# Mackey Machine and Duality of Gerbes on Orbifolds

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Aarhus University, December, 2010

In this talk, we explain a duality conjecture about gerbes on orbifolds using groupoids and noncommutative geometry. The "groupoid Mackey machine" provides a powerful approach to study this conjecture.

This is a joint work with Hsian-hua Tseng.

**Plan:**

1. Example of group extension

2. Gerbes and groupoids

3. Duality of gerbes on orbifolds

**Part I: Group Extension**

We review the Mackey machine on finite groups. Such an idea of induced representations goes back to Frobenius and Schur.

## $G$ Extension of a finite group $Q$

Consider an exact sequence of finite groups

$$1 \to G \xrightarrow{i} H \xrightarrow{j} Q \to 1. \tag{1}$$

Choose a section $s : Q \to H$. As $G$ is normal in $H$, the group $H$ acts on $G$ by conjugation. The section $s$ defines a $Q$ "action" $\alpha$ on $G$ by

$$\alpha(q)(g) = s(q)gs(q)^{-1}.$$

Generally this is not a real group action because $\alpha(q_1) \circ \alpha(q_2) \neq \alpha(q_1 q_2)$.

Define $c : Q \times Q \to G$ by $c(q_1, q_2) = s(q_1)s(q_2)s(q_1q_2)^{-1}$. The the failure of $\alpha$ being a group action can be computed by

$$\alpha(q_1) \circ \alpha(q_2) = Ad_{c(q_1,q_2)} \circ \alpha(q_1q_2).$$

## Mackey machine I

Mackey machine provides an algorithm to determine $H$ representations in terms of representations of $G$ and $Q$.

Let $V$ be a representation of $H$. The group $G$ as a normal subgroup of $H$ also acts on $V$. As $G$ is finite, the space $V$ as a $G$ representation is naturally decomposed into a direct sum of irreducible $G$ representations, i.e.

$$V = \bigoplus_{\rho \in \widehat{G}} V_\rho, \qquad (2)$$

where $\widehat{G}$ is the set of isomorphism classes of irreducible representations of $G$ and $V_\rho$ a subspace of $V$ is a direct sum of $G$ sub-representations of $V$ which are isomorphic to $\rho$.

## Mackey machine II

As $G$ is a normal subgroup of $H$, the group $H$ acts on $G$ by conjugation. Accordingly, $H$ acts on $\widehat{G}$ the set of isomorphism classes of $G$ irreducible representations. This induces a $Q = H/G$ action on $\widehat{G}$ as $G$ acts on $\widehat{G}$ trivially.

Let $\chi_\rho$ be the orthogonal projection from $V$ to $V_\rho$. Then we have

$$T_h \chi_\rho T_h^{-1} = \chi_{h(\rho)}.$$

If $V$ is an $H$ irreducible representation, then the elements in $\widehat{G}$ appearing in $V$ forms an orbit of the $Q$ action on $\widehat{G}$.

Let $\theta$ be an orbit of the $Q$ action on $\widehat{G}$. Choose $\rho \in \theta$, and let $Q_\theta$ be the stabilizer subgroup of $\rho$. Irreducible representations of $H$ associated to $\theta$ are one to one correspondent to $\tau_\theta$-twisted representations of the group $Q_\theta$.

## $U(1)$ **valued cocycle**

Choose $\rho \in \theta$. The group $Q_\theta$ keeps $\rho$ invariant, i.e. $q(\rho) = \rho$, for $q \in Q_\theta$.

Write $V_\rho$ as $E_\rho \otimes F_\rho$, where $E_\rho$ is a $G$ irreducible representation of the isomorphism class $\rho$. Since $T_q \chi_\rho T_q^{-1} = \chi_{q(\rho)} = \chi_\rho$ for $q \in Q_\theta$, the group $Q_\theta$ preserves $V_\rho$, i.e. $T_q(V_\rho) \subset V_\rho$ for $q \in Q_\theta$.

As $G$ action on $E_\rho$ is irreducible, we have that $T_q$ acts on $V_\rho = E_\rho \otimes F_\rho$ diagonally, i.e. $T_q = E_q \otimes F_q$ for $q \in Q_\theta$. It is easy to check that $c(q_1, q_2) E_{s(q_1 q_2)} E_{q_2}^{-1} E_{q_1}^{-1}$ commutes with the $G$ action on $E_\rho$. Therefore, $\tau_\theta(q_1, q_2) = c(q_1, q_2) E_{s(q_1 q_2)} E_{q_2}^{-1} E_{q_1}^{-1}$ is an element of $U(1)$.

The group $Q_\theta$ acts $F_\rho$ with cocycle $\tau_\theta$.

**Duality suggested by Mackey machine**

The Mackey machine suggests the following Morita equivalence result.

The group algebra $\mathbb{C}H$ is Morita equivalent to the algebra

$$\bigoplus_{\theta} \mathbb{C}_{\tau_\theta} Q_\theta.$$

Namely, the category of $H$ modules is isomorphic to the category of $\bigoplus_{\theta} \mathbb{C}_{\tau_\theta} Q_\theta$ modules.

Geometrically, this suggests that there is some duality between the group $H$ and the action groupoid $\widehat{G} \rtimes Q$ with a $U(1)$-valued groupoid 2-cohomology class $[\tau]$.

**A toy example**

When $Q$ is the trivial group, then $H = G$. Our Morita equivalence result states that the group algebra $\mathbb{C}G$ is Morita equivalent to $C(\widehat{G})$, the algebra of functions on $\widehat{G}$.

This is a corollary of the classical result that $\mathbb{C}G$ is isomorphic to

$$\bigoplus_{\rho \in \widehat{G}} \mathrm{End}\, E_\rho. \tag{3}$$

In particular, if $G$ is abelian, then $\mathbb{C}G$ is isomorphic to $C(\widehat{G})$ via the Fourier transform.

Our discussion today is a generalization of the classical "Pontryagin duality" on abelian groups to groupoids.

**Part II: Gerbes and Groupoids**

We explain the groupoid approach to gerbes and the duality conjecture of gerbes on orbifolds.

## Brief review of principal $G$-bundles

A principal $G$-bundle $P$ over a manifold $M$ can be described using Čech cocycles. Choose a nice open covering $\{U_\alpha\}$ of $M$.

On every open chart, $P|_{U_\alpha} \cong U_\alpha \times G$. The collection $\{U_\alpha \times G\}$ glues together via a $G$-valued 1-cocycle $g_{\alpha\beta} : U_\alpha \times G|_{U_\alpha \cap U_\beta} \to U_\beta \times G|_{U_\alpha \cap U_\beta}$, i.e.

$$g_{\beta\gamma} \circ g_{\alpha\beta} = g_{\alpha\gamma}.$$

Consider the Čech groupoid presentation of $M$, $\coprod_{\alpha,\beta} U_\alpha \cap U_\beta \rightrightarrows \coprod_\alpha U_\alpha$. The data $\{g_{\beta\alpha}\}$ defines a $G$-valued 1-cocycle on this Čech groupoid.

## $G$-gerbe

The notion of gerbe was introduced by Giraud in algebraic geometry in his study of nonabelian cohomology. Roughly speaking, a $G$-gerbe over a space $X$ is a $BG$-bundle over $X$.

More precisely, let $\{U_\alpha\}$ be a nice open covering of $X$. We consider the following data:

$\varphi_{\alpha\beta} \in Aut(G)$ for each double overlap $U_{\alpha\beta} := U_\alpha \cap U_\beta$, and

$g_{\alpha\beta\gamma} \in G$ for each triple overlap $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$,

so that the following constraints are satisfied:

$$\varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} = \mathrm{Ad}_{g_{\alpha\beta\gamma}} \circ \varphi_{\alpha\beta}, \quad \text{on } U_{\alpha\beta\gamma},$$
$$g_{\beta\gamma\delta} g_{\alpha\beta\delta} = \varphi_{\gamma\delta}(g_{\alpha\beta\gamma}) g_{\alpha\gamma\delta}, \quad \text{on } U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta.$$

Here $\mathrm{Ad}_g : G \to G$ denotes the conjugation by $g$.

## Groupoid extensions and $G$-gerbes

Example: Group extension

A $G$ gerbe on a manifold $M$ can be realized by a groupoid extension of the Čech groupoid,

$$\coprod_{\alpha} U_\alpha \times G \longrightarrow \mathcal{G} \longrightarrow \coprod_{\alpha,\beta} U_\alpha \cap U_\beta \rightrightarrows \coprod_{\alpha} U_\alpha. \tag{4}$$

In general, if we represent a space $X$ as the quotient space of a groupoid $\mathfrak{Q} \rightrightarrows \mathfrak{Q}_0$, then a $G$-gerbe on $X$ can be presented by a groupoid extension

$$\mathfrak{G} \longrightarrow \mathfrak{H} \longrightarrow \mathfrak{Q} \rightrightarrows \mathfrak{Q}_0,$$

where $\mathfrak{G}$ is a principal $G$-bundle over $\mathfrak{Q}_0$.

## Continuous trace $C^*$-algebras and Dixmier-Douady class

Let $\mathcal{H}$ be a separable Hilbert space, and $K(\mathcal{H})$ be the algebra of compact operators on $\mathcal{H}$. Let $V$ be a bundle of $K(\mathcal{H})$ algebras.

Let $Aut(K(\mathcal{H}))$ be the automorphism group of $K(\mathcal{H})$, and $U(\mathcal{H})$ be the group of unitary operators on $\mathcal{H}$. We have the following exact sequence

$$U(1) \longrightarrow U(\mathcal{H}) \longrightarrow Aut(K(\mathcal{H})).$$

A bundle of $K(\mathcal{H})$ algebras is classified by the associated cohomology class $[c] \in H^1(M, \underline{Aut(K(\mathcal{H}))}) \cong H^2(M, \underline{U(1)}) \cong H^3(M, \mathbb{Z})$, which is called the Dixmier-Douady class. Such a bundle corresponds to a $U(1)$-gerbe on $M$.

## $G$-gerbes on an orbifold

Let $H$ act on a smooth manifold $M$ such that $G$ acts on $M$ trivially. Accordingly, the group $Q = H/G$ acts on $M$. We have the following exact sequence of action groupoids,

$$M \times G \rightarrow M \rtimes H \rightarrow M \rtimes Q. \tag{5}$$

The above groupoid extension defines a $G$-gerbe $\mathcal{Y}$ on the orbifold $\mathcal{B} = M/Q$.

In general, to construct all $G$-gerbes on an orbifold $\mathcal{B}$, we need to present $\mathcal{B}$ by a proper étale groupoid $\mathfrak{Q}$. A $G$-gerbe over $\mathcal{B}$ can be presented as a $G$-extension of the groupoid $\mathfrak{Q}$. Morita equivalent $G$-extensions define isomorphic $G$-gerbes on $\mathcal{B}$.

## A dual orbifold $\widehat{\mathcal{Y}}$ with a $U(1)$-gerbe

As is explained in the group extension case, choose a section $s: Q \to H$. Then $Q$ acts on $\widehat{G}$, the set of isomorphism classes of $G$ irreducible representations.

Define a new groupoid $\widehat{\mathfrak{Q}} = (\widehat{G} \times M) \rtimes Q$. The quotient space of the groupoid $\widehat{\mathfrak{Q}}$ is an orbifold $\widehat{\mathcal{Y}}$, which has a natural decomposition into components with respect to the $Q$ orbits on $\widehat{G}$.

The cocycle $\tau$ on the groupoid $\widehat{G} \rtimes Q$ defined using the Mackey machine induces a groupoid cocycle $\tau$ on the groupoid $\widehat{\mathfrak{Q}}$. The cocycle $\tau$ defines a $U(1)$-gerbe on the orbifold $\widehat{\mathcal{Y}}$, which is actually a torsion.

The Mackey machine suggests that there is a duality between the $G$-gerbe $\mathcal{Y}$ over $\mathcal{B}$ and $U(1)$-gerbe $\tau$ on the dual orbifold $\widehat{\mathcal{Y}}$.

## Duality conjecture

We have introduced a $G$-gerbe $\mathcal{Y}$ over an orbifold $\mathcal{B} = M/Q$ and a dual orbifold $\hat{\mathcal{Y}}$ together with a $U(1)$-gerbe $\tau$.

Inspired from string theory on orbifolds, Hellerman, Henriques, Pantev, and Sharpe conjectured that conformal field theories on the $G$-gerbe $\mathcal{Y}$ are equivalent to the corresponding conformal field theories on $\hat{\mathcal{Y}}$ *twisted by the B-field* $\tau$.

Our viewpoint toward this conjecture is that it suggests the following principle:

*Geometry/topology of the $G$-gerbe $\mathcal{Y}$ is equivalent to geometry/topology of $\hat{\mathcal{Y}}$ twisted by $\tau$.*

## Part III: Duality of gerbes on orbifolds

We explain a number of mathematical results realizing the principle of duality between gerbes on orbifolds. The key tool we use is the Mackey machine.

## $G$-gerbes on $BQ$

When the manifold is a point, then the groupoid extension is reduced to a group extension,

$$G \longrightarrow H \longrightarrow Q.$$

In this example, $\mathcal{B} = [pt/Q]$, $\mathcal{Y} = [pt/H]$. And $\widehat{\mathcal{Y}} = [\widehat{G}/Q]$, and the cocycle $\tau$ is defined by the Mackey machine.

**Proposition 1** *The group algebra $\mathbb{C}H$ is Morita equivalent to the twisted groupoid algebra $C(\widehat{G} \rtimes Q, \tau)$.*

The Morita bimodule as vector space is isomorphic to

$$M = \oplus_{[\rho]\in\widehat{G}} E_\rho \times Q.$$

## Morita equivalence of groupoid algebras

The following results are inspired from the Mackey machine on group extensions, and can be viewed as generalizations of the Mackey machine on groupoid extensions.

**Theorem 2** *The groupoid algebra of the groupoid $M \rtimes H$ is Morita equivalent to the $\tau$-twisted groupoid algebra of the groupoid $(\widehat{G} \times M) \rtimes Q$.*

Furthermore, if $M$ is equipped with a $Q$ invariant symplectic form, then we consider a $Q$ invariant deformation quantization $\mathcal{A}^{((\hbar))}(M)$ of the algebra of smooth functions on $M$.

**Theorem 3** *The crossed product algebra $\mathcal{A}^{((\hbar))}(M) \rtimes H$ is Morita equivalent to the $\tau$ twisted crossed product algebra $\mathcal{A}^{((\hbar))}(\widehat{G} \times M) \rtimes_{\tau} Q$.*

**Categories of sheaves**

On $\mathcal{Y}$, we consider the category of sheaves. On $\hat{\mathcal{Y}}$, we consider the category of $\tau$-twisted sheaves.

**Theorem 4** *The category of sheaves on $\mathcal{Y}$ is equivalent to the category of $\tau$-twisted sheaves on $\hat{\mathcal{Y}}$.*

Key Idea: Use the twisted sheaf $\mathcal{E}_G = \oplus_{\rho \in \widehat{G}} E_\rho$ on $\hat{\mathcal{Y}}$.

Theorem 2-Theorem 4 show that the noncommutative geometry of the gerbe $\mathcal{Y}$ is same to the noncommutative geometry of the dual orbifold $\hat{\mathcal{Y}}$ twisted by $\tau$.

**Corollaries about $K$-theory**

Morita equivalent $C^*$-algebras have isomorphic $K$-groups.

The $K$-theory of the crossed product algebra $C(M) \rtimes H$ is isomorphic to the $H$-equivariant $K$-theory on $M$, i.e. the geometric $K$-theory of the orbifold $\mathcal{Y}$.

The $K$-theory of the $\tau$-twisted groupoid algebra of $(\hat{G} \times M) \rtimes Q$ is isomorphic to the $\tau$-twisted $K$-theory of the orbifold $\widehat{\mathcal{Y}}$.

**Corollary 5** *The $K$-theory of the $G$-gerbe $\mathcal{Y}$ is isomorphic to the $\tau$-twisted $K$-theory of the orbifold $\widehat{\mathcal{Y}}$.*

**Inertia orbifold**

Let $M$ be a smooth manifold and $\Gamma$ be a finite group acting on $M$ by diffeomorphisms. We consider the orbifold $X = M/\Gamma$.

Let $M^\gamma$ be the $\gamma$-fixed point submanifold of $M$ for $\gamma \in \Gamma$. The group $\Gamma$ acts on $\coprod_\gamma M^\gamma$ by $\alpha(\gamma, x) = (\alpha\gamma\alpha^{-1}, \alpha(x))$ for $x \in M^\gamma$. The associated inertia orbifold $IX$ is the quotient space $(\coprod_\gamma M^\gamma)/\Gamma$.

For every $\gamma \in \Gamma$, we define a locally constant function $\ell$ on $M^\gamma$ by the codimension of $M^\gamma \hookrightarrow M$. $\ell$ is invariant under the $\Gamma$ action on $\coprod_\gamma M^\gamma$. So $\ell$ defines a function on the inertia orbifold $IX$.

**Age function**

If $M$ is equipped with a $\Gamma$ equivariant almost complex structure, $\gamma$ acts on $T_xM$ for $x \in M^\gamma$ by an isomorphism of a complex vector.

As $\gamma$ is of finite order, $T_xM$ splits into a sum of eigen-spaces of $\gamma$ action, i.e. $T_xM = \oplus_{k=0}^{r-1} V_k$, where $\gamma$ acts on $V_k$ with eigenvalue $\exp(2\pi\sqrt{-1}k/r)$.

The age function on $\bigsqcup_\gamma M^\gamma$ is defined to be

$$age(\gamma, x) = \sum_{k=0}^{r-1} \frac{k}{r} \dim(V_i) \in \mathbb{Q}.$$

The age function is invariant under the $\Gamma$ action on $\bigsqcup_\gamma M^\gamma$, and hence defines a function on the inertia orbifold $IX$.

**Twisted de Rham cohomology**

Let $\tau$ define a flat $U(1)$-gerbe on an orbifold $X = M/\Gamma$. $\tau$ defines a flat line bundle over $IX$.

Consider the trivial line bundle $C$ on $\coprod_\gamma M^\gamma$. Define an action of $\Gamma$ on $C$ by $\alpha(\xi)(\gamma, x) = \xi(\alpha\gamma\alpha^{-1}, \alpha(x))\tau(\alpha,\gamma)\tau(\alpha\gamma\alpha^{-1}, \alpha)^{-1}$, for $\xi \in \Gamma(C)$. The quotient of $C$ by $\Gamma$ defines a flat line bundle $\mathcal{L}_\tau$ over $IX$.

The $\tau$-twisted de Rham cohomology $H^\bullet(IX, \tau; \mathbb{C})$ of $IX$ is defined to be the de Rham cohomology on $IX$ with coefficient in $\mathcal{L}_\tau$.

## Cohomology

The above Morita equivalence between algebras implies that their corresponding Hochschild and cyclic cohomology are isomorphic.

The Hochschild cohomology of $\mathcal{A}^{((\hbar))}(M) \rtimes H$ isomorphic to the cohomology of $IX$ with a degree shift $\ell$. (Pflaum-Posthuma-T-T)

**Proposition 6** *The Hochschild cohomology of the $\tau$-twisted groupoid algebra $\mathcal{A}^{((\hbar))}(\widehat{G} \times M) \rtimes_\tau Q$ is computed by $H^{\bullet-\ell}(IX, \tau; \mathbb{C}((\hbar)))$.*

**Theorem 7**

$$H^{\bullet-\ell}(I\mathcal{Y}; \mathbb{C}((\hbar))) \simeq H^{\bullet-\ell}(I\widehat{\mathcal{Y}}, \tau; \mathbb{C}((\hbar)))$$

## More on group extension

Morita equivalent algebras have isomorphic centers. Using the Morita equivalence bimodule, we obtain an isomorphism between the center of $\mathbb{C}H$ and the center of the twisted groupoid algebra $C(\widehat{G} \rtimes Q, \tau)$.

The center of $\mathbb{C}H$ consists of class functions. The center of the twisted groupoid algebra $C(\widehat{G} \rtimes Q, \tau)$ is a direct sum of the center of the twisted group algebra $\mathbb{C}_{\tau_\theta} Q_\theta$, where $\theta$ runs over orbits of the $Q$ action on $\widehat{G}$. The center of the twisted group algebra $\mathbb{C}_{\tau_\theta} Q_\theta$ consists of $\tau_\theta$-regular class functions.

Under the isomorphism of centers, we are able to identify the canonical trace on the center of $\mathbb{C}H$ with the following trace on the center of $C(\widehat{G} \rtimes Q, \tau)$, $\sum_\theta \frac{\dim(E_\theta)^2}{|G|^2} \mathrm{tr}_\theta$ .

**Gromov-Witten theory on $BH$**

Hence, we have an isomorphism of Frobenius algebras defined by the centers of the algebras $\mathbb{C}H$ and $C(\widehat{G} \rtimes Q, \tau)$ together with the traces.

Based on this isomorphism, we are able to prove that the Gromov-Witten theory on $BH$ is isomorphic to the direct sum of $\tau_\theta$-twisted Gromov-Witten theory on $BQ_\theta$.

## Morita bimodule and Hochschild cohomology

The Morita equivalence bimodule between the crossed product algebra $\mathcal{A}^{((\hbar))}(M) \rtimes H$ and the $\tau$ twisted crossed product algebra $\mathcal{A}^{((\hbar))}(\widehat{G} \times M) \rtimes_\tau Q$ is realized as

$$\mathcal{M} := \Gamma(\widetilde{\mathcal{A}}^{((\hbar))} \otimes \mathcal{V}_G) \times Q.$$

Such an explicit expression of the Morita equivalence bimodule leads to an explicit formula on the corresponding Hochschild cochain complexes. Such a formula gives rises to an explicit formula for the isomorphism

$$I : H^{\bullet - \ell}(I\mathcal{Y}; \mathbb{C}) \simeq H^{\bullet - \ell}(I\widehat{\mathcal{Y}}, \tau; \mathbb{C})$$

$$I(\alpha)([\rho], q) = \sum_g \frac{1}{\dim(E_\rho)} \alpha(g, q) \operatorname{tr}(\rho(g) {T_q^{[\rho]}}^{-1}).$$

**Chen-Ruan orbifold cup product structure**

The explicit formula of $I$ leads to the following improvement of Theorem 7.

**Theorem 8** *There is an isomorphism*

$$H^{\bullet - age}(I\mathcal{Y}, \mathbb{C}) \simeq H^{\bullet - age}(I\widehat{\mathcal{Y}}, \tau, \mathbb{C}).$$

*of* graded $\mathbb{C}$-algebras.

This serves as the first step toward the isomorphism of Gromov-Witten potentials.

**Outlook**

1. Generalizations to more general $G$-gerbes or more general spaces.

2. Comparison of Gromov-Witten theories.

3. Possible applications to combinatorics problem on counting the number of conjugacy classes in a finite group.