}{2i\pi}\oint_{\mathcal{A}_i} ydx = \epsilon_i = \text{fill.fractions} \to \gamma \end{cases}$$

Generalizations: 2-matrix model, chain of matrices, +ext. field.
In all those cases, if $Z = \exp\left(\sum_g N^{2-2g}\mathcal{F}_{g_{\mathrm{M.M}}}\right)$,
then $\exists\, \mathcal{S}$ such that $\mathcal{F}_{g_{\mathrm{M.M}}} = F_g(\mathcal{S})$.

# Application: discrete surfaces

$\mathbb{M}_g$ = set of discrete surfaces (= maps) obtained by gluing $n_3$ triangles, $n_4$ quadrangles,..., and of genus $g$ and with $v$ vertices.



$$\mathcal{F}_{g\,\text{discr. surf.}} := \sum_{\Sigma\in\mathbb{M}_g} t_3^{n_3(\Sigma)}\, t_4^{n_4(\Sigma)} \dots t_d^{n_d(\Sigma)} \, \frac{t^{v(\Sigma)}}{\#\mathrm{Aut}(\Sigma)}$$

Theorem 2: Tutte's equations $\Rightarrow \mathcal{F}_{g\,\text{discr. surf.}} = F_g(\mathcal{S})$:

$$\mathcal{S} = \begin{cases} x(z) = \gamma(z+z^{-1}) + \alpha \\ y(z) = \sum_k u_k(z^k - z^{-k}) \\ \mathcal{C} = \mathbb{P}^1 \\ x(z) - \sum_k t_{k+1} x(z)^k = \sum_{k=1}^{d-1} u_k(z^k + z^{-k})\,, \quad u_1 = \frac{t}{\gamma} \end{cases}$$

Kontsevich integral:

$$Z_{\text{Kontsevich}}(\Lambda) = \int dM \, e^{-N \mathrm{Tr} \frac{M^3}{3} - M\Lambda^2} = e^{\sum N^{2-2g} \mathcal{F}_g} \, , \qquad t_k = \frac{1}{N} \mathrm{Tr} \, \Lambda^{-k}$$

$$\mathcal{F}_g = \sum_n \frac{2^{2-2g-n}}{n!} \sum_{d_1,\dots,d_n} \prod_{i=1}^{n} t_{2d_i+1} \, (2d_i - 1)!! \int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i}$$

Theorem 3: Scwinger-Dyson eqs imply: $\mathcal{F}_g = F_g(\mathcal{S}_{\text{Kontsevich}})$:

$$\mathcal{S}_{\text{Kontsevich}} = \begin{cases} x(z) = z + \frac{1}{2N} \mathrm{Tr} \, \frac{1}{\Lambda(z-\Lambda)} = z - \frac{1}{2} \sum_{k=0}^{\infty} t_{k+2} z^k \\ y(z) = z^2 + t_1 \end{cases}$$

Theorem 3bis: Double scaling limit of discrete surfaces:

$\mathcal{F}_{\text{Discr. Surf.}g} \sim (t - t_c)^{(2-2g)\frac{p+2}{p+1}} \mathcal{F}_{\text{DSL}}(p,2)_g$, with
$\mathcal{F}_{\text{DSL}}(p,2)_g = F_g(\mathcal{S}_{\text{DSL}}(p,2)) =$ "Liouville + $(p,2)$":

$$\mathcal{S}_{\text{DSL}}(p,2) = \begin{cases} x(z) = z^2 + \tilde{t}_1 \\ y(z) = -\frac{1}{2} \sum_{k=1}^{p} \tilde{t}_{k+2} z^k \end{cases}$$

Kontsevich integral:

$$Z_{\mathrm{Kontsevich}}(\Lambda) = \int dM\, \mathrm{e}^{-N \mathrm{Tr} \frac{M^3}{3} - M\Lambda^2} = \mathrm{e}^{\sum N^{2-2g} \mathcal{F}_g} \, , \qquad t_k = \frac{1}{N} \mathrm{Tr}\, \Lambda^{-k}$$

$$\mathcal{F}_g = \sum_n \frac{2^{2-2g-n}}{n!} \sum_{d_1,\dots,d_n} \prod_{i=1}^n t_{2d_i+1} (2d_i - 1)!! \int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^n \psi_i^{d_i}$$

Theorem 3: Scwinger-Dyson eqs imply: $\mathcal{F}_g = F_g(\mathcal{S}_{\mathrm{Kontsevich}})$:

$$\mathcal{S}_{\mathrm{Kontsevich}} = \begin{cases} x(z) = z + \frac{1}{2N} \mathrm{Tr} \frac{1}{\Lambda(z-\Lambda)} = z - \frac{1}{2} \sum_{k=0}^{\infty} t_{k+2} z^k \\ y(z) = z^2 + t_1 \end{cases}$$

Theorem 3bis: Double scaling limit of discrete surfaces:

$\mathcal{F}_{\mathrm{Discr.\ Surf.}g} \sim (t - t_c)^{(2-2g)\frac{p+2}{p+1}} \mathcal{F}_{\mathrm{DSL}}(p,2)_g$, with

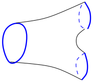
$\mathcal{F}_{\mathrm{DSL}}(p,2)_g = F_g(\mathcal{S}_{\mathrm{DSL}}(p,2))$ = "Liouville + $(p,2)$":

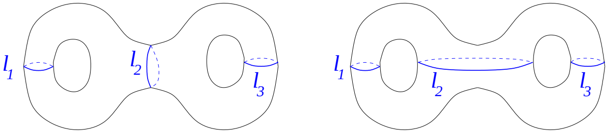
$$\mathcal{S}_{\mathrm{DSL}}(p,2) = \begin{cases} x(z) = z^2 + \tilde{t}_1 \\ y(z) = -\frac{1}{2} \sum_{k=1}^{p} \tilde{t}_{k+2} z^k \end{cases}$$

# Weil-Petersson volumes

$\mathcal{M}_g$= set of Riemann surfaces of genus $g \geq 2$.
$\rightarrow$ decompose into $2g - 2$ "pairs of pants"
$\rightarrow$ $3g - 3$ geodesic lengths $l_i$, and gluing angles $\vartheta_i$= flat coordinates on $\mathcal{M}_g$.

$$\mathcal{F}_{\mathrm{W.P}g} = \mathrm{Vol}(\mathcal{M}_g) = \int_{\mathcal{M}_g} \prod_i dl_i \wedge d\vartheta_i .$$

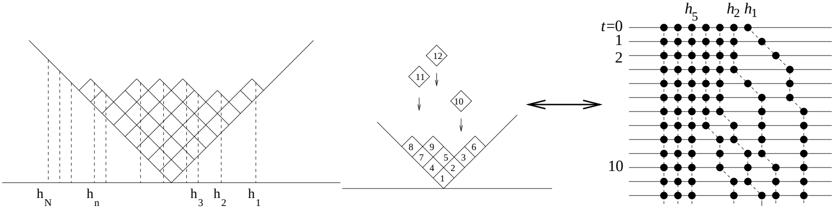





Theorem 4: $\mathcal{F}_{\mathrm{W.P}g} = F_g(\mathcal{S})$:

$$\text{Spectral curve: } \mathcal{S} = \begin{cases} x(z) = z^2 \\ y(z) = \frac{-1}{\pi} \sin(2\pi z) \end{cases}$$

Remark: recursions for $W_n^{(g)}$ $\Leftrightarrow$ Mirzakhani's recursions.



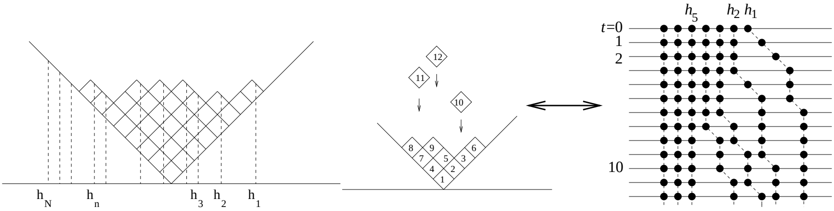

Plancherel Partitions $\leftrightarrow$ Growing crystal $\leftrightarrow$ T.A.S.E.P.-like

$$Z(Q, t_k) = \sum_{\lambda} \hbar^{-2|\lambda|} \left(\frac{\dim(\lambda)}{|\lambda|!}\right)^2 e^{-\frac{1}{\hbar}\sum_k \frac{t_k}{k}\hbar^k C_k(\lambda)}$$

$C_k(\lambda) = k^{\text{th}}$ Casimir $= \sum_i h_i^k$

Question: $\ln Z = \sum_{g=0}^{\infty} \hbar^{2g-2} \mathcal{F}_{\text{Plancherel}\, g}(t_k)$

# Application: partitions, Crystal growth, TASEP




Plancherel Partitions $\leftrightarrow$ Growing crystal $\leftrightarrow$ T.A.S.E.P.-like

$$Z(Q,t_k)=\sum_\lambda \hbar^{-2|\lambda|}\left(\frac{\dim(\lambda)}{|\lambda|!}\right)^2 e^{-\frac{1}{\hbar}\sum_k \frac{t_k}{k}\hbar^k C_k(\lambda)}$$

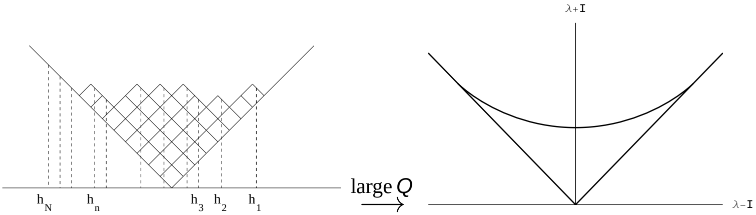
$C_k(\lambda)=k^{\text{th}}$ Casimir $=\sum_i h_i^k$

Question: $\ln Z=\sum_{g=0}^\infty \hbar^{2g-2}\mathcal{F}_{\text{Plancherel}\,g}(t_k)$

Spectral curve = "limit shape"

Spectral curve: $\mathcal{S}(t_k) = \begin{cases} x(z) = z + z^{-1} \\ y(z) = \ln(z) \\ \\ \sum_k t_{k+1} x(z)^k = \sum_k u_k (z^k + z^{-k}) \end{cases}$

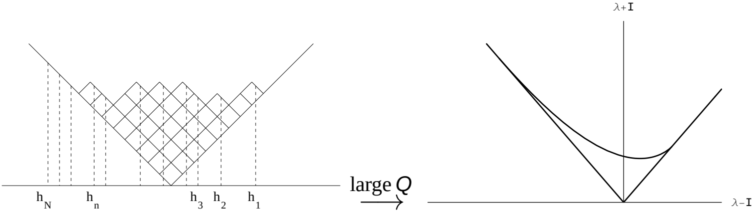
**Theorem 5 :** $\mathcal{F}_{\text{Plancherel}\, g}(t_k) = F_g(\mathcal{S}(t_k))$



Spectral curve = "limit shape"

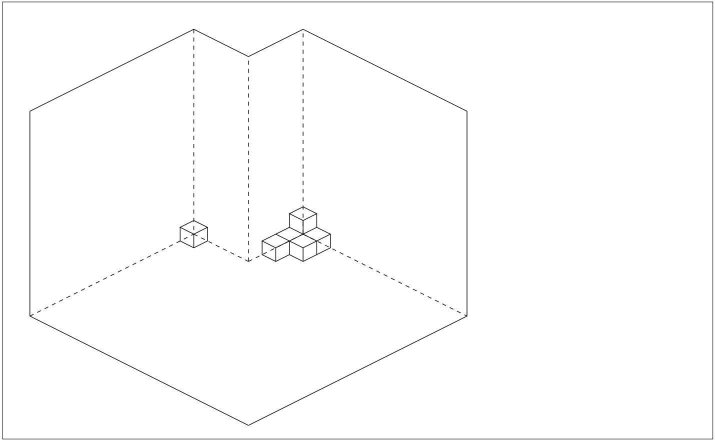
Spectral curve: $\mathcal{S}(t_k) = \begin{cases} x(z) = \mathrm{e}^{-u_0}\,(z + z^{-1} - u_1) \\ y(z) = \ln(z) + \sum_k u_k (z^k - z^{-k}) \\ \\ \sum_k t_{k+1} x(z)^k = \sum_k u_k (z^k + z^{-k}) \end{cases}$

**Theorem 5 :** $\mathcal{F}_{\mathrm{Plancherel}\,g}(t_k) = F_g(\mathcal{S}(t_k))$



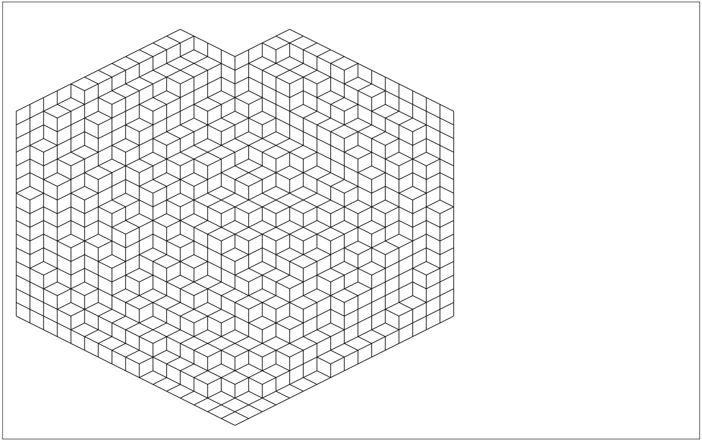
# Plane partitions

- $Z = \#\{3D \text{ partitions}\}, \quad \ln Z = \sum_g \text{size}^{2-2g}\, \mathcal{F}_g.$



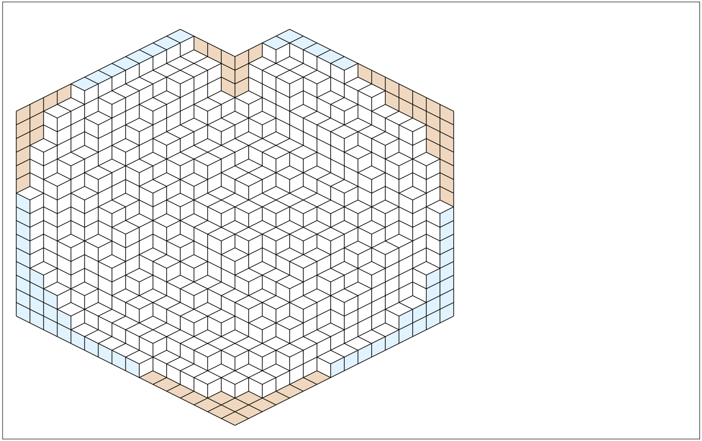
# Plane partitions

- $Z = \#\{3D \text{ partitions}\}, \quad \ln Z = \sum_g \text{size}^{2-2g} \mathcal{F}_g.$



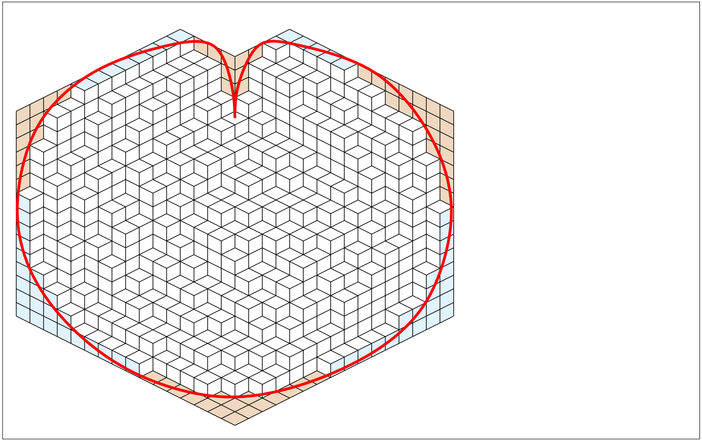
# Plane partitions

- $Z = \#\{3D \text{ partitions}\}, \quad \ln Z = \sum_g \text{size}^{2-2g}\, \mathcal{F}_g.$



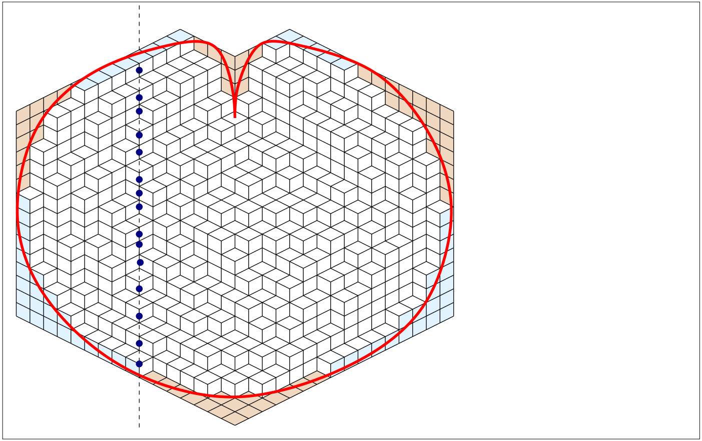
# Plane partitions

- $Z = \#\{3D \text{ partitions}\}, \quad \ln Z = \sum_g \text{size}^{2-2g}\, \mathcal{F}_g.$



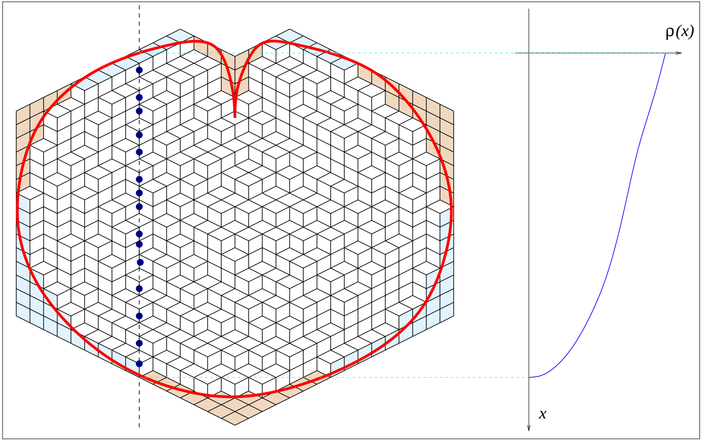
# Plane partitions

- $Z = \#\{3D \text{ partitions}\}, \quad \ln Z = \sum_g \text{size}^{2-2g} \mathcal{F}_g.$



# Plane partitions

- $Z = \#\{3D \text{ partitions}\}, \quad \ln Z = \sum_g \text{size}^{2-2g} \mathcal{F}_g.$

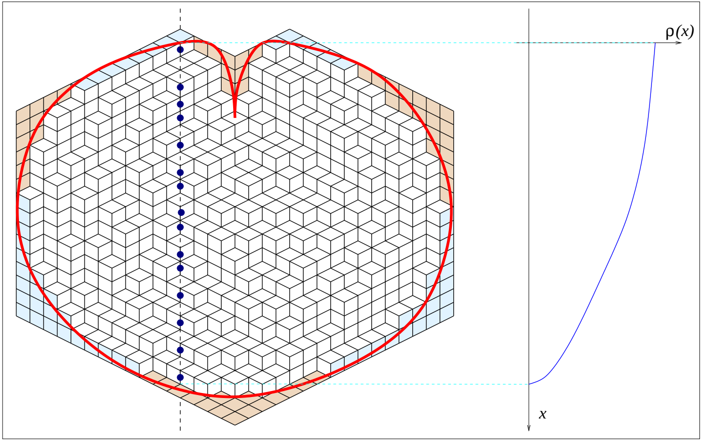



Conjecture:

$\mathcal{F}_g = F_g$(Stieljes transf. of limit density along a vertical line)

# Plane partitions

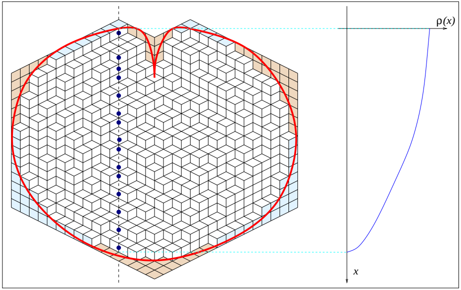
- $Z = \#\{3D \text{ partitions}\}$, $\quad \ln Z = \sum_g \text{size}^{2-2g}\, \mathcal{F}_g$.




Conjecture:

$\mathcal{F}_g = F_g$(Stieljes transf. of limit density along a vertical line)

- $Z = \#\{3D \text{ partitions}\}, \quad \ln Z = \sum_g \text{size}^{2-2g}\, \mathcal{F}_g.$




Conjecture:

$\mathcal{F}_g = F_g$(Stieltjes transf. of limit density along a vertical line)

Idea of a proof: $Z$=matrix integral, which implies that it satisfies the topological recursion. Problem: show that $W_1^{(0)}$ =Kenyon-Okounkov-Sheffield curve (limit shape) ?

- Let $\mathfrak{X}$ a 3D Calabi-Yau manifold with toric symmetry
- Gromov-Witten: $\mathcal{N}_{g,d}(\mathfrak{X}) =$ "# of ways of embedding a Riemann surface of genus $g$ into $\mathfrak{X}$, with homology class $d$, and passing through given points".
- Generating function: $\mathcal{F}_g = \sum_d \mathcal{N}_{g,d}(\mathfrak{X})\, Q^d$.
- String theory: $\mathcal{F}_g =$ amplitude of a closed string of genus $g$ in target space $\mathfrak{X}$.
- Conjecture [Mariño 2006, BKMP 2008]:

$$\mathcal{F}_g = F_g(\text{mirror}\,\mathfrak{X})$$

Proved in [E. Orantin 2012], using combinatorics of localization graphs.

# Topological strings - Gromov-Witten

• Let $\mathfrak{X}$ a 3D Calabi-Yau manifold with toric symmetry

• Gromov-Witten: $\mathcal{N}_{g,d}(\mathfrak{X}) =$ "$\#$ of ways of embedding a Riemann surface of genus $g$ into $\mathfrak{X}$, with homology class $d$, and passing through given points".

• Generating function: $\mathcal{F}_g = \sum_d \mathcal{N}_{g,d}(\mathfrak{X})\, Q^d$.

• String theory: $\mathcal{F}_g =$ amplitude of a closed string of genus $g$ in target space $\mathfrak{X}$.

• Conjecture [Mariño 2006, BKMP 2008]:

$$\mathcal{F}_g = F_g(\text{mirror}\,\mathfrak{X})$$

Proved in [E. Orantin 2012], using combinatorics of localization graphs.

# Topological strings - Gromov-Witten

- Let $\mathfrak{X}$ a 3D Calabi-Yau manifold with toric symmetry
- Gromov-Witten: $\mathcal{N}_{g,d}(\mathfrak{X}) =$ "$\#$ of ways of embedding a Riemann surface of genus $g$ into $\mathfrak{X}$, with homology class $d$, and passing through given points".
- Generating function: $\mathcal{F}_g = \sum_d \mathcal{N}_{g,d}(\mathfrak{X})\, Q^d$.
- String theory: $\mathcal{F}_g =$ amplitude of a closed string of genus $g$ in target space $\mathfrak{X}$.
- Conjecture [Mariño 2006, BKMP 2008]:

$$\mathcal{F}_g = F_g(\text{mirror}\,\mathfrak{X})$$

Proved in [E. Orantin 2012], using combinatorics of localization graphs.

# Topological strings - Gromov-Witten

- Let $\mathfrak{X}$ a 3D Calabi-Yau manifold with toric symmetry
- Gromov-Witten: $\mathcal{N}_{g,d}(\mathfrak{X}) =$ "# of ways of embedding a Riemann surface of genus $g$ into $\mathfrak{X}$, with homology class $d$, and passing through given points".
- Generating function: $\mathcal{F}_g = \sum_d \mathcal{N}_{g,d}(\mathfrak{X})\, Q^d$.
- String theory: $\mathcal{F}_g =$ amplitude of a closed string of genus $g$ in target space $\mathfrak{X}$.
- Conjecture [Mariño 2006, BKMP 2008]:

$\mathcal{F}_g = F_g(\text{mirror}\,\mathfrak{X})$

Proved in [E. Orantin 2012], using combinatorics of localization graphs.

# Topological strings - Gromov-Witten

• Let $\mathfrak{X}$ a 3D Calabi-Yau manifold with toric symmetry

• Gromov-Witten: $\mathcal{N}_{g,d}(\mathfrak{X}) =$ "$\#$ of ways of embedding a Riemann surface of genus $g$ into $\mathfrak{X}$, with homology class $d$, and passing through given points".

• Generating function: $\mathcal{F}_g = \sum_d \mathcal{N}_{g,d}(\mathfrak{X})\, Q^d$.

• String theory: $\mathcal{F}_g =$ amplitude of a closed string of genus $g$ in target space $\mathfrak{X}$.

• Conjecture [Mariño 2006, BKMP 2008]:

$$\mathcal{F}_g = F_g(\mathrm{mirror}\,\mathfrak{X})$$

Proved in [E. Orantin 2012], using combinatorics of localization graphs.

- Let $\mathfrak{X}$ a 3D Calabi-Yau manifold with toric symmetry

- Gromov-Witten: $\mathcal{N}_{g,d}(\mathfrak{X}) =$ "$\#$ of ways of embedding a Riemann surface of genus $g$ into $\mathfrak{X}$, with homology class $d$, and passing through given points".

- Generating function: $\mathcal{F}_g = \sum_d \mathcal{N}_{g,d}(\mathfrak{X})\, Q^d$.

- String theory: $\mathcal{F}_g =$ amplitude of a closed string of genus $g$ in target space $\mathfrak{X}$.

- Conjecture [Mariño 2006, BKMP 2008]:

$$\mathcal{F}_g = F_g(\text{mirror}\,\mathfrak{X})$$

Proved in [E. Orantin 2012], using combinatorics of localization graphs.

## Knot invariants

Conjecture [Dijkgraaf-Fuji 2010+Borot, E. 2012]: Let $J_N(\mathrm{Knot}, q) =$ colored Jones polynomial of Knot, in the double scaling limit:

$$N \to \infty, \quad N \ln q = x = O(1)$$

$$\text{Then}: \ \ln J_N(\mathrm{Knot}, q) \sim \frac{1}{2} \sum_{g=0}^{\infty} \sum_{n+m=1}^{\infty} \frac{(\ln q)^{2-2g-n-m}\,\Theta^{(m)}}{n!m!} \times \overbrace{\int^{x+\bar{x}} \cdots \int^{x+\bar{x}}}^{n} \overbrace{\oint_B \cdots \oint_B}^{m} W_{g,n+m}[\mathcal{S}]$$

where $\mathcal{S}$ =character variety of the knot ($A-$polynomial). Leading order ($W_{0,1}$-term) coincides with the famous "volume conjecture" [Kashaev 97+ $\cdots$]

Ex: figure of 8 knot

$$\mathcal{S} = \{(x, y) \,|\, \mathrm{e}^y + \mathrm{e}^{-y} = \mathrm{e}^{2x} - \mathrm{e}^x - 2 - \mathrm{e}^{-x} + \mathrm{e}^{-2x}\}$$

Liouville 2D-CFT, with $c = 1 + 6Q^2$, $Q = b + 1/b$.

Liouville action: $\mathcal{S} = \int \left(\partial\phi\bar\partial\phi + \mu\, \mathrm{e}^{2b\phi} + QR\phi\right)$.
compute $n$-point correlation functions of vertex operators of dimensions $\Delta_i = \alpha_i(Q - \alpha_i)$:

$$Z(\{\Delta_i, z_i\}) = \left\langle \prod_{i=1}^{n} \mathrm{e}^{\alpha_i \phi(z_i)} \right\rangle$$

as a large $Q$ expansion (heavy limit), with $\eta_i = \frac{\alpha_i}{Q} = O(1)$:

$$\ln Z \underset{Q\to\infty}{\sim} \sum_{g=0}^{\infty} Q^{2-2g} F_g(\{\eta_i, z_i\})$$

We propose that:

$$F_g = \mathcal{F}_g[S] \quad , \quad S = \{y = T(x)\} = \text{classical stress energy tensor}$$

We proved [Chekhov, E, Ribault 2012] it for the $n = 3$-point function, and to order $Q^0$, and give heuristic arguments (Ward identities) towards the general case.

# Conclusion

Main results:

- Topological recursion is very universal: it appears in many problems of enumerative geometry and integrable systems (more applications: coloured surfaces, CFT, ...).
- They are easy to compute, easy to compare.
- topological recursion provides a unifying geometric framework.

Further prospects:

- Find more applications, and prove them...
- New topological recursion for Boundaries (started with Orantin, to be continued)...
- non-orientable surfaces, quantum deformation... (started with Bergère, Chekhov, Marchal), $\rightarrow$ quantum Riemannian geometry...

# More applications

- Airy, Tracy Widom (1,2), $c = -2$

$y = \sqrt{x}$

- Liouville + Minimal model $(p, q)$, $c = 1 - 6(p-q)^2/pq$

$x = \mathrm{Pol}_q(z), y = \mathrm{Pol}_p(z)$.

- Liouville + pure gravity $(3, 2)$, $c = 0$

$x = z^2 - 2, y = z^3 - 3z$.

- Liouville + Ising $(4, 3)$, $c = 1/2$

$x = z^3 - 3z, y = z^4 - 4z^2 + 2$.

- Liouville + Unitary models $(q+1, q)$, $c = 1 - 6/q(q+1)$

$x = T_q(z), y = T_{q+1}(z)$ (Tchebychev polynomials).

- Kontsevich integral, times $t_k = \frac{1}{N} \mathrm{Tr}\, \Lambda^{-k}$

$x(z) = z^2 + t_1, y(z) = z - \frac{1}{2} \sum_k t_{k+2} z^k$,

- Weil-Petersson volumes

$x(z) = z^2, y(z) = \frac{1}{2\pi} \sin(2\pi z)$

- Seiberg-Witten

$x(z) = \wp(z), y(z) = \wp'(z)$.