Functoriality of quantization: an operator-algebraic approach

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Paths that cross

1. **Symplectic geometry and geometric quantization:**

   Guillemin–Sternberg (-Dirac) conjecture '$[Q, R] = 0$'

   'Geometric quantization commutes with symplectic reduction'

   Reformulation in terms of equivariant index theory (Bott)

   Defined and proved for **compact** groups and manifolds

   (Noncompact examples: Gotay, Vergne, Paradan, Hochs, . . .)

2. **Operator algebras and equivariant K-theory:**

   Baum–Connes conjecture $\mu_r : K_\bullet^G(\underline{E}G) \xrightarrow{\cong} K_\bullet(C_r^*(G))$

   Interesting for **noncompact** groups G (and **proper** actions)

3. **Functoriality of quantization**

   Can symplectic data be 'neatly' mapped into operator data?

   Are geometric and deformation quantization perhaps related?

The Janus faces of quantization

1. Heisenberg (1925): classical observables $\rightsquigarrow$ matrices
2. Schrödinger (1926): classical states $\rightsquigarrow$ wave functions
3. von Neumann (1932): unification through Hilbert space

   matrices $\rightarrow$ operators, wave functions $\rightarrow$ vectors

1. Classical observables form Poisson algebra

   Quantum observables form C*-algebra $\Rightarrow$ first face:

   'Deformation' quantization: Poisson algebra $\rightsquigarrow$ C*-algebra

2. Classical states form Symplectic manifold (M, ω) $\Rightarrow$ 2nd face:

   Geometric quantization: symplectic manifold $\rightsquigarrow$ Hilbert space

- Classically: state space (M, ω) determines observables $C^\infty(M)$
- Quantumly: C*-algebra A determines state space $S(A)$ (or $P(A)$)
- But this doesn't help (very much) for quantization!

Key examples of quantization

1. **'Strict' deformation quantization (Rieffel)**

   Lie group G, Lie algebra $\mathfrak{g}$, Poisson mfd $\mathfrak{g}^*$: for $X \in \mathfrak{g}$,
   $\hat{X} \in C^\infty(\mathfrak{g}^*)$ defined by $\hat{X}(\theta) = \theta(X)$, $\{\hat{X}, \hat{Y}\} = \widehat{[X, Y]}$

   Quantization of Poisson algebra $C^\infty(\mathfrak{g}^*)$ is C*-algebra $C^*(G)$

2. **Traditional geometric quantization (Kostant, Souriau)**

   compact symplectic manifold (M, ω) such that $[\omega] \in H^2(M, \mathbb{Z})$
   $\Rightarrow$ $\mathbb{C}$-line bundle $L \to M$ plus connection ∇^L with $F(\nabla^L) = 2\pi i\omega$
   $\Rightarrow$ almost complex structure J s.t. $g(\xi, \eta) = \omega(\xi, J\eta)$ is metric
   $\Rightarrow$ Hilbert space $Q(M, \omega, J) = \{s \in \Gamma(L) \mid \nabla^L_{J\xi - i\xi} s = 0, \xi \in \mathbf{X}(M)\}$

3. **'Postmodern' geometric quantization**

   $Q_B(M, \omega, J) := \pi_*([L]) = \mathrm{index}(\not{D}^L) \equiv \dim(\ker(\not{D}^L_+)) - \dim(\ker(\not{D}^L_-))$
   $\not{D}^L$ is Spin^c Dirac operator on M defined by J coupled to L
   "A definition of quantization that is apparently due to Bott"

Guillemin-Sternberg-Bott conjecture

Quantization after reduction w.r.t. $G \circlearrowright M$ ($G\&M$ compact!):

1. **Symplectic reduction:** $(M/\!/G, \omega_G)$
2. **Geometric quantization à la Bott:** $Q_B(M/\!/G, \omega_G) = \mathrm{index}(\not{D}^{L/\!/G})$

Reduction after quantization ($G\&M$ compact!):

1. **Equivariant geometric quantization à la Bott:**
   $Q_B(M, \omega) = \mathrm{index}_G(\not{D}^L) = [\ker(\not{D}^L_+)] - [\ker(\not{D}^L_-)] \in R(G)$
2. **Quantum reduction:** $R(G) \to \mathbb{Z}$, $[V] - [W] \mapsto \dim(V^G) - \dim(W^G)$

So $Q_B(M, \omega)^G = \dim((\ker(\not{D}^L_+)^G) - \dim((\ker(\not{D}^L_-)^G)$

$\Rightarrow [Q, R] = 0$ **reads:** $\boxed{\dim((\ker(\not{D}^L_+)^G) - \dim((\ker(\not{D}^L_-)^G) = \mathrm{index}(\not{D}^{L/\!/G})}$

- **Proved by many people in mid 1990s (Meinrenken, . . .)**

! G and M noncompact: $\dim(\ker(\not{D}^L_\pm)) = \infty$, $\dim((\ker(\not{D}^L_\pm)^G) = 0$, **in** L^2

Noncompact groups and manifolds

For noncompact G and M need substantial reformulation of G-S-B conjecture, under assumptions: $G \circlearrowright M$ proper and M/G compact

Compact $\rightsquigarrow$ noncompact dictionary

- Representation ring $R(G)$ $\rightsquigarrow$ K-theory group $K_0(C^*(G))$
- {Operator data $(H, \not{D}, U(G))$} $\rightsquigarrow$ K-homology group $K_0^G(M)$
- Equivariant index $\rightsquigarrow$ assembly map $\mu_M^G : K_0^G(M) \to K_0(C^*(G))$
- Quantum reduction $R(G) \to \mathbb{Z}$ $\rightsquigarrow$ map $K_0(C^*(G)) \xrightarrow{x \mapsto x^G} K_0(\mathbb{C}) \cong \mathbb{Z}$

  (induced by map $C^*(G) \to \mathbb{C}$ determined by trivial rep of G)

$\Rightarrow$ Generalized G-S-B conjecture: $\left(\mu_M^G\left(\left[\not{D}^L\right]\right)\right)^G = \text{index}\left(\not{D}^{L/\!/G}\right)$

Proved by Hochs–Landsman (2008) if G contains cocompact discrete normal subgroup, general proof by Mathai–Zhang (2010) through reduction to proof of compact case by Tian–Zhang (1998)

Baum–Connes–Higson assembly map

- **K-homology of M: abelian group $K_0(M) = \{[H, F, \pi]_{\sim h}\}$, where:**
  1. **$H = H_+ \oplus H_-$ is separable $\mathbb{Z}_2$-graded Hilbert space**
  2. **$F \in B(H)$ odd operator, $F_\pm : H_\pm \to H_\mp$**
  3. **$\pi : C_0(M) \to B(H)$ is even representation, $\pi_\pm(f) : H_\pm \to H_\pm$**
  4. **$[\pi(f), F] \in K(H)$, $f \in C_0(M)$, i.e. F 'almost' intertwines $\pi_\pm$**
  5. **$\pi(f)(F^2 - 1) \in K(H)$, $f \in C_0(M)$, i.e. $F_\pm$ 'locally' Fredholm**
- **Equivariant K-homology $K_0^G(M)$: add proper G-action on M, $C_0(M)$-covariant rep $U(G)$ on H, commuting with F mod $K(H)$**
- **K-theory of $\hat{G}$: $K_0(C^*(G)) = \{[E_1] - [E_2]\}$, E_i f.g.p. $C^*(G)$-modules**

Description of (unreduced) assembly map $\mu_M^G : K_0^G(M) \to K_0(C^*(G))$:

1. **Unitary G-rep U on H turns H into (Hilbert) $C^*(G)$-module: for G unimodular, right $C^*(G)$-action is $\pi(f) = \int_G dg\, f(g^{-1})U(g)$**
2. **$K_0^G(M) \ni [H, F, \pi, U(G)]_{\sim h} \mapsto [\ker(F'_+)] - [\ker(F'_-)] \in K_0(C^*(G))$**

Proof of $\left(\mu_M^G\left(\left[\not{D}^L\right]\right)\right)^G = \operatorname{index}\left(\not{D}^{L//G}\right)$

- (M,ω) symplectic G-manifold $\Rightarrow$ almost complex structure J
- Spinor bundle $\mathcal{S}_J \to M$ for G-inv. Spin^c-structure defined by J
- $E = \mathcal{S}_J \otimes L$ with $L \to M$ prequantization line bundle w.r.t. ω
- $\not{D}^L : \Gamma(E) \to \Gamma(E)$ is G-inv. Spin^c Dirac operator coupled to L

1. $G \circlearrowright M$ cocompact $\Rightarrow \exists$ compact $Y \subset M$ such that $G \cdot Y = M$
2. Pick $c \in C_c^\infty(M)$ s.t. $Y \subset \operatorname{supp}(c)$, $\int_G dg\, c(g^{-1}x)^2 = 1\ \forall x \in M$
3. $L_c^2(E)^G = c \cdot L_{\mathrm{loc}}^2(E)^G \subset L^2(E)$, $H_c^1(E)^G = c \cdot H_{\mathrm{loc}}^1(E)^G \subset H^1(E)$
4. $\not{D}_c^L \equiv [L_c^2(E)^G] \circ \not{D}^L \circ [H_c^1(E)^G] : H_c^1(E)^G \to L_c^2(E)^G$ is Fredholm

$$\left(\mu_M^G\left(\left[\not{D}^L\right]\right)\right)^G \overset{\text{Bunke}}{=} \operatorname{index}(\not{D}_c^L) \overset{\text{MZ}}{=} \dim(\ker_{\Gamma(E)}(\not{D}_+^L)^G) - \dim(\ker_{\Gamma(E)}(\not{D}_-^L)^G)$$

$$\overset{\text{MZ}}{=} \operatorname{index}(\not{D}^{L/G}) \overset{\text{TZ}}{=} \operatorname{index}(\not{D}^{L//G}) \text{ by localization to } \Phi^{-1}(0)$$

Functorial quantization

- 'Explains' generalized Guillemin-Sternberg-Bott conjecture as a special instance of functoriality of quantization
- Unifies the Janus faces of quantization into a functor Q

1. Domain of Q: category of (quantizable) 'regular dual pairs'
   (a) (integrable) Poisson manifolds as objects
   (b) (regular) Weinstein dual pairs $[P_1 \leftarrow M \rightarrow P_2]_{\cong}$ as arrows
2. Codomain of Q: Kasparov's category KK
   (a) (separable) C*-algebras as objects
   (b) $[$graded Hilbert bimodules 'with operator'$]_{\sim h}$ as arrows
3. Hypothetical quantization functor (based on examples only)
   (a) Deformation quantization: $P_i \rightsquigarrow$ C*-algebra A_i
   (b) Geometric quantization: $M \rightsquigarrow$ '[Spin^c Dirac operator $\not{D}^L]_?$'
   (c) Functorial quantization: $[P_1 \leftarrow M \rightarrow P_2] \rightsquigarrow [\not{D}^L] \in KK(A_1, A_2)$

Category of Weinstein dual pairs

- $P_1^- \xleftarrow{\phi_1} M \xrightarrow{\phi_2} P_2$ (M symplectic, P_i Poisson mfds, ϕ_i complete Poisson maps) forms **dual pair** if $\{\phi_1^* f_1, \phi_2^* f_2\} = 0$, $f_i \in C^\infty(P_i)$
- Poisson mfd P is **integrable** if P is base mfd of some symplectic groupoid $\Gamma(P)$ (unique/$\cong$ if s-connected & s-simply connected)
- Poisson map $M \xrightarrow{\phi} P$ is **integrable** if ϕ is base map of some symplectic groupoid action on M ($\Rightarrow P$ integrable, ϕ complete)
- Dual pair $P_1^- \xleftarrow{\phi_1} M \xrightarrow{\phi_2} P_2$ is **regular** if both ϕ_i are integrable and $P_1^- \xleftarrow{\phi_1} M$ is principal $\Gamma(P_2)$-bundle (cf. Moerdijk)
- Iso classes of regular dual pairs form arrows of category:
  Product of $P_1^- \leftarrow M \rightarrow P$ and $P \leftarrow N \rightarrow P_2$ is symplectic reduction of coisotropic constraint $C = M \times_P N \subset M \times N$ (Xu)

$\Rightarrow$ Category of regular dual pairs $\simeq$ category of s-simply connected symplectic groupoids with right principal symplectic bibundles

Kasparov's category KK

- **Category KK has C*-algebras as objects and homotopy classes of $\mathbb{Z}_2$-graded Hilbert bimodules 'with operator' as arrows:**

1. A-B **Hilbert bimodule** $E = E_- \oplus E_+$ **has** B**-valued inner product such that** $\langle a^*\psi, \phi\rangle = \langle \psi, a\phi\rangle$, $a \in A$, **and** $\langle \psi, \phi b\rangle = \langle \psi, \phi\rangle b$, $b \in B$
2. **Odd operator** $F : E \to E$ **almost intertwines even** A**-action**
3. F **is locally Fredholm w.r.t.** A**-action on** E **(relative to** $K(E)$**)**
4. A-$C([0,1], B)$ **Hilbert bimodules give notion of homotopy** h
5. **Abelian group** $KK(A, B) = \{[A, B, E, F]_{\sim h}\}$ **(**E **countably gen./**B**)**
6. **Kasparov product** $KK(A, B) \times KK(B, C) \to KK(A, C)$ **gives arrow composition in** KK **(functorial in every way)**
7. $KK(A, \mathbb{C}) =: K^0(A)$ **defines** K**-homology,** $K_0(M) \equiv K^0(C_0(M))$
8. $KK(\mathbb{C}, B) \cong K_0(B)$ **through** $[\mathbb{C}, B, E, F]_{\sim h} \mapsto [\ker(F'_+)] - [\ker(F'_-)]$

Examples of functorial quantization

1. **Symplectic manifold M yields dual pair $pt \leftarrow M \rightarrow pt$**

   (a) **Deformation quantization:** $pt \rightsquigarrow \mathbb{C}$

   (b) **Geometric quantization:** $(M, \omega) \rightsquigarrow [\not{D}^L]_?$

   (c) **Functorial quantization:** $(pt \leftarrow M \rightarrow pt) \rightsquigarrow [\not{D}^L] \in KK(\mathbb{C}, \mathbb{C})$

   Identification $KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$ **identifies** $[\not{D}^L] \cong \mathrm{index}(\not{D}^L)$

2. **Hamiltonian group action $G \circlearrowright M$ generated by momentum map $\Phi : M \rightarrow \mathfrak{g}^*$ yields dual pair $pt \leftarrow M \xrightarrow{\Phi} \mathfrak{g}^*$ (assume G connected)**

   (a) **Deformation quantization:** $pt \rightsquigarrow \mathbb{C}$, $\mathfrak{g}^* \rightsquigarrow C^*(G)$

   (b) **Geometric quantization:** $(M, \omega) \rightsquigarrow [\not{D}^L]_?$

   (c) **Functorial quantization:** $(pt \leftarrow M \rightarrow \mathfrak{g}^*) \rightsquigarrow [\not{D}^L] \in KK(\mathbb{C}, C^*(G))$

   $KK(\mathbb{C}, C^*(G)) \cong K_0(C^*(G))$ **identifies** $[\not{D}^L] \cong \mu_M^G([\not{D}^L]_{K_0^G(M)})$

3. $(\mathfrak{g}^* \hookleftarrow 0 \rightarrow pt) \rightsquigarrow [\not{D} = 0] \in KK(C^*(G), \mathbb{C})$, **trivial action** $C^*(G) \circlearrowright \mathbb{C}$

Guillemin-Sternberg-Bott revisited

1. **Composition $\circ$ of dual pairs reproduces symplectic reduction:**

$$(pt \leftarrow M \rightarrow \mathfrak{g}^*) \circ (\mathfrak{g}^* \hookleftarrow 0 \rightarrow pt) \cong pt \leftarrow M//G \rightarrow pt$$

2. **Kasparov product reproduces quantum reduction:**

$$x_{KK(\mathbb{C},C^*(G))} \times_{KK} [\not{D} = 0]_{KK(C^*(G),\mathbb{C})} = x^G \in KK(\mathbb{C},\mathbb{C})$$

i.e. map $K_0(C^*(G)) \xrightarrow{x \mapsto x^G} \mathbb{Z}$ **given as product in category** KK

3. **Recall:**

$$\begin{aligned} \mathbf{Q}(pt \leftarrow M//G \rightarrow pt) &= \mathrm{index}(\not{D}^{L//G}) \\ \mathbf{Q}(pt \leftarrow M \rightarrow \mathfrak{g}^*) &= \mu_M^G([\not{D}^L]_{K_0^G(M)}) \\ \mathbf{Q}(\mathfrak{g}^* \hookleftarrow 0 \rightarrow pt) &= [\not{D} = 0]_{KK(C^*(G),\mathbb{C})} \end{aligned}$$

$\Rightarrow$ **Functoriality of quantization map Q gives G-S-B conjecture:**

$$\mathbf{Q}(pt \leftarrow M \rightarrow \mathfrak{g}^*) \circ \mathbf{Q}(\mathfrak{g}^* \hookleftarrow 0 \rightarrow pt) = \mathbf{Q}(pt \leftarrow M//G \rightarrow pt)$$

is the same as $\mu_M^G\left(\left[\not{D}^L\right]\right)^G = \mathrm{index}\left(\not{D}^{L//G}\right)$