From the Hitchin component to opers

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“Opers versus non-abelian Hodge” arXiv 1607.02172

Joint work with:

Olivia Dumitrescu

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Andy Neitzke

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Initial question at AIM workshop:

What is the precise relationship between quantum curves and the Hitchin moduli space?

Overview

Let C be a compact Riemann surface, genus ≥ 2

$G_{\mathbb{C}}$ be a complex simple Lie group.

$K_C^{1/2}$ be a fixed choice of a square root of $K_C = (\mathcal{T}^*)^{1,0}\, C$.

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Given a **holomorphic quadratic differential**, we can make

- **a Higgs bundle in the Hitchin section** $\subset$ Higgs bundles $(\mathcal{E}, \varphi)$
- **an oper** $\subset$ holomorphic $SL(2, \mathbb{C})$-connections $(\mathcal{E}, \nabla)$

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Explain how these objects are related.

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Outline:

1. An oper
2. A Higgs bundle in the Hitchin section
3. Relation between these two objects
4. Relation between oper and quantum curve?

An oper

Opers (Holomorphic Schrödinger operators)

Fix $\hbar \in \mathbb{C}^\times$.

Locally: $q_2 = p(z)dz^2 \rightsquigarrow$ differential operator, $L_\hbar$

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Globally: To interpret $L_\hbar$ globally,

- View it as

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$$L_\hbar = \partial_z^2 - \tfrac{1}{\hbar^2} p(z) \qquad \nabla_\hbar^{\mathrm{oper}} = d + \frac{1}{\hbar} \begin{pmatrix} 0 & p(z) \\ 1 & 0 \end{pmatrix} dz$$

$$L_\hbar f = 0 \qquad \Leftrightarrow \qquad \nabla_\hbar^{\mathrm{oper}} \begin{pmatrix} -\hbar \partial_z f \\ f \end{pmatrix} = 0$$

for $f \in H^0\left(C, K_C^{-1/2}\right)$

Opers

Oper (Definition)

An $SL(2,\mathbb{C})$-**oper** is a triple $(\mathcal{E}, \nabla, F_\bullet)$ where

- $\mathcal{E}$ is holomorphic vector bundle (rank 2, $\mathrm{Det}(\mathcal{E}) \cong \mathcal{O}$)
- ∇ is a holomorphic flat connection on $\mathcal{E}$,
- with "compatible" filtration $F_\bullet = \{0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 = \mathcal{E}\}$ of $\mathcal{E}$ by holomorphic subbundles.

An oper from $q_2 \in H^0(C, K_C^2)$, $\hbar \in \mathbb{C}^\times$

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  - For $\hbar = 0$, $\mathcal{E}_0 = K_C^{1/2} \oplus K_C^{-1/2}$.
  - For $\hbar \in \mathbb{C}^\times$, $\mathcal{E}_\hbar$ are all isomorphic.

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$(\mathcal{E}_\hbar, \nabla_\hbar^{\mathrm{oper}})$ is a holomorphic $SL(2,\mathbb{C})$-connection.

A Higgs bundle in the Hitchin section

Higgs bundles

Higgs bundles (Definition)

An $SL(2,\mathbb{C})$-Higgs bundle is a pair $(\mathcal{E},\varphi)$ where

- $\mathcal{E}$ is a holomorphic vector bundle (rank 2, $\mathrm{Det}(\mathcal{E})\cong\mathcal{O}$)
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A Higgs bundle from $q_2 \in H^0(C, K_C^2)$

- $\mathcal{E}_0 = K_C^{1/2} \oplus K_C^{-1/2}$
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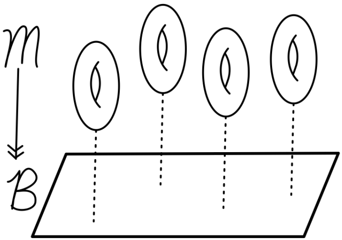
  $\mathbf{1}$ represents the identity map $K_C^{1/2} \to K_C^{-1/2} \otimes K_C \cong K_C^{1/2}$.
  $\mathbf{q_2}$ represents a map $K_C^{-1/2} \to K_C^{1/2} \otimes K_C \cong K_C^{3/2}$.
- Locally, if $q_2 = p(z)\mathrm{d}z^2$, $\varphi = \begin{pmatrix} 0 & p(z) \\ 1 & 0 \end{pmatrix} \mathrm{d}z$

Hitchin integrable system

The moduli space of Higgs bundles is a complex integrable system.



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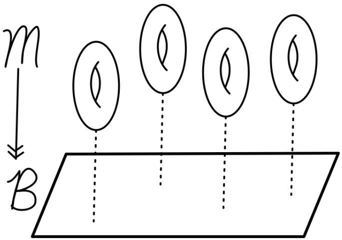

Spectral interpretation of fibration:

- Point of $\mathcal{B}$ is spectral curve, Σ.
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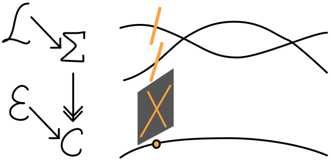

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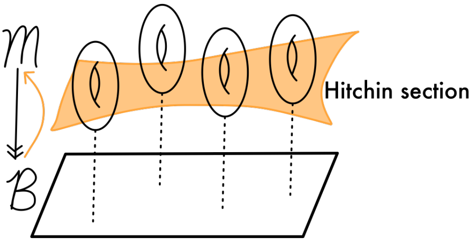

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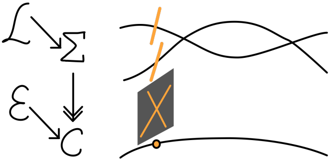

Relation between these two objects

Two objects from $q_2 = p(z)\mathrm{d}z^2$

Recall, there is a $\mathbb{C}$-family of extensions

$$0 \to K_C^{1/2} \to \mathcal{E}_\hbar \to K_C^{-1/2} \to 0.$$

A Higgs bundle in the Hitchin section $(\mathcal{E}, \varphi)$ from $q_2 \in H^0(C, K_C^2)$

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An oper $(\mathcal{E}, \nabla)$ from $q_2 \in H^0(C, K_C^2)$, $\hbar \in \mathbb{C}^\times$.

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Aim: Explain how these two objects are related.

Two avatars of the Hitchin moduli space

Data:	• C, Riemann surface	$\rightsquigarrow \mathcal{M}$,
	• $E \to C$, complex vector bundle	$SL(2, \mathbb{C})$-Hitchin moduli space

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Two avatars of $\mathcal{M}$:

- $\zeta = 0$ Moduli space of stable Higgs bundles $\{(\mathcal{E} = (E, \overline{\partial}_E), \varphi)\}/\sim$
- $\zeta \in \mathbb{C}^\times$ Moduli space of irreducible flat $SL(2,\mathbb{C})$-connections $\{(\mathcal{E} = (E, \nabla^{0,1}), \nabla)\}/\sim$

Two objects from $q_2 = p(z)\mathrm{d}z^2$

A Higgs bundle in the Hitchin section $(\mathcal{E}, \varphi)$ from $q_2 \in H^0(C, K_C^2)$

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Non-abelian Hodge correspondence

$\hbar \in \mathbb{C}^{\times}$

Moduli space of Higgs bundles $\mathcal{M}_{\zeta=0}$:

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$\hbar \in \mathbb{C}^\times, R > 0$

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Moduli space of solutions of Hitchin's equations, $\mathcal{M}$:

Theorem (Hitchin, Simpson)

Given a stable Higgs bundle $(\mathcal{E}, \varphi)$, there is a hermitian metric h on $\mathcal{E}$ solving Hitchin's equations $F_{D(\overline{\partial}_{\mathcal{E}}, h)} + \left[\varphi, \varphi^{\dagger_h}\right] = 0$.

- $(\mathcal{E}_0, R\varphi, h_R)$, solution of Hitchin's equations

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Moduli space of flat connections, $\mathcal{M}_{\zeta\in\mathbb{C}^\times}$:

- $\nabla(R,\zeta) = \zeta^{-1}R\varphi + D_{(\overline{\partial}_{\mathcal{E}}, h_R)} + \zeta R\varphi^{\dagger_{h_R}}$, flat $SL(2,\mathbb{C})$ connection.

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Conjecture (Gaiotto, arXiv:1403.6137)

Fix $\frac{\zeta}{R} = \hbar$. Then $\lim_{R\to 0} \nabla(R, R\hbar)$ is gauge equivalent to $\nabla_\hbar^{\mathrm{oper}}$.

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Theorem (DFKMNN)

Gaiotto's conjecture is true for $G_{\mathbb{C}} = SL(2,\mathbb{C})$.

Moreover, it's true for any simple complex Lie group, $G_{\mathbb{C}}$.

Idea of Proof

Theorem (DFKMMN)

Fix $\frac{\zeta}{R} = \hbar$. Then $\nabla_\hbar^{\mathrm{oper}}$ is gauge equivalent to

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Major difficulty: Understand h_R as $R \to 0$.

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$q_2 = 0$:

$$h_R = \begin{pmatrix} \left(\frac{\lambda}{R}\right)^{-1} & \\ & \left(\frac{\lambda}{R}\right) \end{pmatrix}$$

where $g = \lambda(z)\mathrm{d}z\mathrm{d}\overline{z}$ is the metric of constant curvature -4 on C.

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$$\begin{aligned}
\nabla_\hbar &= \lim_{R\to 0} \frac{1}{\hbar}\varphi + D_{(\overline{\partial}_{\mathcal{E}_0}, h_R)} + R^2\hbar\varphi^{\dagger_{h_R}} \\
&= \frac{1}{\hbar}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d - \partial_z \log\lambda \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\mathrm{d}z + \hbar\begin{pmatrix} 0 & \lambda^2 \\ 0 & 0 \end{pmatrix}\mathrm{d}\overline{z} \\
&= g \circ \left(d + \frac{1}{\hbar}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\mathrm{d}z \right) \circ g^{-1}, \qquad g = \begin{pmatrix} 1 & \hbar\partial_z \log\lambda \\ & 1 \end{pmatrix}.
\end{aligned}$$

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Fix $\frac{\zeta}{R} = \hbar$. Then $\nabla_\hbar^{\mathrm{oper}}$ is gauge equivalent to

$$\lim_{R\to 0} \nabla(R, R\hbar) = \lim_{R=0} \frac{1}{\hbar}\varphi + D_{(\bar\partial_{\mathcal{E}_0}, h_R)} + R^2\hbar\varphi^{\dagger_{h_R}}.$$

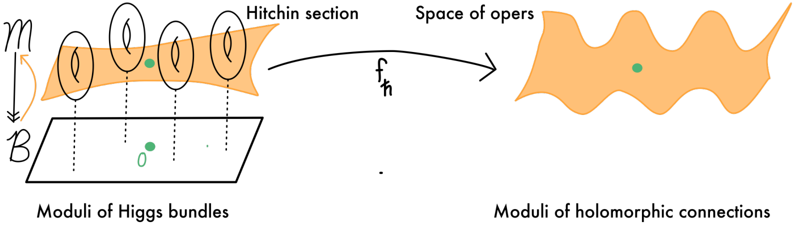
Major difficulty: Understand h_R as $R \to 0$.

$q_2 = p(z)\mathrm{d}z^2 \neq 0$: Expand around known solution. Prove $f_R = O(R^4)$.

$$h_R = \begin{pmatrix} \left(\frac{\lambda}{R} e^{f_R}\right)^{-1} & \\ & \left(\frac{\lambda}{R} e^{f_R}\right) \end{pmatrix}$$

$$\begin{aligned} \nabla_\hbar &= \lim_{R\to 0} \frac{1}{\hbar}\varphi + D_{(\bar\partial_{\mathcal{E}_0}, h_R)} + R^2\hbar\varphi^{\dagger_{h_R}} \\ &= \frac{1}{\hbar}\begin{pmatrix} 0 & p(z) \\ 1 & 0 \end{pmatrix} + d - \partial_z \log\lambda \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \mathrm{d}z + \hbar \begin{pmatrix} 0 & \lambda^2 \\ 0 & 0 \end{pmatrix} \mathrm{d}\bar z \\ &= g \circ \left(d + \frac{1}{\hbar}\begin{pmatrix} 0 & p(z) \\ 1 & 0 \end{pmatrix} \mathrm{d}z \right) \circ g^{-1}, \qquad g = \begin{pmatrix} 1 & \hbar\partial_z \log\lambda \\ & 1 \end{pmatrix}. \end{aligned}$$

Canonical biholomorphic map



Relation to quantum curves?

Quantum curves

Example: Airy

$C = \mathbb{C}P^1$ with holomorphic coordinate z on $\mathbb{C} = \mathbb{C}P^1 - \{\infty\}$.

Spectral curve:

$\Sigma = \{\lambda : \lambda^2 - z\mathrm{d}z^2 = 0\} \subset K_C$ $\qquad$ $\mathrm{genus}(\Sigma) = 0$

Quantum curve: $\lambda \leftrightarrow \hbar \frac{d}{dz}$

$L = \hbar^2 \frac{d}{dz} - z\mathrm{d}^2.$

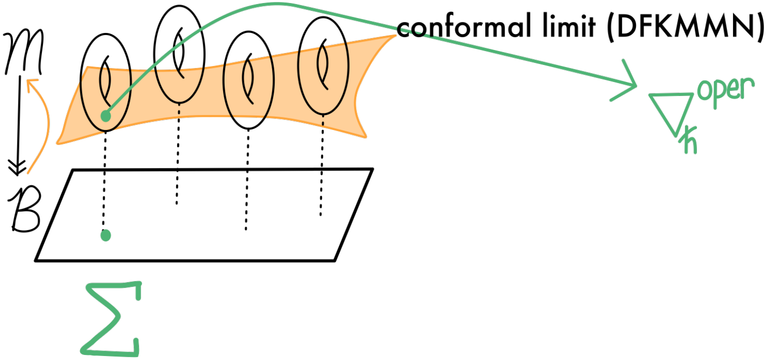
$$\nabla_\hbar^{qc} = d + \begin{pmatrix} & z \\ 1 & \end{pmatrix} \mathrm{d}z$$

Quantum curve from topological recursion:

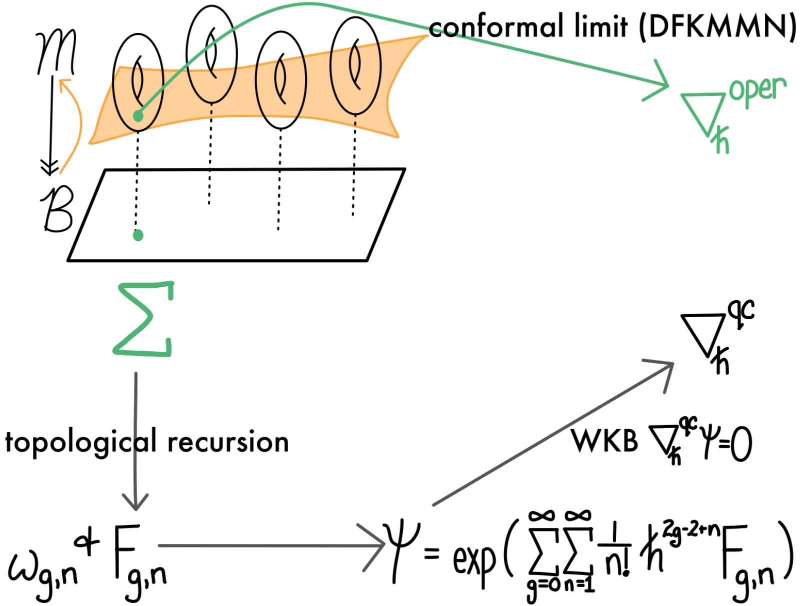
- [Eynard-Orantin] "the topological recursion" (matrix models)
- [Dumitrescu-Mulase] "differential recursion"

Only works when $C = \mathbb{C}P^1$.

Relation between oper and quantum curve?



Relation between oper and quantum curve?



Thank you!