SINGULAR REDUCTION, QUANTUM REDUCTION, AND COHERENT STATES QUANTIZATION

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PLAN OF THE PRESENTATION

- **Regular reduction**
- **Singular point reduction**
- **Singular orbit reduction**
- **Regular cotangent bundle reduction**
- **Singular cotangent bundle reduction**
- **Banach Poisson manifolds**
- **Banach Lie-Poisson spaces**
- **Quantum reduction**
- **Coherent states quantization**

ABSTRACT SYMMETRY REDUCTION

$\Phi : G \times M \to M$ proper Lie group action
$\mathfrak{X}(M)^G := \{X \in \mathfrak{X}(M) \mid \Phi_g^* X = X, \forall g \in G\}$
$X \in \mathfrak{X}(M)^G \Longleftrightarrow F_t \circ \Phi_g = \Phi_g \circ F_t, \forall g \in G$

Law of conservation of isotropy: $M_H := \{m \in M \mid G_m = H\}$ is preserved by F_t. In general, M_H not closed. $\iota_H : M_H \hookrightarrow M$ inclusion.

$N(H) := \{g \in G \mid gH = Hg\}$ normalizer of H in G. $N(H)/H$ acts freely and properly on M_H; $\pi_H : M_H \to M_H/(N(H)/H)$ projection.

X induces a unique H-**isotropy type reduced vector field** X^H on $M_H/(N(H)/H)$ by

$$X^H \circ \pi_H = T\pi_H \circ X \circ \iota_H,$$

whose flow F_t^H is given by

$$F_t^H \circ \pi_H = \pi_H \circ F_t \circ \iota_H.$$

SYMPLECTIC POINT REDUCTION

$\Phi : G \times M \to M$ left action; infinitesimal generator of $\xi \in \mathfrak{g} = \mathrm{Lie}(G)$

$$\xi_M(m) := \left.\frac{d}{dt}\right|_{t=0} \Phi_{\exp t\xi}(m) \in T_mM, \qquad \xi_M \in \mathfrak{X}(M).$$

Φ free proper $\Rightarrow M/G$ manifold and $\pi : M \to M/G$ submersion

(M, ω) symplectic manifold. Hamiltonian vector field $X_H \in \mathfrak{X}(M)$ is defined by $\mathbf{d}H = \omega(X_H, \cdot)$ and the associated Poisson bracket by $\{F, H\} := \omega(X_F, X_H)$. Hamilton's equations are:

$$\dot{m}(t) = X_H(m(t)) \Longleftrightarrow \frac{d}{dt}F(m(t)) = \{F, H\}(m(t)), \forall F \in C^\infty(M).$$

(M, ω) symplectic manifold, $\Phi : G \times M \to M$ free proper symplectic action ($\Phi_g^*\omega = \omega$, $\forall g \in G$). Then M/G is a Poisson manifold uniquely characterized by the condition that $\pi : M \to M/G$ is a Poisson map: $\{f, h\}_{M/G} \circ \pi = \{f \circ \pi, h \circ \pi\}_M$, $\forall f, h \in C^\infty(M/G)$.

What are its leaves? The symplectic point of orbit reduced spaces.

(M, ω), G connected Lie group with Lie algebra $\mathfrak{g}$,
$\Phi : G \times M \to M$ proper symplectic action

$\mathbf{J} : M \to \mathfrak{g}^*$ **momentum map:** $X_{\mathbf{J}^\xi} = \xi_M, \forall \xi \in \mathfrak{g}$, $\mathbf{J}^\xi := \langle \mathbf{J}, \xi \rangle \in C^\infty(M)$.
$\mathbf{J}$ is **equivariant** if $\mathbf{J}(g \cdot m) = \mathrm{Ad}^*_{g^{-1}} \mathbf{J}(m)$, for all $g \in G$, $m \in M$.

Equivariant $\mathbf{J} : M \to \mathfrak{g}^*_+$ is a Poisson map, where

$$\{f, h\}_\pm(\mu) := \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu}\right] \right\rangle, \quad \forall f, h \in C^\infty(\mathfrak{g}^*)$$

$$\mathbf{D}f(\mu) \cdot \nu = \left\langle \nu, \frac{\delta f}{\delta \mu} \right\rangle, \quad \forall \nu \in \mathfrak{g}^*$$

Noether's Theorem: If $h \in C^\infty(M)^G$, then $\mathbf{J} \circ F_t = \mathbf{J}$, where F_t is the flow of X_h.

Bifurcation Lemma: range $(T_m \mathbf{J}) = (\mathfrak{g}_m)^\circ$.

Fiber Lemma: ker $(T_m \mathbf{J}) = (\mathfrak{g} \cdot m)^\omega$.

Existence: The obstruction is the vanishing of the map

$$\mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \ni [\xi] \longmapsto \left[\mathbf{i}_{\xi_M}\omega\right] \in H^1(M;\mathbb{R})$$

Infinitesimal equivariance: $(\mathfrak{g},[\cdot,\cdot]) \ni \xi \longmapsto \mathbf{J}^\xi \in (C^\infty(M),\{\cdot,\cdot\})$
Lie algebra homomorphism, i.e.,

$$\mathbf{J}^{[\xi,\eta]} = \{\mathbf{J}^\xi,\mathbf{J}^\eta\},\ \forall \xi,\eta \in \mathfrak{g} \iff T_z\mathbf{J}\left(\xi_M(z)\right) = -\operatorname{ad}^*_\xi \mathbf{J}(z),\ \forall \xi \in \mathfrak{g}$$

G connected: infinitesimal equivariance equivalent to equivariance.
Among all possible choices of momentum maps for a given action, there is at most one that is infinitesimally equivariant.
If G is compact $\mathbf{J}$ can be chosen G-equivariant.

Sufficient conditions: Assume $H^1(\mathfrak{g};\mathbb{R}) = H^2(\mathfrak{g};\mathbb{R}) = 0$.
By **Whitehead lemmas**, this holds if $\mathfrak{g}$ is semisimple.

Non-equivariance or **Souriau one-cocycle** associated to $\mathbf{J}$:

$$\sigma : G \ni g \longmapsto \mathbf{J}(\Phi_g(m)) - \operatorname{Ad}^*_{g^{-1}}(\mathbf{J}(m)) \in \mathfrak{g}^*$$

Suppose that M is connected. Then:

(i) The definition of σ does not depend on $m \in M$.

(ii) σ is a $\mathfrak{g}^*$-valued one-cocycle on G for the coadjoint representation: $\sigma(gh) = \sigma(g) + \mathrm{Ad}^*_{g^{-1}} \sigma(h)$ for all $g, h \in G$.

Affine coadjoint action of G on $\mathfrak{g}^*$ with cocycle σ:

$$\Theta : G \times \mathfrak{g}^* \ni (g, \mu) \longmapsto \mathrm{Ad}^*_{g^{-1}} \mu + \sigma(g) \in \mathfrak{g}^*$$

$\mathbf{J} : M \to \mathfrak{g}^*$ is equivariant with respect to the symplectic action Φ on M and the affine action Θ on $\mathfrak{g}^*$.

Affine orbits $\mathcal{O}_\mu$ are symplectic with G-invariant symplectic structure

$$\omega^{\pm}_{\mathcal{O}_\mu}(\nu)(\xi_{\mathfrak{g}^*}(\nu), \eta_{\mathfrak{g}^*}(\nu)) = \pm\langle \nu, [\xi, \eta]\rangle \mp \Sigma(\xi, \eta)$$

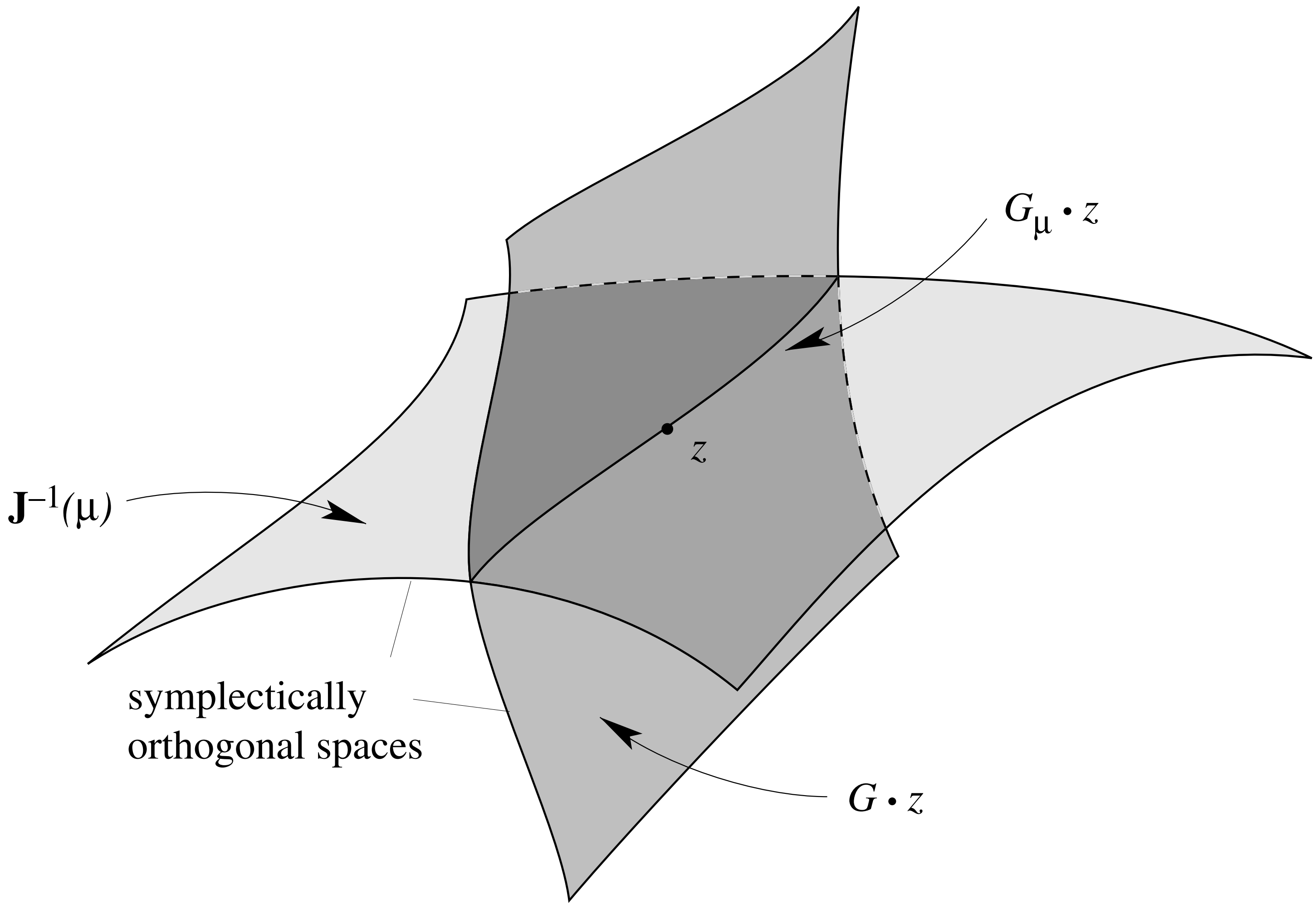
Infinitesimal non-equivariance two-cocycle $\Sigma \in Z^2(\mathfrak{g}, \mathbb{R})$:

$$\Sigma : \mathfrak{g} \times \mathfrak{g} \ni (\xi, \eta) \longmapsto \mathbf{d}\widehat{\sigma}_\eta(e) \cdot \xi \in \mathbb{R}$$

where $\widehat{\sigma}_\eta : G \to \mathbb{R}$ is defined by $\widehat{\sigma}_\eta(g) = \langle \sigma(g), \eta \rangle$.

Reduction Lemma:

$$\mathfrak{g}_{\mathbf{J}(m)} \cdot m = \mathfrak{g} \cdot m \cap \ker T_m \mathbf{J} = \mathfrak{g} \cdot m \cap (\mathfrak{g} \cdot m)^{\omega}.$$



Momentum maps and isotropy type manifolds

Fix $m \in M$. Then M_{G_m} is a symplectic submanifold of M.

This follows from the following result.

H compact Lie group and (V, ω) symplectic representation space. Then V^H is a symplectic subspace of V.

WARNING: $M_{(H)} := \{m \in M \mid G_m \text{ conjugate to } H\}$ not symplectic.

$M_{G_m}^m$ connected component of M_{G_m} containing m

$$N(G_m)^m := \{n \in N(G_m) \mid n \cdot z \in M_{G_m}^m \text{ for all } z \in M_{G_m}^m\}.$$

$N(G_m)^m$ is a closed subgroup of $N(G_m)$, $N(G_m)^m \supset N(G_m)_\circ$, the connected component of the identity. So $N(G_m)^m$ is also open $\Rightarrow$ Lie$(N(G_m)^m) =$ Lie$(N(G_m))$. Also $H \subset N(G_m)^m$. In addition, $(N(G_m)/G_m)^m = N(G_m)^m/G_m$ so that

$$\text{Lie}\left(N(G_m)^m/G_m\right) = \text{Lie}\left(N(G_m)/G_m\right).$$

$L^m := N(G_m)^m/G_m$ acts freely properly symplectically on $M^m_{G_m}$ by $\Psi(nG_m, z) := n \cdot z$.

The free proper symplectic action of $L^m := N(G_m)^m/G_m$ on $M^m_{G_m}$ has a momentum map $\mathbf{J}_{L^m} : M^m_{G_m} \to (\mathrm{Lie}(L^m))^*$

$$\mathbf{J}_{L^m}(z) := \Lambda(\mathbf{J}|_{M^m_{G_m}}(z) - \mathbf{J}(m)), \quad z \in M^m_{G_m}.$$

In this expression $\Lambda : (\mathfrak{g}_m^\circ)^{G_m} \to (\mathrm{Lie}(L^m))^*$ denotes the natural L^m-equivariant isomorphism given by

$$\left\langle \Lambda(\beta), \left.\frac{d}{dt}\right|_{t=0} (\exp t\xi)\, G_m \right\rangle = \langle \beta, \xi \rangle,$$

for any $\beta \in (\mathfrak{g}_m^\circ)^{G_m}$, $\xi \in \mathrm{Lie}(N(G_m)^m) = \mathrm{Lie}(N(G_m))$.

The non-equivariance one-cocycle $\tau : M^m_{G_m} \to (\mathrm{Lie}(L^m))^*$ of the momentum map $\mathbf{J}_{L^m}$ is given by the map

$$\tau(l) = \Lambda(\sigma(n) + n \cdot \mathbf{J}(m) - \mathbf{J}(m)).$$

So, even if $\mathbf{J}$ is equivariant ($\sigma = 0$), $\mathbf{J}_{L^m}$ is not, in general.

MARSDEN-WEINSTEIN POINT REDUCTION

- $\Phi : G \times M \to M$ free proper symplectic action.
- The action admits a momentum map $\mathbf{J} : M \to \mathfrak{g}^*$.
- M is connected; if $\mathbf{J}$ equivariant, this is not needed.
- Then any $\mu \in \mathbf{J}(M) \subset \mathfrak{g}^*$ is a regular value of $\mathbf{J}$: by the Bifurcation Lemma, $\mathbf{J}$ is a submersion on an open subset of M. So $\mathbf{J}^{-1}(\mu)$ is a G_μ-invariant (equivariance of $\mathbf{J}$) closed embedded submanifold of M.
- $G_\mu := \{g \in G \mid \mathrm{Ad}^*_{g^{-1}} \mu + \sigma(g) = \mu\}$

(i) Then on $M_\mu := \mathbf{J}^{-1}(\mu)/G_\mu$ there is a unique symplectic from ω_μ characterized by: $\pi_\mu^* \omega_\mu = \iota_\mu^* \omega$.

$\iota_\mu : \mathbf{J}^{-1}(\mu) \hookrightarrow M$ inclusion,
$\pi_\mu : \mathbf{J}^{-1}(\mu) \to \mathbf{J}^{-1}(\mu)/G_\mu$ projection.

(ii) Flow F_t of X_h, $h \in C^\infty(M)^G$, leaves the connected components of $\mathbf{J}^{-1}(\mu)$ invariant, commutes with the G-action, so it induces a flow F_t^μ on M_μ by

$$\pi_\mu \circ F_t \circ i_\mu = F_t^\mu \circ \pi_\mu .$$

(iii) F_t^μ is Hamiltonian on (M_μ, ω_μ) for the **reduced Hamiltonian** $h_\mu \in C^\infty(M_\mu)$ given by

$$h_\mu \circ \pi_\mu = h \circ i_\mu .$$

(iv) If $h, k \in C^\infty(M)^G$, then $\{h, k\}_\mu = \{h_\mu, k_\mu\}_{M_\mu}$.

Initial submanifold $\iota : N \hookrightarrow M$: (injective) immersion such that for any smooth manifold P, an arbitrary map $g : P \to M$ is smooth if and only if $\iota \circ g : P \to N$ smooth.

This is stronger than injective immersion but weaker than embedding. Typical examples are the leaves of a generalized foliation. For example, any orbit $\mathcal{O}_m$ of a smooth G-action is an initial submanifold diffeomorphic to G/G_m. Let G be a Lie group and H a subgroup; then there is a unique smooth manifold structure on H that makes it into an initial submanifold of G. So Lie subgroups (the inclusion is an immersion) are initial.

ORBIT REDUCTION

(i) On $M_{\mathcal{O}_\mu} := \mathbf{J}^{-1}(\mathcal{O}_\mu)/G$ there is a unique symplectic form $\omega_{\mathcal{O}_\mu}$ characterized by $\iota_{\mathcal{O}_\mu}^*\omega = \pi_{\mathcal{O}_\mu}^*\omega_{\mathcal{O}_\mu} + \mathbf{J}_{\mathcal{O}_\mu}^*\omega_{\mathcal{O}_\mu}^+$. $\iota_{\mathcal{O}_\mu} : \mathbf{J}^{-1}(\mathcal{O}_\mu) \hookrightarrow M$, $\mathbf{J}_{\mathcal{O}_\mu} := \mathbf{J}|_{\mathbf{J}^{-1}(\mathcal{O}_\mu)}$, and $\omega_{\mathcal{O}_\mu}^+$ is the $+$-symplectic structure on the affine orbit $\mathcal{O}_\mu$. $(M_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ is the **symplectic orbit reduced space**.

(ii) $h \in C^\infty(M)^G$. The flow F_t of X_h leaves the connected components of $\mathbf{J}^{-1}(\mathcal{O}_\mu)$ invariant and commutes with the G-action, so it induces a flow $F_t^{\mathcal{O}_\mu}$ on $M_{\mathcal{O}_\mu}$, determined by $\pi_{\mathcal{O}_\mu} \circ F_t \circ \iota_{\mathcal{O}_\mu} = F_t^{\mathcal{O}_\mu} \circ \pi_{\mathcal{O}_\mu}$.

(iii) Vector field generated by the flow $F_t^{\mathcal{O}_\mu}$ on $(M_{\mathcal{O}_\mu}, \omega_{\mathcal{O}_\mu})$ is Hamiltonian with associated **reduced Hamiltonian** $h_{\mathcal{O}_\mu} \in C^\infty(M_{\mathcal{O}_\mu})$ defined by $h_{\mathcal{O}_\mu} \circ \pi_{\mathcal{O}_\mu} = h \circ \iota_{\mathcal{O}_\mu}$. X_h and $X_{h_{\mathcal{O}_\mu}}$ are $\pi_{\mathcal{O}_\mu}$-related.

(iv) $h, k \in C^\infty(M)^G \Rightarrow \{h, k\} \in C^\infty(M)^G$ and $\{h, k\}_{\mathcal{O}_\mu} = \{h_{\mathcal{O}_\mu}, k_{\mathcal{O}_\mu}\}_{M_{\mathcal{O}_\mu}}$, where $\{\cdot, \cdot\}_{M_{\mathcal{O}_\mu}}$ denotes the Poisson bracket associated to the symplectic form $\omega_{\mathcal{O}_\mu}$ on $M_{\mathcal{O}_\mu}$.

(v) M_μ and $M_{\mathcal{O}_\mu}$ are symplectomorphic.

Problems with the symplectic reduction procedure

1.) Momentum map may not exist. This is particularly acute in the Poisson case. Any Poisson action that admits a momentum map must preserve the leaves. This is too strong! Many examples.

2.) Where is conservation of isotropy? The momentum map does NOT get it. $\mathbf{J}^{-1}(\mu)$ are not the smallest invariant sets. Reduction completely ignores this point.

3.) M_μ is not a smooth manifold.

4.) If G is discrete, the momentum map is zero. What is reduction in that case?

5.) For G-manifolds $M_{(H)}$ are the natural objects and M_H appear as technical tools. For symplectic geometry, M_H is natural and there is no reason to work with $M_{(H)}$ that has no structure.

Issues 2-4 are related to bifurcation problems.

POISSON REDUCTION

Fernandes-Ortega-Ratiu [2009]

$\Phi : G \times P \to P$ proper Poisson action, $H \subset G$ an isotropy subgroup, $N(H)$ its normalizer.

P_H is a Lie-Dirac submanifold of P with Poisson bracket given by

$$\{f, h\}_{M_H} := \{\overline{f}, \overline{h}\}|_{M_H}, \quad \forall f, h \in C^\infty(M_H)$$

where $\overline{f}, \overline{h} \in C^\infty(M)^H$ are arbitrary H-invariant extensions of f, h.
Natural action of $L(H) := N(H)/H$ on P_H is proper free Poisson so $M_H/L(H)$ is Poisson: char. projection $M_H \to M_H/L(H)$ is Poisson.

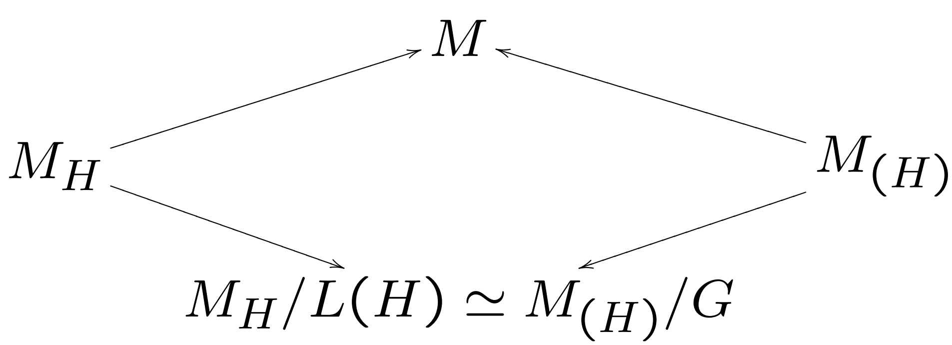
$N \subset P$ is **Lie-Dirac** if $\exists E \subset TM|_N$ vector subbundle such that E° is a Lie subalgebroid of $T^*M \iff$ the vector subbundle E is a coisotropic submanifold of the tangent Poisson manifold TP.

$M = \bigsqcup M_{(H)}$, $M/G = \bigsqcup M_{(H)}/G$, $\pi : M \to M/G$.
$C^\infty(M/G) := \{f \in C^0(M/G) \mid f \circ \pi \in C^\infty(M)^G\}$

X a topological space. A **Poisson stratification** of X is a stratification $\mathcal{S} = \{S_i\}_{i \in I}$ into Poisson manifolds S_i with a Poisson algebra $(C^\infty(P), \{\,,\}_X)$ such that all inclusions $s_i : S_i \hookrightarrow X$ are Poisson: $\{f, h\}_X \circ s_i = \{f \circ s_i, h \circ s_i\}_{S_i}$, $\forall f, h \in C^\infty(X)$, $i \in I$.

$\Phi : G \times P \to P$ proper Poisson action, H an isotropy subgroup. Then $M_H/L(H) \to M_{(H)}/G$ (the quotient of the inclusion) is a diffeomorphism thereby endowing $M_{(H)}/G$ with a Poisson structure. If $H_1, H_2 \in (H)$, the Poisson structures on $M_{(H)}/G$ induced by $M_{H_1}/L(H_1)$ and $M_{H_2}/L(H_2)$ coincide. Can characterize Poisson structure on $M_{(H)}/G$ in terms of Dirac geometry.

Diagram of Dirac manifolds (relevant structures are Dirac)



inclusions are backward and projections are forward Dirac maps.

SINGULAR POINT REDUCTION

Sjamaar[1990], SjLe[1991], BaLe[1997], OrRa[2004]

$\mathbf{J} : (M, \omega) \to \mathfrak{g}^*$, M connected or $\mathbf{J}$ equivariant,

$\sigma(g) := \mathbf{J}(g \cdot z) - \mathrm{Ad}^*_{g^{-1}} \mathbf{J}(z)$, $\quad \sigma : G \to \mathfrak{g}^*$, Souriau one-cocycle

$\Theta(g, \nu) := \mathrm{Ad}^*_{g^{-1}} \nu + \sigma(g)$, affine action

M_H^m connected component of M_H containing m,

$H := G_m$, $\quad \mathbf{J}(m) = \mu$, $\quad G_\mu$ is the Θ-isotropy at μ

(i) $\mathbf{J}^{-1}(\mu) \cap (G_\mu \cdot M_H^m)$ is embedded in M.
(ii) $M_\mu^{(H)} := [\mathbf{J}^{-1}(\mu) \cap (G_\mu \cdot M_H^m)]/G_\mu$ has a unique quotient manifold structure such that

$$\pi_\mu^{(H)} : \mathbf{J}^{-1}(\mu) \cap (G_\mu \cdot M_H^m) \longrightarrow M_\mu^{(H)}$$

is a surjective submersion.

(iii) $\exists! \omega_\mu^{(H)} \in \Omega^2(M_\mu^{(H)})$ symplectic, characterized by

$$\iota_\mu^{(H)\,*}\omega = \pi_\mu^{(H)\,*}\omega_\mu^{(H)},$$

$\iota_\mu^{(H)} : \mathbf{J}^{-1}(\mu) \cap (G_\mu \cdot M_H^m) \hookrightarrow M$ inclusion. $(M_\mu^{(H)}, \omega_\mu^{(H)})$ are the **singular symplectic point strata**.

(iv) $h \in C^\infty(M)^G$. Flow F_t of X_h leaves the connected components of $\mathbf{J}^{-1}(\mu) \cap (G_\mu \cdot M_H^m)$ invariant and commutes with the G_μ-action, so it induces flow F_t^μ on $M_\mu^{(H)}$ characterized by

$$\pi_\mu^{(H)} \circ F_t \circ \iota_\mu^{(H)} = F_t^\mu \circ \pi_\mu^{(H)}.$$

(v) F_t^μ is the flow of $X_{h_\mu^{(H)}} \in \mathfrak{X}(M_\mu^{(H)}, \omega_\mu^{(H)})$ for the **reduced Hamiltonian** $h_\mu^{(H)} : M_\mu^{(H)} \to \mathbb{R}$ given by $h_\mu^{(H)} \circ \pi_\mu^{(H)} = h \circ \iota_\mu^{(H)}$.
X_h and $X_{h_\mu^{(H)}}$ are $\pi_\mu^{(H)}$-related.

(vi) $h, k \in C^\infty(M)^G \Rightarrow \{h, k\} \in C^\infty(M)^G$, $\{h, k\}_\mu^{(H)} = \{h_\mu^{(H)}, k_\mu^{(H)}\}_{M_\mu^{(H)}}$, where $\{\cdot, \cdot\}_{M_\mu^{(H)}}$ is the Poisson bracket induced by the symplectic structure on $M_\mu^{(H)}$.

Sjamaar point reduction principle

GOAL: Realize the strata as usual reduced spaces.

• Remember that $m \in M$ is fixed, $H := G_m$, $\mu := \mathbf{J}(m)$.

• $N(H)^m := \{n \in N(H) \mid n \cdot M_H^m \subset M_H^m\}$.

$N(H)^m$ is open hence closed in $N(H)$. Also $H \subset N(H)^m$. Thus $\mathrm{Lie}(N(H)^m/H) = \mathrm{Lie}(N(H)/H) =: \mathfrak{l}$

• $L^m := N(H)^m/H$ acts freely properly and symplectically on M_H^m with momentum map

$$\mathbf{J}_{L^m} : M_H^m \ni z \longmapsto \Lambda(\mathbf{J}|_{M_H^m}(z) - \mu) \in (\mathrm{Lie}(L^m))^*$$

• $\Lambda : (\mathfrak{g}_m^\circ)^H \to (\mathrm{Lie}(L^m))^*$, L^m-equivariant isomorphism

$$\left\langle \Lambda(\beta), \left.\frac{d}{dt}\right|_{t=0} (\exp t\xi)H \right\rangle = \langle \beta, \xi \rangle,$$

$\beta \in (\mathfrak{g}_m^\circ)^H$, $\xi \in \mathrm{Lie}(N(H)^m) = \mathrm{Lie}(N(H))$

• $\mathfrak{g}_m^\circ$ denotes the annihilator of $\mathfrak{g}_m$ in $\mathfrak{g}^*$

• $(\mathfrak{g}_m^\circ)^H$ are the H-fixed points in $\mathfrak{g}_m^\circ$

• Non-equivariance 1-cocycle τ of $\mathbf{J}_{L^m}$: $\forall l = nH \in L^m$, $\forall n \in N(H)^m$

$$\tau : L^m \ni l \longmapsto \Lambda(\sigma(n) + n \cdot \mu - \mu) \in (\mathrm{Lie}(L^m))^*.$$

(i) $\pi_\mu^{(H)} : \mathbf{J}^{-1}(\mu) \cap (G_\mu \cdot M_H^m) \to M_\mu^{(H)} := [\mathbf{J}^{-1}(\mu) \cap (G_\mu \cdot M_H^m)]/G_\mu$ smooth bundle with fiber G_μ/H and structure group $N_{G_\mu}(H)^m/H$.

(ii) $(M_H^m)_0 := \mathbf{J}_{L^m}^{-1}(0)/L_0^m = [\mathbf{J}^{-1}(\mu) \cap M_H^m]/(N_{G_\mu}(H)^m/H)$; $L_0^m \neq L^m$, in general since the action is affine.

(iii) $\pi_0 : \mathbf{J}_{L^m}^{-1}(0) \to (M_H^m)_0$ is a principal L_0^m-bundle. G_μ/H is a right $(N_{G_\mu}(H)^m/H)$-space and $\mathbf{J}^{-1}(\mu) \cap M_H^m$ is a left $(N_{G_\mu}(H)^m/H)$-space. Associated bundle with fiber G_μ/H

$$G_\mu/H \times_{N_{G_\mu}(H)^m/H} \left(\mathbf{J}^{-1}(\mu) \cap M_H^m\right) \longrightarrow [\mathbf{J}^{-1}(\mu) \cap M_H^m]/(N_{G_\mu}(H)^m/H).$$

is G_μ-symplectomorphic to

$$\pi_\mu^{(H)} : \mathbf{J}^{-1}(\mu) \cap (G_\mu \cdot M_H^m) \longrightarrow M_\mu^{(H)}, \qquad \text{i.e.,}$$

- $G_\mu/H \times_{N_{G_\mu}(H)^m/H} \left(\mathbf{J}^{-1}(\mu) \cap M_H^m\right) \cong_{G_\mu} \mathbf{J}^{-1}(\mu) \cap (G_\mu \cdot M_H^m)$ diffeo.
- $(M_H^m)_0 = \mathbf{J}_{L^m}^{-1}(0)/L_0^m = (\mathbf{J}^{-1}(\mu) \cap M_H^m)/(N_{G_\mu}(H)^m/H)$ is symplectomorphic to $M_\mu^{(H)}$.

Stratification theorems

$\{\mathbf{J}^{-1}(\mu) \cap (G_\mu \cdot M_H^m) \mid \mathbf{J}(m) = \mu\}$ Whitney (B) stratification of $\mathbf{J}^{-1}(\mu)$.

$\{M_\mu^{(H)} \mid (H)\}$ is a symplectic Whitney (B) stratification of the cone space $M_\mu := \mathbf{J}^{-1}(\mu)/G_\mu$.

Each connected component of M_μ contains a unique open stratum that is connected, open, and dense in the connected component of M_μ that contains it.

SINGULAR ORBIT REDUCTION

$(M, \omega, G, \mathbf{J} : M \to \mathfrak{g}^*)$, M connected or $\mathbf{J}$ equivariant
affine action Θ associated to group one-cocyle $\sigma : G \to \mathfrak{g}^*$
induced $\Sigma \in Z^2(\mathfrak{g}; \mathbb{R})$

$$\Sigma(\xi, \eta) := \mathbf{J}^{[\xi,\eta]}(z) - \{\mathbf{J}^\xi, \mathbf{J}^\eta\}(z), \quad z \in M$$

Affine Lie-Poisson structure on $\mathfrak{g}^*$

$$\{f, g\}^{\Sigma}_{\pm}(\mu) := \pm \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}\right]\right\rangle \mp \Sigma\left(\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}\right)$$

Orbit symplectic form

$$\omega^{\pm}_{\mathcal{O}_\mu}(\nu)(\xi_{\mathfrak{g}^*}(\nu), \eta_{\mathfrak{g}^*}(\nu)) = \pm\langle \nu, [\xi, \eta]\rangle \mp \Sigma(\xi, \eta),$$

where $\xi_{\mathfrak{g}^*}(\nu) := -\mathrm{ad}^*_\xi \nu + \Sigma(\xi, \cdot)$.

What topology should we put on $\mathbf{J}^{-1}(\mathcal{O}_\mu)$? For $\mathbf{J}^{-1}(\mu)$ we took the induced topology. Wrong now! We shall take the initial topology induced by the map $\mathbf{J}_{\mathcal{O}_\mu} := \mathbf{J}|_{\mathbf{J}^{-1}(\mathcal{O}_\mu)} : \mathbf{J}^{-1}(\mathcal{O}_\mu) \to \mathcal{O}_\mu$, where $\mathcal{O}_\mu \approx G/G_\mu$ comes with its own manifold structure. Then

$$f : G \times_{G_\mu} \mathbf{J}^{-1}(\mu) \ni [g, z] \longmapsto g \cdot z \in \mathbf{J}^{-1}(\mathcal{O}_\mu)$$

is a homeomorphism. Consistent with regular case.

(i) $G \cdot (\mathbf{J}^{-1}(\mu) \cap M_H^m)$ is an initial submanifold of M

$$T_z\left(G \cdot (\mathbf{J}^{-1}(\mu) \cap M_H^m)\right) = \\ \operatorname{span}\{\xi_M(z) + X_f(z) \mid \xi \in \mathfrak{g},\, f \in C^\infty(M)^G\}.$$

(ii) $M_{\mathcal{O}_\mu}^{(H)} := [G \cdot (\mathbf{J}^{-1}(\mu) \cap M_H^m)]/G$ has a unique quotient differentiable structure such that the projection

$$\pi_{\mathcal{O}_\mu}^{(H)} : G \cdot (\mathbf{J}^{-1}(\mu) \cap M_H^m) \longrightarrow M_{\mathcal{O}_\mu}^{(H)}$$

is a surjective submersion.

(iii) $\exists!$ symplectic form $\omega_{\mathcal{O}_\mu}^{(H)} \in \Omega^2(M_{\mathcal{O}_\mu}^{(H)})$ characterized by

$$\iota_{\mathcal{O}_\mu}^{(H)\,*}\omega = \pi_{\mathcal{O}_\mu}^{(H)\,*}\omega_{\mathcal{O}_\mu}^{(H)} + \mathbf{J}_{\mathcal{O}_\mu}^{(H)\,*}\omega_{\mathcal{O}_\mu}^{+},$$

$\iota_{\mathcal{O}_\mu}^{(H)} : G \cdot (\mathbf{J}^{-1}(\mu) \cap M_H^m) \hookrightarrow M$ inclusion, $\mathbf{J}_{\mathcal{O}_\mu}^{(H)} : G \cdot (\mathbf{J}^{-1}(\mu) \cap M_H^m) \to \mathcal{O}_\mu$ is obtained by restriction of the momentum map

(iv) $h \in C^\infty(M)^G$. Flow F_t of X_h leaves connected components of $G \cdot (\mathbf{J}^{-1}(\mu) \cap M_H^m)$ invariant and commutes with the G-action, so it induces a flow $F_t^{\mathcal{O}_\mu}$ on $M_{\mathcal{O}_\mu}^{(H)}$ given by $\pi_{\mathcal{O}_\mu}^{(H)} \circ F_t \circ \iota_{\mathcal{O}_\mu}^{(H)} = F_t^{\mathcal{O}_\mu} \circ \pi_{\mathcal{O}_\mu}^{(H)}$.

(v) $F_t^{\mathcal{O}_\mu}$ is the flow of $X_{h_{\mathcal{O}_\mu}^{(H)}} \in \mathfrak{X}(M_{\mathcal{O}_\mu}^{(H)}, \omega_{\mathcal{O}_\mu}^{(H)})$ relative to the **reduced Hamiltonian** $h_{\mathcal{O}_\mu}^{(H)} : M_{\mathcal{O}_\mu}^{(H)} \to \mathbb{R}$ defined by $h_{\mathcal{O}_\mu}^{(H)} \circ \pi_{\mathcal{O}_\mu}^{(H)} = h \circ \iota_{\mathcal{O}_\mu}^{(H)}$.

X_h and $X_{h_{\mathcal{O}_\mu}^{(H)}}$ are $\pi_{\mathcal{O}_\mu}^{(H)}$-related.

(vi) $h, k \in C^\infty(M)^G \Rightarrow \{h, k\} \in C^\infty(M)^G$, $\{h, k\}_{\mathcal{O}_\mu}^{(H)} = \{h_{\mathcal{O}_\mu}^{(H)}, k_{\mathcal{O}_\mu}^{(H)}\}_{M_{\mathcal{O}_\mu}^{(H)}}$,

$\{\cdot, \cdot\}_{M_{\mathcal{O}_\mu}^{(H)}}$ Poisson bracket induced by the symplectic form on $M_{\mathcal{O}_\mu}^{(H)}$.

Sjamaar orbit reduction principle

GOAL: Realize the strata as usual reduced spaces.

(i) $$\pi_{\mathcal{O}_\mu}^{(H)} : G \cdot (\mathbf{J}^{-1}(\mu) \cap M_H^m) \longrightarrow M_{\mathcal{O}_\mu}^{(H)} = [G \cdot (\mathbf{J}^{-1}(\mu) \cap M_H^m)]/G$$

defines a smooth fiber bundle with fiber G/H and structure group $N(H)^m/H$. Recall that $N(H)^m$ is the open and hence closed subgroup of $N(H)$ that leaves M_H^m invariant (as a set) and $\mathbf{J}(m) = \mu$.

(ii) $L^m := N(H)^m/H$ acts freely properly and canonically on M_H^m. It admits an associated momentum map

$$\mathbf{J}_{L^m} : M_H \ni z \longmapsto \Lambda(\mathbf{J}|_{M_H^m}(z) - \mu) \in \mathfrak{l}^*.$$

The regular orbit reduced space $(M_H^m)_{\mathcal{O}_0}$ at the affine orbit corresponding to $0 \in \mathfrak{l}^*$ is given by

$$\begin{aligned}(M_H^m)_{\mathcal{O}_0} &= \mathbf{J}_{L^m}^{-1}(\mathcal{O}_0)/L^m \\ &= \left[\mathbf{J}^{-1}(N(H)^m \cdot \mu) \cap M_H^m\right] /(N(H)^m/H)\end{aligned}$$

(iii) $\pi_{\mathcal{O}_0} : \mathbf{J}^{-1}_{L^m}(\mathcal{O}_0) \to (M^m_H)_{\mathcal{O}_0}$ is a principal L^m-bundle. G/H is a right $(N(H)^m/H)$-space and $\mathbf{J}^{-1}(N(H)^m \cdot \mu) \cap M^m_H$ is a left $(N(H)^m/H)$-space. The associated bundle

$$G/H \times_{N(H)^m/H} \left(\mathbf{J}^{-1}(N(H)^m \cdot \mu) \cap M^m_H\right) \longrightarrow \left[\mathbf{J}^{-1}(N(H)^m \cdot \mu) \cap M^m_H\right] / (N(H)^m/H).$$

is G-symplectomorphic to

$$\pi^{(H)}_{\mathcal{O}_\mu} : G \cdot (\mathbf{J}^{-1}(\mu) \cap M^m_H) \longrightarrow M^{(H)}_{\mathcal{O}_\mu}, \quad \text{i.e.,}$$

- $G/H \times_{N(H)^m/H} \left(\mathbf{J}^{-1}(N(H)^m \cdot \mu) \cap M^m_H\right)$ is G-diffeomorphic to $G \cdot (\mathbf{J}^{-1}(\mu) \cap M^m_H)$
- $(M^m_H)_{\mathcal{O}_0} = \mathbf{J}^{-1}_{L^m}(\mathcal{O}_0)/L^m =$

$$\left[\mathbf{J}^{-1}(N(H)^m \cdot \mu) \cap M^m_H\right] / (N(H)^m/H)$$

is symplectomorphic to $M^{(H)}_{\mathcal{O}_\mu}$.

Stratification theorems

$l_\mu : \mathbf{J}^{-1}(\mu) \hookrightarrow \mathbf{J}^{-1}(\mathcal{O}_\mu)$ inclusion and $L_\mu : \mathbf{J}^{-1}(\mu)/G_\mu \to \mathbf{J}^{-1}(\mathcal{O}_\mu)/G$ be defined by the commutative diagram

$$\begin{array}{ccc} \mathbf{J}^{-1}(\mu) & \xrightarrow{l_\mu} & \mathbf{J}^{-1}(\mathcal{O}_\mu) \\ {\scriptstyle \pi_\mu}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{O}_\mu}} \\ \mathbf{J}^{-1}(\mu)/G_\mu & \xrightarrow{L_\mu} & \mathbf{J}^{-1}(\mathcal{O}_\mu)/G. \end{array}$$

Consider $\mathbf{J}^{-1}(\mu)/G_\mu$ as a smooth symplectically stratified topological space. Then:

(i) $\left\{ M_{\mathcal{O}_\mu}^{(H)} \mid (H) \right\}$ is a smooth symplectic Whitney (B) stratification on $\mathbf{J}^{-1}(\mathcal{O}_\mu)/G$.

(ii) L_μ is a homeomorphism of smooth symplectic Whitney (B) stratified spaces.

$$
\begin{array}{ccc}
\mathbf{J}^{-1}(\mu) & \xrightarrow{\quad l_\mu \quad} & \mathbf{J}^{-1}(\mathcal{O}_\mu) \\
\Big\downarrow \pi_\mu & & \Big\downarrow \pi_{\mathcal{O}_\mu} \\
\mathbf{J}^{-1}(\mu)/G_\mu & \xrightarrow{\quad L_\mu \quad} & \mathbf{J}^{-1}(\mathcal{O}_\mu)/G \\
\Big\uparrow & & \Big\uparrow \\
\mathbf{J}^{-1}(\mu) \cap (G_\mu \cdot M_H^m)/G_\mu & \xrightarrow{\quad L_\mu^{(H)} \quad} & G \cdot (\mathbf{J}^{-1}(\mu) \cap M_H^m)/G \\
\Big\uparrow f_\mu^{(H)} & & \Big\uparrow f_{\mathcal{O}_\mu}^{(H)} \\
\mathbf{J}_{L^m}^{-1}(0)/L_0^m & \xrightarrow{\quad L_0 \quad} & \mathbf{J}_{L^m}^{-1}(\mathcal{O}_0)/L^m \\
\Big\| & & \Big\| \\
\left[\mathbf{J}^{-1}(\mu) \cap M_H^m\right] / \left(N_{G_\mu}(H)^m/H\right) & & \left[\mathbf{J}^{-1}(N(H)^m \cdot \mu) \cap M_H^m\right] / \left(N(H)^m/H\right)
\end{array}
$$

$l_\mu : \mathbf{J}^{-1}(\mu) \hookrightarrow \mathbf{J}^{-1}(\mathcal{O}_\mu)$ is the inclusion

L_μ is an isomorphism of symplectic cone (hence Whitney (B)) stratified spaces; in particular it is a homeomorphism

$L_0 : \mathbf{J}^{-1}_{L^m}(0)/L^m_0 \to \mathbf{J}^{-1}_{L^m}(\mathcal{O}_0)/L^m$ and
$L^{(H)}_\mu : \mathbf{J}^{-1}(\mu) \cap (G_\mu \cdot M^m_H)/G_\mu \to G \cdot \left(\mathbf{J}^{-1}(\mu) \cap M^m_H\right)/G$
are symplectomorphisms

$L^{(H)}_\mu$ is the restriction of L_μ to the stratum determined by $H := G_m$

$\pi_\mu : \mathbf{J}^{-1}(\mu) \to \mathbf{J}^{-1}(\mu)/G_\mu$ and $\pi_{\mathcal{O}_\mu} : \mathbf{J}^{-1}(\mathcal{O}_\mu) \to \mathbf{J}^{-1}(\mathcal{O}_\mu)/G$
are projections onto quotient manifolds

$f^{(H)}_\mu : \mathbf{J}^{-1}_{L^m}(0)/L^m_0 \to \mathbf{J}^{-1}(\mu) \cap (G_\mu \cdot M^m_H)/G_\mu$ and
$f^{(H)}_{\mathcal{O}_\mu} : \mathbf{J}^{-1}_{L^m}(\mathcal{O}_0)/L^m \to G \cdot \left(\mathbf{J}^{-1}(\mu) \cap M^m_H\right)/G$
are the Sjamaar principle diffeomorphisms

Second pair of upwards pointing arrows are the inclusions of the stratum in the ambient stratified space

COTANGENT BUNDLE REDUCTION

Regular case: G acts freely and properly on Q and by lift on T^*Q.

Problem: Relate the reduced space $(T^*Q)_\mu \cong (T^*Q)_{\mathcal{O}_\mu}$ to $T^*(Q/G_\mu)$ and $T^*(Q/G)$. There is obviously a problem because

$$\begin{aligned}\dim\left(T^*(Q/G_\mu)\right) &= 2\dim Q - 2\dim G_\mu \\ &\geq 2\dim Q - \dim G - \dim G_\mu = \dim\left((T^*Q)_\mu\right) \\ &\geq 2\dim Q - 2\dim G = \dim\left(T^*(Q/G)\right)\end{aligned}$$

So one expects two theorems: an embedding theorem into $T^*(Q/G_\mu)$ and a fibration theorem over $T^*(Q/G)$. In addition, there should be a Poisson theorem giving explicit expressions to the bracket on $(T^*Q)/G$. This is indeed the case and we will discuss these results.

Equivariant momentum map $\mathbf{J} : T^*Q \to \mathfrak{g}^*$

$$\langle \mathbf{J}(\alpha_q), \xi \rangle = \alpha_q(\xi_Q(q)), \quad \alpha_q \in T^*_qQ,\ \xi \in \mathfrak{g}$$

Form reduced manifold $((T^*Q)_\mu, \omega_\mu)$, $(T^*Q)_\mu = \mathbf{J}^{-1}(\mu)/G_\mu$.

Embedding version

- $\pi_{Q,G_\mu} : Q \to Q_\mu := Q/G_\mu$ surjective submersion

- G_μ-momentum map $\mathbf{J}^\mu : T^*Q \to \mathfrak{g}_\mu^*$, $\mathbf{J}^\mu(\alpha_q) = \mathbf{J}(\alpha_q)|_{\mathfrak{g}_\mu}$.

Note: $\mathbf{J}^{-1}(\mu) \subset (\mathbf{J}^\mu)^{-1}(\mu')$ where $\mu' := \mu|_{\mathfrak{g}_\mu} \in \mathfrak{g}_\mu^*$

Since the action is free and proper, μ and μ' are regular values.

μ' is G_μ-invariant but μ is not

Assume there is a G_μ-invariant $\alpha_\mu \in \Omega^1(Q)$ such that:
(H1) $\alpha_\mu(Q) \subset (\mathbf{J}^\mu)^{-1}(\mu')$, or
(H2) $\alpha_\mu(Q) \subset \mathbf{J}^{-1}(\mu)$

$\xi \in \mathfrak{g}_\mu,\ q \in Q \Rightarrow (\mathbf{i}_{\xi_Q}\alpha_\mu)(q) = \langle \mathbf{J}(\alpha_\mu(q)), \xi\rangle \overset{(H1)}{=} \langle \mu', \xi\rangle \Rightarrow \mathbf{i}_{\xi_Q}\alpha_\mu$ is a constant function on Q, so $\mathbf{i}_{\xi_Q}\mathbf{d}\alpha_\mu = \pounds_{\xi_Q}\alpha_\mu - \mathbf{d}\mathbf{i}_{\xi_Q}\alpha_\mu = 0 \Rightarrow \exists! \beta_\mu \in \Omega^2(Q_\mu)$ closed such that $\pi_{Q,G_\mu}^*\beta_\mu = \mathbf{d}\alpha_\mu$.
Let $B_\mu := \pi_{Q_\mu}^*\beta_\mu \in \Omega^2(T^*Q_\mu)$, where $\pi_{Q_\mu} : T^*Q_\mu \to Q_\mu$

(i) If hypothesis **(H1)** holds, then there is a symplectic embedding

$$\varphi_\mu : ((T^*Q)_\mu, \omega_\mu) \to (T^*Q_\mu, \omega_{\text{can}} - B_\mu)$$
$$\left\langle \varphi_\mu([\alpha_q]_\mu), T_q\pi_{Q,G_\mu}(v_q) \right\rangle = \langle \alpha_q - \alpha_\mu(q), v_q \rangle, \quad \forall v_q \in T_qQ$$

onto a submanifold of T^*Q_μ covering the base $Q_\mu = Q/G_\mu$.

(ii) φ_μ is a symplectic diffeomorphism if and only if $\mathfrak{g} = \mathfrak{g}_\mu$.

(iii) If **(H2)** holds, then the image of φ_μ is the vector subbundle $[T\pi_{Q,G_\mu}(V)]^\circ \subset T^*Q_\mu$, where $V \subset TQ$ is the vector subbundle with fibers $V_q = \{\xi_Q(q) \mid \xi \in \mathfrak{g}\}$, and $^\circ$ denotes the annihilator.

(iv) If **(H1)** holds and $\mathcal{A}^\mu \in \Omega^1(Q; \mathfrak{g}_\mu)$ is a connection on the principal G_μ-bundle $\pi_{Q,G_\mu} : Q \to Q/G_\mu$, take $\alpha_\mu := \langle \mu', \mathcal{A}^\mu \rangle \in \Omega^1(Q)$. Then $\mathbf{d}\alpha_\mu = \langle \mu', \mathcal{B}^\mu \rangle$, $\mathcal{B}^\mu \in \Omega^2(Q; \mathfrak{g}_\mu)$ is the curvature of $\mathcal{A}^\mu$.

(iv) If **(H2)** holds and $\mathcal{A} \in \Omega^1(Q; \mathfrak{g})$ is a connection on the principal G-bundle $\pi_{Q,G} : Q \to Q/G$, take $\alpha_\mu := \langle \mu, \mathcal{A} \rangle \in \Omega^1(Q)$. Then $\mathbf{d}\alpha_\mu = \langle \mu, \mathcal{B} \rangle + \langle \mu, [\mathcal{A}, \mathcal{A}] \rangle = \langle \mu, \mathbf{d}\mathcal{A} \rangle$, $\mathcal{B} \in \Omega^2(Q; \mathfrak{g})$ is the curvature of $\mathcal{A}$.

The mechanical connection

$(Q, \langle\langle\, , \rangle\rangle)$ Riemannian manifold. G acts freely properly by isometries on Q. Horizontal bundle $H :=$ metric orthogonal to vertical bundle V; defines the **mechanical connection**. Its connection one-form?

The momentum map can be interpreted as $\mathbf{J} : TQ \to \mathfrak{g}^*$, $\langle \mathbf{J}(v_q), \xi \rangle = \left\langle\left\langle v_q, \xi_Q(q) \right\rangle\right\rangle$, $\forall \xi \in \mathfrak{g}$. $H_q = \{v_q \in T_qQ \mid \mathbf{J}(v_q) = 0\}$.

Locked inertia tensor: $\mathbb{I}(q) : \mathfrak{g} \to \mathfrak{g}^*$, $\langle \mathbb{I}(q)\eta, \zeta \rangle = \left\langle\left\langle \eta_Q(q), \zeta_Q(q) \right\rangle\right\rangle$, $\forall \eta, \zeta \in \mathfrak{g}$. Since the action is free, $\mathbb{I}(q)$ is invertible.

Mechanical connection one-form $\mathcal{A} \in \Omega^1(Q; \mathfrak{g})$ is

$$\mathcal{A}(q)(v_q) = \mathbb{I}(q)^{-1}(\mathbf{J}(v_q)).$$

Then $\alpha_\mu = \langle \mu, \mathcal{A} \rangle$ is characterized by (Smale [1970])

$$\|\alpha_\mu(q)^\sharp\|^2 = \inf\{\|\beta_q^\sharp\|^2 \mid \beta_q \in \mathbf{J}^{-1}(\mu) \cap T_q^*Q\}.$$

Bundle version

Adjoint bundle: $\tilde{\mathfrak{g}} := Q \times_G \mathfrak{g} := (Q \times \mathfrak{g})/G$ for the G-action $(g,(q,\xi)) \in G \times (Q \times \mathfrak{g}) \mapsto (g \cdot q, \mathrm{Ad}_g \xi) \in Q \times \mathfrak{g}$. The vector bundle $\pi_{\tilde{\mathfrak{g}}} : \tilde{\mathfrak{g}} \to Q/G$, $\pi_{\tilde{\mathfrak{g}}}(g,\xi) := \pi_{Q,G}(q) = [q]$ is a **Lie algebra bundle:** that is, its fibers are Lie algebras defining the bracket:

$$[\pi_{\tilde{\mathfrak{g}}}(g,\xi), \pi_{\tilde{\mathfrak{g}}}(g,\eta)] := \pi_{\tilde{\mathfrak{g}}}(g,[\xi,\eta]), \quad \forall g \in G, \ \forall \xi,\eta \in \mathfrak{g}$$

Defines a Lie bracket on every fiber; smooth as a function of $[q]$.

$\pi : Q \to Q/G$ principal G-bundle, $\mathcal{A} \in \Omega^1(Q;\mathfrak{g})$ connection.

Vector bundle isomorphisms over Q/G (CeMaRa[2001])

$$\alpha_{\mathcal{A}} : TQ/G \ni [v_q] \longmapsto T_q\pi(v_q) \oplus [q, \mathcal{A}(q)(v_q)] \in T(Q/G) \oplus \tilde{\mathfrak{g}}$$

$$\left(\alpha_{\mathcal{A}}^{-1}\right)^* : T^*Q/G \ni [\alpha_q] \longmapsto \mathrm{hor}_q^* \alpha_q \oplus [q, \mathbf{J}(\alpha_q)] \quad \in T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$$

$\mathrm{hor}_q : T_{[q]}(Q/G) \to T_qQ$ is the horizontal lift map.

Poisson bracket on $T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$: for $f,h \in C^\infty\left(T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*\right)$

$$\{f,h\}(x,y,\bar{\mu}) = \frac{\partial f}{\partial x}\frac{\partial h}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial h}{\partial x} + \left\langle \bar{\mu}, \widetilde{\mathcal{B}}\left(\frac{\partial f}{\partial y}, \frac{\partial h}{\partial y}\right)\right\rangle + \left\langle \bar{\mu}, \left[\frac{\partial h}{\partial \bar{\mu}}, \frac{\partial f}{\partial \bar{\mu}}\right]\right\rangle$$

- First two terms: symbolic for the canonical bracket on $T^*(Q/G)$
- $\partial f/\partial y \in T(Q/G)$ and $\partial f/\partial \bar{\mu} \in \widetilde{\mathfrak{g}}$ are fiber derivatives
- $\widetilde{\mathcal{B}}$ is the curvature thought of as a $\widetilde{\mathfrak{g}}$-valued two-form on Q/G

MoMaRa[1984], Montgomery[1986], CeMaPeRa[2003]

The leaves are the orbit reduced spaces $(T^*Q)_{\mathcal{O}} = \mathbf{J}^{-1}(\mathcal{O})/G$.

$$(\alpha_{\mathcal{A}}^{-1})^*(\mathbf{J}^{-1}(\mathcal{O})/G) = T^*(Q/G) \times_{Q/G} \widetilde{\mathcal{O}} \cong (V^\circ \times \mathcal{O})/G \longrightarrow T^*(Q/G),$$

associated bundle $\widetilde{\mathcal{O}} = (Q \times \mathcal{O})/G$ for coadjoint action of G on $\mathcal{O}$. The symplectic form is $\omega_{\mathrm{can}} - \beta$, where $\omega_{\mathrm{can}} \in \Omega^2(T^*(Q/G))$ is the canonical symplectic form and $\beta \in \Omega^2(\widetilde{\mathcal{O}})$ is the unique two-form determined by $\pi_{\widetilde{\mathcal{O}}}^*\beta = \mathbf{d}\alpha + \pi_2^*\omega_{\mathcal{O}}^+$, $\pi_{\widetilde{\mathcal{O}}} : Q \times \mathcal{O} \to \widetilde{\mathcal{O}}$, $\pi_2 : V^\circ \times \mathcal{O} \to \mathcal{O}$, $\alpha \in \Omega^1(Q \times \mathcal{O})$ is defined by $\alpha(q,\nu)(v_q, -\operatorname{ad}_\xi^*\nu) := \langle \nu, \mathcal{A}(q)(v_q)\rangle$ and

$$\mathbf{d}\alpha(q,\nu)((u_q, -\operatorname{ad}_\xi^*\nu), (v_q, -\operatorname{ad}_\eta^*\nu))$$
$$= -\langle \nu, [\mathcal{A}(q)(u_q), \eta]\rangle - \langle \nu, [\xi, \mathcal{A}(q)(v_q)]\rangle + \langle \nu, [\xi,\eta]\rangle + \langle \nu, \mathcal{B}(q)(u_q, v_q)\rangle$$

MaPe[2000], Zaalani[1999], CuSn[1999]

Extreme case 1: $Q = G \Rightarrow Q/G =$point $\Rightarrow$

$$\widetilde{\mathcal{O}} = (Q \times \mathcal{O})/G = g \times \mathcal{O})/G = \mathcal{O}$$

and $\omega_{\text{can}} - \beta = \omega_{\bar{\mathcal{O}}}$.

Extreme case 2: G commutative $\Rightarrow$ fibers of $\widetilde{\mathcal{O}}$ collapse and we are left with $T^*(Q/G)$. The symplectic form is magnetic: $\omega_{\text{can}} - \langle \mu, \mathcal{B} \rangle$, where $\mathcal{O} = \{\mu\}$.

Singular cotangent bundle reduction is not a finished subject yet. The results presented below are due to PeRoSo[2007], PeRo2006], Rodriguez[2006]. Complete results are for reduction at 0. The study in complete generality, i.e., reduction at an arbitrary $\mu \in \mathfrak{g}^*$, is in PeRaRo[2011], almost finished.

SINGULAR COTANGENT BUNDLE REDUCTION

- $(T^*Q)_0$ is a stratified symplectic space.

- Q/G has an orbit type stratification

$$Q/G = \bigsqcup_{(K) \in I(G,Q)} Q_{(K)}/G$$

$I(G,Q)$ lattice of isotropy subgroups of G-action on Q

- $\mathbf{J}^{-1}(0)/G$ has an orbit type stratification

$$\mathbf{J}^{-1}(0)/G = \bigsqcup_{(L) \in I(G,\mathbf{J}^{-1}(0))} \left(\mathbf{J}^{-1}(0) \cap (T^*Q)_{(L)} \right)/G$$
$$=: \bigsqcup_{(L) \in I(G,\mathbf{J}^{-1}(0))} (T^*Q)_0^{(L)}$$

$I(G,\mathbf{J}^{-1}(0))$ lattice of isotropy subgroups of G-action on $\mathbf{J}^{-1}(0)$

QUESTION: Is $\mathbf{J}^{-1}(0)/G \to Q/G$ a stratified fibration? NO! The **coisotropic stratification** appears here. Explain it.

Standard notation: $(K) \preceq (H) \Leftrightarrow H$ conjugate to a subgroup of K

- $\mathbf{J}^{-1}(0) \cap T_q^*Q = (\mathfrak{g} \cdot q)^\circ \Rightarrow \mathbf{J}^{-1}(0)|_{Q_{(H)}} \subset (T^*Q)|_{Q_{(H)}}$ vector subbundle. It carries a G-action.

- $\left(\mathbf{J}^{-1}(0)|_{Q_{(H)}}\right)_{(K)} \neq \varnothing$ if and only if $(K) \in I(G, Q)$ and $(H) \preceq (K)$

- $\left(\mathbf{J}^{-1}(0)|_{Q_{(H)}}\right)_{(K)}$ is a fiber bundle over $Q_{(H)}$. Its quotient

$$S_{(H)\preceq(K)} := \left(\mathbf{J}^{-1}(0)|_{Q_{(H)}}\right)_{(K)} \Big/ G = \frac{\mathbf{J}^{-1}(0) \cap (T^*Q)_{(H)} \cap (T^*Q)|_{(K)}}{G}$$

is called a **seam** from H to K if $(H) \prec (K)$ (Schmah[2002]).

- $\mathbf{J}^{-1}(0)/G \to Q/G$ restricts to a fibration $S_{(H)\preceq(K)} \to Q_{(H)}/G$.

- $(L) \in I(G, \mathbf{J}^{-1}(0)) \Longleftrightarrow (L) \in I(G, Q)$.
- Therefore, stratum $(T^*Q)_0^{(L)} \neq \varnothing$ if and only if $(L) \in I(G, Q)$.
- The symplectic stratum $(T^*Q)_0^{(L)}$ admits the following partition:

$$(T^*Q)_0^{(L)} = \bigsqcup_{(H) \preceq (L)} S_{(H) \preceq (L)} = \left(\mathbf{J}_{(L)}^{-1}(0)/G \right) \bigsqcup \bigsqcup_{(H) \prec (L)} S_{(H) \prec (L)}$$

where $S_{(H) \preceq (L)}$ is the seam from H to L and $\mathbf{J}_{(L)} : T^*(Q_{(L)}) \to \mathfrak{g}^*$ is the momentum map for the lifted action of G on $Q_{(L)}$.

- So we get the *finer* stratification

$$\mathbf{J}^{-1}(0)/G = \bigsqcup_{(H) \preceq (L) \in I(G,Q)} S_{(H) \preceq (L)}$$

refining the symplectic stratification: $S_{(H) \preceq (L)} \subseteq (\mathbf{J}^{-1}(0) \cap (T^*Q)_{(H)}/G$.

- $S_{(H) \preceq (L)}$ is a coisotropic submanifold of $(\mathbf{J}^{-1}(0) \cap (T^*Q)_{(L)}/G$.
- $S_{(L) \preceq (L)} = \mathbf{J}_{(L)}^{-1}(0)/G$ is open and dense in $(T^*Q)_0^{(L)}$, projection $(T^*Q)_0^{(L)} \to \overline{Q_{(L)}/G} =: \overline{Q^{(L)}}$ is a stratified surjective submersion.

• $(T^*Q)_0 \to Q/G$ is a stratified surjective submersion relative to the coisotropic stratification on $(T^*Q)_0$ and the usual orbit type stratification on Q/G.

• The reduced symplectic form $\omega_0^{(L)}$ on $(T^*Q)_0^{(L)}$ coincides with the symplectic form on $\mathbf{J}_{(L)}^{-1}(0)/G$ obtained by reduction of $T^*(Q_{(L)})$ and, conversely, this reduced symplectic form on $\mathbf{J}_{(L)}^{-1}(0)/G$ has a unique extension to $(T^*Q)_0^{(L)}$ which is $\omega_0^{(L)}$.

• The open dense stratum $S_{(L)\preceq(L)} = \mathbf{J}_{(L)}^{-1}(0)/G$ is a maximal symplectic submanifold of the symplectic stratum $\left((T^*Q)_0^{(L)}, \omega_0^{(L)}\right)$ and is symplectomorphic to the canonical cotangent bundle $\left(T^*Q^{(L)}, \omega_L\right)$.

ROUGHLY: Each stratum of the singular reduced space $\mathbf{J}^{-1}(0)/G$ is the union of a cotangent bundle with coisotropic seams. So, $\mathbf{J}^{-1}(0)/G$ is a bunch of cotangent bundles glued together by the coisotropic seams.

What happens for $\mathbf{J}^{-1}(\mu)/G_\mu$?

The first question to be answered is: How do the orbit type stratifications on Q relate to those on T^*Q. Since $G_{\alpha_q} \subset G_q$ and $G_{\alpha_q} \neq G_q$, in general, even the isotropy lattices do not coincide.

$L \subset G$ closed subgroup. Then $(L) \in I(G, T^*Q) \iff \exists (H_1), (H_2) \in I(G, Q)$, $(K) \in I(H_2, \mathfrak{h}_2^\circ)$ such that $(H_1) \succeq (H_2)$ and $L = H_1 \cap K$.

The next step is to stratify all concepts in the bundle version of cotangent bundle reduction; e.g., have singular connections, etc. Replace the Sternberg construction

$$\begin{array}{ccc} V^\circ & \longrightarrow & Q \\ \downarrow & & \downarrow \\ T^*(Q/G) & \longrightarrow & Q/G. \end{array}$$

by

$$\begin{array}{ccc} \widetilde{Q} & \longrightarrow & N^*Q \\ \downarrow & & \downarrow \\ \mathbf{J}^{-1}(0)/G & \longrightarrow & Q/G. \end{array}$$

BANACH POISSON MANIFOLDS

$(P, \{\cdot,\cdot\})$ **Poisson manifold:** $C^\infty(P)$ has a Lie algebra bracket $\{\cdot,\cdot\}$ satisfying the Leibniz property in each factor.

Joint work with Anatol Odzijewicz

What happens in infinite dimensions? By the Leibniz property, the value of the Poisson bracket at a given point $p \in P$ depends only on the differentials $\mathbf{d}f(p)$, $\mathbf{d}g(p) \in T_p^*P$ which implies that there is a smooth section ϖ of the vector bundle $\bigwedge^2 T^{**}P$ satisfying

$$\{f, g\} = \varpi(\mathbf{d}f, \mathbf{d}g).$$

So, for each $p \in P$ the map $\varpi_p : T_p^*P \times T_p^*P \to \mathbb{R}$ is a continuous bilinear antisymmetric map that depends smoothly on the base point p. Let $\sharp : T^*P \to T^{**}P$ be the bundle map covering the identity defined by $\sharp_p(\mathbf{d}h(p)) := \varpi(\cdot, \mathbf{d}h)(p)$, that is, $\sharp_p(\mathbf{d}h(p))(\mathbf{d}g(p)) = \{g, h\}(p)$, for any locally defined functions g and h.

$X_f := \varpi(\cdot, \mathbf{d}f) = \sharp(\mathbf{d}f)$, or, as a derivation, $X_f = \{\cdot, f\}$ is a smooth section of $T^{**}P$ and hence is not, in general, a vector field on P. In analogy with the finite dimensional case, we want X_f to be the Hamiltonian vector field defined by the function f. In order to achieve this, we are forced to make the assumption that the Poisson bracket on P satisfies the condition $\sharp(T^*P) \subset TP \subset T^{**}P$.

A **Banach Poisson manifold** is a pair $(P, \{\cdot, \cdot\})$ consisting of a smooth Banach manifold and a bilinear operation $\{\cdot, \cdot\}$ satisfying the following conditions:

(i) $(C^\infty(P), \{\cdot, \cdot\})$ is a Lie algebra;

(ii) $\{\cdot, \cdot\}$ satisfies the Leibniz identity on each factor;

(iii) the vector bundle map $\sharp : T^*P \to T^{**}P$ covering the identity satisfies $\sharp(T^*P) \subset TP$.

By (iii), every $h \in C^\infty(P)$ defines a **Hamiltonian vector field** by

$$X_h[f] := \langle \mathbf{d}f, X_h \rangle = \{f, h\}$$

where f is an arbitrary smooth locally defined function on P.

$\varphi : (P_1, \{\,,\}_1) \to (P_2, \{\,,\}_2)$ is **canonical** or **Poisson** if

$$\varphi^*\{f, g\}_2 = \{\varphi^* f, \varphi^* g\}_1$$

for any two smooth locally defined functions f and g on P_2. Using (iii), this is equivalent to

$$X_f^2 \circ \varphi = T\varphi \circ X_{f\circ\varphi}^1$$

for any smooth locally defined function f on P_2. Therefore, the flow of a Hamiltonian vector field is a Poisson map and Hamilton's equations in Poisson bracket formulation are valid.

Banach symplectic manifolds

If the continuous linear map $v_p \in T_pP \mapsto \omega(p)(v_p, \cdot) \in T_p^*P$ is

- injective, ω is said to be **weak**,
- bijective, ω is said to be **strong**.

Any strong symplectic Banach manifold (P, ω) is a Banach Poisson manifold: for any $f \in C^\infty(P)$ there exists a vector field X_f such that $\mathbf{d}f = \omega(X_f, \cdot)$. The Poisson bracket is defined by $\{f, g\} = \omega(X_f, X_g) = \langle \mathbf{d}f, X_g \rangle$, thus $\sharp \mathbf{d}f = X_f$, so $\sharp(T^*P) \subset TP$.

If (P, ω) is weak, this is not possible. So we generalize here the notion of strong symplectic manifold. In the weak case, a different approach must be taken by restricting the function spaces and working with vector fields that are only densely defined.

Classical Poisson reduction holds for Banach Poisson manifolds (with the usual condition but taking the closure of a non-direct sum).

BANACH LIE-POISSON SPACES

$(\mathfrak{b}, \{\cdot,\cdot\})$ a Poisson manifold with $\mathfrak{b}$ a Banach space and $\mathfrak{b}^* \subset C^\infty(\mathfrak{b})$ is a Banach Lie algebra under the Poisson bracket operation.

$\mathfrak{b}$ is a Banach Lie-Poisson space if and only if $\mathfrak{b}^*$ is a Banach Lie algebra $(\mathfrak{b}^*, [\cdot,\cdot])$ satisfying $\mathrm{ad}_x^* \mathfrak{b} \subset \mathfrak{b} \subset \mathfrak{b}^{**}$ for all $x \in \mathfrak{b}^*$. Moreover, the Poisson bracket of $f, g \in C^\infty(\mathfrak{b})$ is given by

$$\{f, g\}(b) = \langle [\mathbf{D}f(b), \mathbf{D}g(b)], b\rangle, \qquad b \in \mathfrak{b},$$

where $\mathbf{D}$ is the Fréchet derivative. If $h \in C^\infty(P)$,

$$X_h(b) = -\,\mathrm{ad}^*_{\mathbf{D}h(b)}\, b.$$

A **morphism** between two Banach Lie-Poisson spaces $\mathfrak{b}_1$ and $\mathfrak{b}_2$ is a continuous linear map $\phi : \mathfrak{b}_1 \to \mathfrak{b}_2$ that preserves the Poisson bracket structure, that is,

$$\{f \circ \phi, g \circ \phi\}_1 = \{f, g\}_2 \circ \phi$$

for any $f, g \in C^\infty(\mathfrak{b}_2)$; ϕ is also called a **linear Poisson map**.

Example: Operator Algebra Brackets. $\mathcal{H}$ complex Hilbert space.

- $\mathfrak{S}(\mathcal{H}) = L^1(\mathcal{H})$, **trace class operators**
- $\mathfrak{H}\mathfrak{S}(\mathcal{H}) = L^2(\mathcal{H})$, **Hilbert-Schmidt operators**
- $\mathfrak{K}(\mathcal{H})$, **compact operators**
- $\mathfrak{B}(\mathcal{H}) = L^\infty(\mathcal{H})$, **bounded operators**

They form involutive Banach algebras. $\mathfrak{S}(\mathcal{H})$, $\mathfrak{H}\mathfrak{S}(\mathcal{H})$, $\mathfrak{K}(\mathcal{H})$ are self adjoint ideals in $\mathfrak{B}(\mathcal{H})$.

$$\mathfrak{S}(\mathcal{H}) \subset \mathfrak{H}\mathfrak{S}(\mathcal{H}) \subset \mathfrak{K}(\mathcal{H}) \subset \mathfrak{B}(\mathcal{H})$$

$$\mathfrak{K}(\mathcal{H})^* \cong \mathfrak{S}(\mathcal{H}), \quad \mathfrak{H}\mathfrak{S}(\mathcal{H})^* \cong \mathfrak{H}\mathfrak{S}(\mathcal{H}), \quad \mathfrak{S}(\mathcal{H})^* \cong \mathfrak{B}(\mathcal{H});$$

the right hand sides are all Banach Lie algebras.

Dualities are implemented by the strongly non-degenerate pairing

$$\langle x, \rho \rangle = \operatorname{trace}(x\rho)$$

where $x \in \mathfrak{S}(\mathcal{H})$, $\rho \in \mathfrak{K}(\mathcal{H})$ for the first isomorphism, $\rho, x \in \mathfrak{H}\mathfrak{S}(\mathcal{H})$ for the second isomorphism, and $x \in \mathfrak{B}(\mathcal{H})$, $\rho \in \mathfrak{S}(\mathcal{H})$ for the third isomorphism.

The Banach spaces $\mathfrak{S}(\mathcal{H})$, $\mathfrak{H}\mathfrak{S}(\mathcal{H})$, and $\mathfrak{K}(\mathcal{H})$ are Banach Lie-Poisson spaces. The Lie-Poisson bracket becomes in this case

$$\{F, H\}(\rho) = \pm \operatorname{trace}([\mathbf{D}F(\rho), \mathbf{D}H(\rho)]\rho)$$

where ρ is an element of $\mathfrak{S}(\mathcal{H})$, $\mathfrak{H}\mathfrak{S}(\mathcal{H})$, or $\mathfrak{K}(\mathcal{H})$, respectively. The bracket $[\mathbf{D}F(\rho), \mathbf{D}H(\rho)]$ denotes the commutator bracket of operators. The Hamiltonian vector field associated to H is given by

$$X_H(\rho) = \pm[\mathbf{D}H(\rho), \rho].$$

Example: Preduals of W^*-algebras. A W^*-algebra is a C^*-algebra $\mathfrak{m}$ with predual Banach space $\mathfrak{m}_*$, i.e., $\mathfrak{m} = (\mathfrak{m}_*)^*$; predual is unique.

Since $\mathfrak{m}^* = (\mathfrak{m}_*)^{**} \Rightarrow \mathfrak{m}_*$ canonically embeds into the Banach space $\mathfrak{m}^*$ dual to $\mathfrak{m}$. Always think of $\mathfrak{m}_*$ as a Banach subspace of $\mathfrak{m}^*$.

A net $\{x_\alpha\}_{\alpha \in A} \subset \mathfrak{m}$ **converges to** $x \in \mathfrak{m}$ **in the** σ**-topology** if $\lim_{\alpha \in A} \langle x_\alpha, b \rangle = \langle x, b \rangle$ for all $b \in \mathfrak{m}_*$. σ-topology is Hausdorff. Alaoglu's theorem states that the unit ball of $\mathfrak{m}$ is compact in the σ-topology.

$\mathfrak{m}_* = \{\alpha \in \mathfrak{m}^* \mid \alpha$ is σ-continuous$\}$ is a Banach Lie-Poisson space.

$\mathfrak{a}$ a C^*-algebra. Then its dual $\mathfrak{a}^*$ is a Banach Lie-Poisson space.

$\mathfrak{m}$ a W^*-algebra, $\mathfrak{m}_*$ its predual. The inclusion $\iota : \mathfrak{m}_* \hookrightarrow \mathfrak{m}^*$ is an injective linear Poisson map and the Poisson structure induced by it from $\mathfrak{m}^*$ coincides with the original Lie-Poisson structure on $\mathfrak{m}_*$.

SYMPLECTIC LEAVES AND COADJOINT ORBITS

A smooth map $f : M \to N$ between Banach manifolds is called a:
(i) **immersion** if for every $m \in M$ the tangent map $T_m f : T_m M \to T_{f(m)} N$ is injective with closed split range;
(ii) **quasi immersion** if for every $m \in M$ the tangent map $T_m f : T_m M \to T_{f(m)} N$ is injective with closed range;
(iii) **weak immersion** if for every $m \in M$ the tangent map $T_m f : T_m M \to T_{f(m)} N$ is injective.

$(P, \{\cdot, \cdot\}_P)$ Banach Poisson manifold. **Characteristic subspace at p:** $S_p := \{X_f(p) \mid f \in C^\infty(U), U$ open in P, $p \in U\} \subset T_p P$; not always closed in $T_p P$. **Characteristic distribution:** $S := \cup_{p \in P} S_p \subset TP$, the **characteristic distribution**; not a subbundle of TP, in general. S is always **smooth:** $\forall v_p \in S_p \subset T_p P$, $\exists$ a locally defined smooth vector field (namely some X_f) whose value at p is v_p.

Assume that the characteristic distribution is integrable. For finite dimensional manifolds this is automatic by the Stefan-Sussmann theorem which is not available in infinite dimensions.

Let $\mathcal{L}$ be a **leaf** of the characteristic distribution, that is,

- $\mathcal{L}$ is a connected smooth Banach manifold,
- the inclusion $\iota : \mathcal{L} \hookrightarrow P$ is a weak injective immersion,
- $T_q\iota(T_q\mathcal{L}) = S_q$ for each $q \in \mathcal{L}$,
- if inclusion $\iota' : \mathcal{L}' \hookrightarrow P$ is another weak immersion satisfying the three conditions above and $\mathcal{L} \subset \mathcal{L}' \Rightarrow \mathcal{L}' = \mathcal{L}$, that is, $\mathcal{L}$ is maximal.

If on $\mathcal{L}$ there is a weak symplectic form $\omega_{\mathcal{L}}$ consistent with the Poisson structure on P, then $\mathcal{L}$ is a **symplectic leaf**.

Poisson structure $\sharp_p : T_p^*P \to T_pP$ induces a bijective continuous map $[\sharp_p] : T_p^*P / \ker \sharp_p \to S_p$. Then $\omega_{\mathcal{L}}$ is **consistent** with $\{\,,\}$ if

$$\omega_{\mathcal{L}}(q)(u_q, v_q) = \varpi(\iota(q))\left(([\sharp_{\iota(q)}]^{-1} \circ T_q\iota)(u_q), ([\sharp_{\iota(q)}]^{-1} \circ T_q\iota)(v_q)\right).$$

So, if consistent $\omega_{\mathcal{L}}$ exists, it is unique.

$\dim P < \infty \Rightarrow$ all leaves are symplectic.

$\dim P = \infty$ open, even for Lie-Poisson.

Let G be a Banach Lie group with Lie algebra $\mathfrak{g}$. Assume that:

(i) $\mathfrak{g}$ admits a predual $\mathfrak{g}_*$;

(ii) the coadjoint action of G on the dual $\mathfrak{g}^*$ leaves the predual $\mathfrak{g}_*$ invariant, that is, $\mathrm{Ad}_g^*(\mathfrak{g}_*) \subset \mathfrak{g}_*$, for any $g \in G$;

(iii) if $\rho \in \mathfrak{g}_*$, $G_\rho := \{g \in G \mid \mathrm{Ad}_g^* \rho = \rho\}$, closed in G, is a Lie subgroup of G (a submanifold of G and not just injectively immersed).

Then the Lie algebra of G_ρ equals $\mathfrak{g}_\rho := \{\xi \in \mathfrak{g} \mid \mathrm{ad}_\xi^* \rho = 0\}$ and the quotient topological space $G/G_\rho := \{gG_\rho \mid g \in G\}$ admits a unique smooth Banach manifold structure making the canonical projection $\pi : G \to G/G_\rho$ a surjective submersion. The manifold G/G_ρ is weakly symplectic relative to the two-form $\omega_\rho \in \Omega^2(G/G_\rho)$ given by

$$\omega_\rho([g])(T_g\pi(T_eL_g\xi), T_g\pi(T_eL_g\eta)) := \langle \rho, [\xi, \eta] \rangle ,$$

$\xi, \eta \in \mathfrak{g}$, $g \in G$, $[g] := \pi(g) = gG_\rho$, and $\langle \cdot, \cdot \rangle : \mathfrak{g}_* \times \mathfrak{g} \to \mathbb{R}$ (or $\mathbb{C}$) is the canonical pairing. The two-form ω_ρ is invariant under the left action of G on G/G_ρ given by $g \cdot [h] := [gh]$, for $g, h \in G$.

$$\iota : [g] \in G/G_\rho \mapsto \mathrm{Ad}^*_{g^{-1}}\rho \in \mathfrak{g}_*$$

is an injective weak immersion. Endow the coadjoint orbit $\mathcal{O} := \{\mathrm{Ad}^*_g \rho \mid g \in G\}$ with the smooth manifold structure making ι into a diffeomorphism. Then $\iota_*(\omega_\rho) = \omega_{\mathcal{O}}$ where

$$\omega_{\mathcal{O}}(\mathrm{Ad}^*_{g^{-1}}\rho)\left(\mathrm{ad}^*_{\mathrm{Ad}_g \xi}\mathrm{Ad}^*_{g^{-1}}\rho, \mathrm{ad}^*_{\mathrm{Ad}_g \eta}\mathrm{Ad}^*_{g^{-1}}\rho\right) = \langle \rho, [\xi, \eta] \rangle,$$

$g \in G$, $\xi, \eta \in \mathfrak{g}$, $\rho \in \mathfrak{g}_*$. Connected components of the coadjoint orbit $\mathcal{O}$ are symplectic leaves of the Banach Lie-Poisson space $\mathfrak{g}_*$.

The following conditions are equivalent:
(a) $\iota : G/G_\rho \to \mathfrak{g}_*$ is an injective immersion;
(b) the characteristic subspace $S_\rho := \{\mathrm{ad}^*_\xi \rho \mid \xi \in \mathfrak{g}\}$ is closed in $\mathfrak{g}_*$;
(c) $S_\rho = \mathfrak{g}_\rho^\circ$, where $\mathfrak{g}_\rho^\circ$ is the annihilator of $\mathfrak{g}_\rho$ in $\mathfrak{g}_*$.
Endow the coadjoint orbit $\mathcal{O} := \{\mathrm{Ad}^*_g \rho \mid g \in G\}$ with the manifold structure making ι a diffeomorphism. Then, under any of the hypotheses (a), (b), (c), the weak symplectic form $\omega_{\mathcal{O}}$ is strong.

Examples in operator algebras with D. Beltiţă and B. Tumpach. Conditions for Kähler orbits. Borel-Weil Theorem via reproducing kernel Hilbert spaces; avoids integration in infinite dimensions.

QUANTUM REDUCTION

Coinduced Poisson structure

$(\mathfrak{b}_1, \{\cdot,\cdot\})$ Banach Lie-Poisson space, $\mathfrak{b}_2$ Banach space, $\pi : \mathfrak{b}_1 \to \mathfrak{b}_2$ linear continuous surjective. Then $\mathfrak{b}_2$ carries the Banach Lie-Poisson structure **coinduced by** $\pi \Leftrightarrow$ range $\pi^* \subset \mathfrak{b}_1^*$ is closed under the Lie bracket $[\cdot,\cdot]_1$ of $\mathfrak{b}_1^*$. The map $\pi^* : \mathfrak{b}_2^* \to \mathfrak{b}_1^*$ is a Banach Lie algebra morphism whose dual $\pi^{**} : \mathfrak{b}_1^{**} \to \mathfrak{b}_2^{**}$ maps $\mathfrak{b}_1$ into $\mathfrak{b}_2$.

Quantum mechanics example

$\rho \in L^1(\mathcal{H})$ a mixed state: $\rho^* = \rho \geq 0$ and trace$(\rho) = \|\rho\|_1 = 1$. Apply a measurement operation corresponding to the discrete orthonormal decomposition of the unit

$$P_n P_m = \delta_{nm} P_n, \qquad \sum_{n=1}^{\infty} P_n = 1.$$

By the von Neumann projection postulate, the density operator ρ of the state before measurement is transformed by the measurement to the density operator $R(\rho)$ given by

$$R(\rho) := \sum_{n=1}^{\infty} P_n \rho P_n, \quad R : L^1(\mathcal{H}) \to L^1(\mathcal{H}) \Rightarrow R^* : L^\infty(\mathcal{H}) \to L^\infty(\mathcal{H})$$

(i) R is a continuous norm one projector: $R^2 = R$ and $\|R\| = 1$;
(ii) R preserves space of states: $\rho^* = \rho > 0 \Rightarrow R(\rho)^* = R(\rho^*) = R(\rho) > 0$;
(iii) range R^* is a Banach Lie subalgebra of $L^\infty(\mathcal{H})$.

Hence $R : L^1(\mathcal{H}) \to \text{range}\, R \subset L^1(\mathcal{H})$ is a Poisson projection coinducing a Poisson bracket on range R.

Conditional expectations

In the theory of quantum physical systems (including statistical physics) the W^*-algebra is the algebra of observables and the norm one positive elements of $\mathfrak{m}_* \subset \mathfrak{m}^*$ are the normal states.

A norm one map $E : \mathfrak{m} \to \mathfrak{m}$ which is idempotent ($E^2 = E$) and maps $\mathfrak{m}$ onto a C^*-subalgebra $\mathfrak{n}$ is called **conditional expectation**. If E is σ-continuous then $\mathfrak{n}$ is a W^*-subalgebra of $\mathfrak{m}$. In that case, the adjoint $E^* : \mathfrak{m}^* \to \mathfrak{m}^*$ preserves $\mathfrak{m}_* \subset \mathfrak{m}^*$ and maps $\mathfrak{m}_*$ onto $\mathfrak{n}_*$.

$R := E^*|_{\mathfrak{m}_*} : \mathfrak{m}_* \to \mathfrak{m}_*$ is a continuous projector; $R^* : \mathfrak{m} \to \mathfrak{m}$. Since range $R^* =$ range $E = \mathfrak{n} \Rightarrow$ range R^* Banach Lie subalgebra $(\mathfrak{n}, [\cdot, \cdot])$ of $(\mathfrak{m}, [\cdot, \cdot])$. So, like in the case of the measurement map discussed earlier, R to coinduces a Banach Lie-Poisson structure on range R.

A **quantum reduction map** is a continuous projector $R : \mathfrak{b} \to \mathfrak{b}$ on a Banach Lie-Poisson space $(\mathfrak{b}, \{\cdot, \cdot\})$ such that range R^* is a Banach Lie subalgebra of $\mathfrak{b}^*$.

Hence R coinduces a Lie-Poisson structure on range R; $R : (\mathfrak{b}, \{\cdot, \cdot\}) \to$ (range $R, \{\cdot, \cdot\}_{\text{range}\, R}$) is a Poisson map.

Example 1: $\mathfrak{m}$ a W^*-algebra, $p \in \mathfrak{m}$ a self-adjoint projector. Then

$$\mathfrak{m} \ni x \mapsto P(x) := pxp \in \mathfrak{m}$$

is a uniformly and σ-continuous projector on $\mathfrak{m}$. Let $P^* : \mathfrak{m}^* \to \mathfrak{m}^*$ be the projector dual to P, i.e., $\langle P^* \mu, x \rangle = \langle \mu, Px \rangle$, $\forall \mu \in \mathfrak{m}^*$, $x \in \mathfrak{m}$.

Since P is σ-continuous $\Rightarrow P^*(\mathfrak{m}_*) \subset \mathfrak{m}_*$. Define $P_* := P^*|_{\mathfrak{m}_*}$, projector on $\mathfrak{m}_*$. Then $(P_*)^* = P$ and range P is a W^*-subalgebra of $\mathfrak{m}$. But $\mathrm{ad}_x^* \mathfrak{m}_* \subset \mathfrak{m}_*$ for $x \in \mathfrak{m} \Rightarrow \mathrm{ad}_x \, \mathrm{range}\, P_* \subset \mathrm{range}\, P_*$, $\forall x \in \mathrm{range}\, P$.

(i) $\|P_*\| = 1$
(ii) $[\mathrm{range}(P_*)]^*$ is a Banach Lie algebra;
(iii) $\mathrm{ad}_x \, \mathrm{range}\, P_* \subset \mathrm{range}\, P_*$, $\forall x \in \mathrm{range}\, P$.
Therefore $P_* : \mathfrak{m}_* \to \mathfrak{m}_*$ is a quantum reduction map.

If $\mathfrak{m} = L^\infty(\mathcal{H})$ and $\mathfrak{m}_* = L^1(\mathcal{H})$ the projector $P_* : L^1(\mathcal{H}) \to L^1(\mathcal{H})$ reduces the mixed state ρ of the quantum system to the state $p\rho p = P_*\rho$ localized on the subspace $L^1(p\mathcal{H}) \subset L^1(\mathcal{H})$. In quantum mechanics the projector $p : \mathcal{H} \to \mathcal{H}$ represents the so called elementary observable or "proposition" ("question") which can have only two alternative outcomes: "yes" or "no". The measurement of the "proposition" p reduces the state ρ to the state $P_*\rho$ and the non-negative number trace$(P_*\rho)$ is the probability of the yes-answer. Since P_* is a projector, the repetition of the measurement does not change the state $P_*\rho$. This is the mathematical expression of the von Neumann reproducing postulate (von Neumann[1955]).

Example 2: $\mathfrak{m}$ a W^*-algebra, $\{p_\alpha\}_{\alpha\in I} \subset \mathfrak{m}$ a family of self-adjoint mutually orthogonal projectors (i.e., $p_\alpha p_\beta = \delta_{\alpha\beta} p_\alpha$ and $p_\alpha^* = p_\alpha$) such that $\sum_{\alpha\in I} p_\alpha = 1$; the index set I is not necessary countable. Define the map $R^* : \mathfrak{m} \to \mathfrak{m}$ by

$$R^*(x) := \sum_{\alpha\in I} p_\alpha x p_\alpha, \quad \forall x \in \mathfrak{m},$$

where the summation is taken in the sense of the σ-topology.

The map $R^* : \mathfrak{m} \to \mathfrak{m}$ is a σ-continuous linear projector with $\|R^*\| = 1$. Moreover, range R^* is a W^*-subalgebra of $\mathfrak{m}$ and hence a Banach Lie subalgebra of $(\mathfrak{m}, [\cdot,\cdot])$. Additionally, $\forall x, y, \in \mathfrak{m}$, one has

$$R^*(R^*(x)R^*(y)) = R^*(x)R^*(y) \quad \text{and} \quad R^*(x^*) = (R^*(x))^*. \quad (1)$$

So $(R^*)^* : \mathfrak{m}^* \to \mathfrak{m}^*$ preserves the predual $\mathfrak{m}_* \subset (\mathfrak{m}_*)^{**} = \mathfrak{m}^*$ and hence $R := (R^*)^*|_{\mathfrak{m}_*}$ is a quantum reduction. Note that one has the splitting $\mathfrak{m} = \text{range}\, R^* \oplus \ker R^*$.

$G(\mathfrak{m})$ the group of invertible elements of a W^*-algebra $\mathfrak{m}$; it is an open subset (in the norm topology) of $\mathfrak{m}$ and is therefore a (real or complex) Banach Lie group whose Lie algebra is $\mathfrak{m}$ relative to the commutator bracket $[\cdot, \cdot]$.

Let $R : \mathfrak{m}_* \to \mathfrak{m}_*$ be a quantum reduction that also satisfies $R^*(1) = 1$ and (1). Then $G(\mathfrak{m}) \cap \operatorname{range} R^* = G(\operatorname{range} R^*)$ is a closed Banach Lie subgroup of $G(\mathfrak{m})$ whose Lie algebra is the Banach Lie subalgebra $\operatorname{range} R^*$.

COHERENT STATES QUANTIZATION

P Banach Poisson manifold, $\mathfrak{b}$ Banach Lie-Poisson space. A **coherent states map** of P into $\mathfrak{b}$ is a Poisson embedding $\mathcal{K} : P \to \mathfrak{b}$ with linearly dense range, i.e., $\overline{\operatorname{span} \operatorname{range} \mathcal{K}} = \mathfrak{b}$.

Standard situation: $\dim P < \infty$, $\mathfrak{b} = \mathfrak{h}^1(\mathcal{H})$ is the Banach space of Hermitian trace class operators on a separable complex Hilbert space $\mathcal{H}$, $\mathcal{K}(p)$ is a rank one orthogonal projector for every $p \in P$. Then $\operatorname{range} \mathcal{K} \subset \mathbb{CP}(\mathcal{H})$ by identifying a rank one projector with the point in projective space determined by its image.

How is $\mathcal{K}$ used in the quantization of a physical system? For the Poisson diffeomorphism $\sigma : P \to P$, we assume that there is a linear Poisson automorphism $\Sigma : \mathfrak{h}^1(\mathcal{H}) \to \mathfrak{h}^1(\mathcal{H})$, such that the diagram of canonical maps commutes:

$$\begin{array}{ccc} P & \xrightarrow{\mathcal{K}} & \mathfrak{h}^1(\mathcal{H}) \\ \sigma \downarrow & & \downarrow \Sigma \\ P & \xrightarrow{\mathcal{K}} & \mathfrak{h}^1(\mathcal{H}) \end{array}$$

By a theorem of Wigner, the automorphism Σ is of the form $\Sigma(\rho) = U\rho U^*$, where U is a unitary or anti-unitary operator on $\mathcal{H}$. Because range $\mathcal{K}$ is linearly dense in $\mathfrak{b}$, if such an automorphism Σ exists, it is necessarily unique. Interpret Σ as the quantization of σ.

$\mathrm{Aut}(\mathfrak{h}^1(\mathcal{H}))$ the linear Poisson isomorphisms of $\mathfrak{h}^1(\mathcal{H})$. The set of all Poisson diffeomorphisms σ for which a Σ as above exists, forms a subgroup $\mathrm{Diff}_{\mathcal{K}}(P, \{\cdot, \cdot\})$ of the Poisson diffeomorphism group $\mathrm{Diff}(P, \{\cdot, \cdot\})$ of P. The map

$$\mathcal{E} : \mathrm{Diff}_{\mathcal{K}}(P, \{\cdot, \cdot\}) \to \mathrm{Aut}(\mathfrak{h}^1(\mathcal{H}))$$

so defined, is a group homomorphism, the **Ehrenfest quantization**.

σ_t flow of the Hamiltonian vector field X_h on M and assume that $\sigma_t \in \mathrm{Diff}_{\mathcal{K}}(P, \{\cdot, \cdot\})$ for all t. It is known that the set of all Hamiltonians h satisfying this condition form a Poisson subalgebra of $C^\infty(P)$. Hence, by Wigner

$$\mathcal{E}(\sigma_t)(\rho) = \exp(itH)\rho\exp(-itH)$$

H self-adjoint operator (unbounded, in general) whose domain includes $\operatorname{span}\mathcal{K}(P)(\mathcal{H})$. This correspondence $\mathcal{Q} : h \mapsto H$ is linear and satisfies the relation

$$\mathcal{Q}(\{h_1, h_2\}) = i[\mathcal{Q}(h_1), \mathcal{Q}(h_2)],$$

that is, $\mathcal{Q}$ is the Lie algebra homomorphism induced by $\mathcal{E}$.
$\mathcal{Q}$ is the **coherent states quantization**.

Odzijewicz [1988], [1992], [1994]: precise relationship between the coherent states map quantization and the Kostant-Souriau geometric quantization as well as $*$-product quantization.

BeBe[2010] have formulated a Weyl calculus for pseudo-differential operators on nilpotent groups. "Reduction commutes with $*$-product quantization for abelian and nilpotent groups". Uses the explicit formula of Gutt. Work in progress.