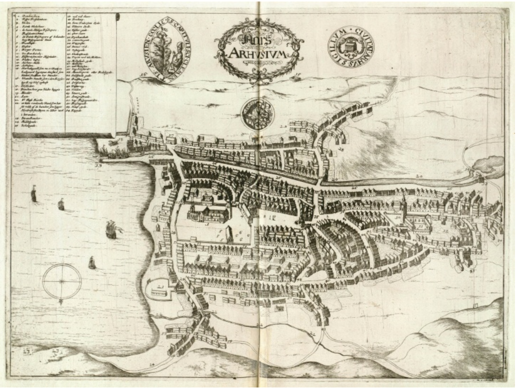
Hitchin 70: Differential Geometry and Quantization

Quaternionic geometry in eight dimensions



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Outline of the talk:

Motivation & background

The special geometry of interest – Sp(2)Sp(1)

- Wolf's symmetric spaces in dim. 8
- Cohomogeneity one approach
- Perturbation theory: flexibility?
- Main result

Outlook: related special geometries

References

Distinguished metrics: parallel vs closed geometries

Many interesting geometries are defined by a differential form Ω (possibly several) with stabiliser $\mathrm{G} \subset \mathrm{SO}(n)$.

Holonomy reduction occurs when

$$\nabla \Omega = 0$$

and often this produces solutions to Einstein's equations.

Obviously, parallelness implies

$$d\Omega = 0$$

but converse is generally false. In such cases, we have natural way of "weakening" holonomy condition.

Question

Can we learn something about, say, privileged metrics by studying such closed (or weakened holonomy) geometries?

Some groups determined by differential forms on $\mathbb{R}^8$

On $\mathbb{R}^8 \cong \mathbb{H}^2$ we have standard hyperKähler triplet of 2-forms

$$\omega_1 = dx^{12} + dx^{34} + dx^{56} + dx^{78}$$
$$\omega_2 = dx^{13} + dx^{42} + dx^{57} + dx^{86}$$
$$\omega_3 = dx^{14} + dx^{23} + dx^{58} + dx^{67}$$

preserved by $\mathrm{Sp}(2) \subset \mathrm{SO}(8)$. From these we can also form family of 4-forms,

$$\Omega_\lambda = \tfrac{1}{2}(\lambda\omega_1^2 + \omega_2^2 + \omega_3^2),$$

with generic stabiliser of dim. 11 ($\supset \mathrm{Sp}(2)\mathrm{U}(1)$) but two notable exceptions:

$$\mathrm{Stab}(\Omega_1) = \mathrm{Sp}(2)\mathrm{Sp}(1) \quad \text{and} \quad \mathrm{Stab}(\Omega_{-1}) = \mathrm{Spin}(7),$$

both maximal in $\mathrm{SO}(8)$.

Holonomy reduction and Einstein metrics

If an 8-manifold M has a *parallel*

- *triplet 2-forms* (pointwise linearly) equivalent to $(\omega_1, \omega_2, \omega_3)$, it is called hyperKähler and has holonomy in $\mathrm{Sp}(2)$;
- *4-form* equivalent to Ω_{-1} it is called a Spin(7)-manifold and has holonomy in $\mathrm{Spin}(7)$;
- *4-form* Ω equivalent to Ω_1 it is called quaternionic Kähler and has holonomy in $\mathrm{Sp}(2)\mathrm{Sp}(1)$.

These groups all appear on Berger's list and, in a sense, represent "fundamental geometries".

First two situations force metric to be Ricci flat. Latter more enigmatic in that metric is Einstein but generally not Ricci flat; positive scalar curvature case proves particularly rigid (indeed, Poon-Salamon showed these spaces are symmetric!)

Deviation from being parallel

Known that $\mathrm{Sp}(2)$and $\mathrm{Spin}(7)$ leave no room for closed geometries (i.e., closed $\Longrightarrow$ parallel).

In $\mathrm{Sp}(2)\mathrm{Sp}(1)$ case, however, Swann characterised the considerable flexibility:

$$\mathrm{Sp}(2)\mathrm{Sp}(1) \circlearrowleft \Lambda^5\mathbb{R}^8 \cong \Lambda^5_8 \oplus \Lambda^5_{16} \oplus \Lambda^5_{32} \ni d\Omega$$

$$\mathrm{Sp}(2)\mathrm{Sp}(1) \circlearrowleft \mathbb{R}^8 \otimes (\mathfrak{sp}(2) \oplus \mathfrak{sp}(1))^\perp \cong \Lambda^5_8 \oplus \Lambda^5_{16} \oplus \Lambda^5_{32} \oplus V_{64} \ni \nabla\Omega$$

(also proved that in dim. $4n \geqslant 12$ closedness of Ω implies quaternionic Kähler, so similar to cases $\mathrm{Sp}(2)$ and $\mathrm{Spin}(7)$)

Closed Sp(2)Sp(1)-structures: local vs. global

Bryant's analysis using EDS:

- a priori overdetermined: $\dim \Lambda^5\mathbb{R}^8 = 56$ first order PDE in $\dim \mathrm{GL}(8,\mathbb{R}) - \dim \mathrm{Sp}(2)\mathrm{Sp}(1) = 51$ unknowns;
- effectively underdetermined: modulo diffeomorphisms, closed $\mathrm{Sp}(2)\mathrm{Sp}(1)$-structures depend on 8 functions of 8 variables.

So locally problem of finding closed $\mathrm{Sp}(2)\mathrm{Sp}(1)$-structures has many solutions!

Question

What about (explicit) examples on compact manifolds?

First attempts

Left-invariant examples exist on nilmanifolds N^8:

- Giovannini & Salamon: on $N = \Gamma\backslash\mathrm{H} \times T^2$ by reducing internal symmetry to $\mathrm{SO}(3) \cong \mathrm{Sp}(2)\mathrm{Sp}(1) \cap \mathrm{SO}(6) \circlearrowleft \mathbb{R}^6$.
- Conti-Madsen: on $N = \Gamma\backslash\mathrm{H} \times S^1$ by reducing internal symmetry to $\mathrm{SO}(4) \cong \mathrm{Sp}(2)\mathrm{Sp}(1) \cap \mathrm{SO}(7) \circlearrowleft \mathbb{R}^7$.

(structure group reductions can be phrased more geometrically in terms of stable forms induced on $\Gamma\backslash\mathrm{H}$)

In above examples N has infinite fundamental group and associated $\mathrm{Sp}(2)\mathrm{Sp}(1)$-metric has negative scalar curvature.

Question (rephrased)

Can we find positive scalar curvature examples on simply-connected manifolds?

(perhaps even on manifolds supporting quaternionic Kähler structure?)

Wolf's positive quaternionic Kähler 4n-manifolds

G compact centreless simple Lie group (e.g., $\mathrm{Sp}(3)/\mathbb{Z}_2$, $\mathrm{SU}(4)/\mathbb{Z}_4$ or G_2).

Pick subalgebra $\mathfrak{sp}(1) \cong \mathfrak{su}(2)$ of $\mathfrak{g}$ coming from highest root. Let

$$\mathfrak{k} = \mathfrak{C}_{\mathfrak{g}}(\mathfrak{sp}(1)) \oplus \mathfrak{sp}(1);$$

$\mathrm{K} = N_{\mathrm{G}}(\mathfrak{sp}(1)) \subseteq \mathrm{Sp}(n)\mathrm{Sp}(1)$ corresponding subgroup of G.
Then we get (irreducible) Riemannian symmetric space

$$\mathrm{G}/\mathrm{K} \quad (e.g., \mathbb{HP}(2),\ \mathrm{Gr}_2(\mathbb{C}^4), \mathrm{G}_2/\mathrm{SO}(4))$$

with compatible positive quaternionic Kähler structure ($s > 0$):

Metric induced by Killing form on $\mathfrak{g}$

Local action of J_1, J_2, J_3 *generated by* $\mathfrak{sp}(1)$

In particular Ω_{eK} can be expressed very explicitly.

Example: $\mathrm{Gr}_2(\mathbb{C}^4)$ in more details

Consider $\mathfrak{su}(4)$ with basis

$$\begin{pmatrix} i &&& \\ & -i && \\ && 0 & \\ &&& 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 && \\ -1 & 0 && \\ && 0 & \\ &&& 0 \end{pmatrix}, \begin{pmatrix} 0 & i && \\ i & 0 && \\ && 0 & \\ &&& 0 \end{pmatrix}, \dots$$

Highest root $\mathfrak{sp}(1)$, generating local action of J_1, J_2, J_3, is spanned by

$$\begin{pmatrix} i &&& \\ & 0 && \\ && 0 & \\ &&& -i \end{pmatrix}, \begin{pmatrix} 0 &&& 1 \\ & 0 && \\ && 0 & \\ -1 &&& 0 \end{pmatrix}, \begin{pmatrix} 0 &&& i \\ & 0 && \\ && 0 & \\ i &&& 0 \end{pmatrix}$$

and its centralizer is determined by

$$\begin{pmatrix} i &&& \\ & -i && \\ && -i & \\ &&& i \end{pmatrix}, \begin{pmatrix} 0 &&& \\ & i && \\ && -i & \\ &&& 0 \end{pmatrix}, \begin{pmatrix} 0 &&& \\ & 0 & 1 & \\ & -1 & 0 & \\ &&& 0 \end{pmatrix}, \begin{pmatrix} 0 &&& \\ & 0 & i & \\ & i & 0 & \\ &&& 0 \end{pmatrix}.$$

Up to overall scaling, an $\mathrm{Sp}(2)\mathrm{Sp}(1)$-adapted frame ("Wolf frame") at the identity coset is then

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}.$$

Cohomogeneous-one SU(3)-action

Note that we have natural inclusions

$$\mathrm{SU}(3) \subset \mathrm{U}(3) \subset \mathrm{Sp}(3), \quad \mathrm{SU}(3) \subset \mathrm{U}(3) \subset \mathrm{SU}(4), \quad \mathrm{SU}(3) \subset \mathrm{G}_2.$$

In particular, we have induced action of $\mathrm{SU}(3)$ on the associated Wolf space

$$\mathbb{HP}(2), \quad \mathrm{Gr}(\mathbb{C}^4), \quad \mathrm{G}_2/\mathrm{SO}(4).$$

Maximal orbits are of codimension $8 - 7 = 1$.

(there are other possible cohom. 1 actions, e.g., for $\mathbb{HP}(2)$ and $\mathrm{Gr}_2(\mathbb{C}^4)$ could consider action of $\mathrm{Sp}(2)$)

Commuting $\mathrm{U}(1)$ generates Killing vector field X on $\mathbb{HP}(2)$ and $\mathrm{Gr}_2(\mathbb{C}^4)$ that will play role later on.

"Missing" commuting $\mathrm{U}(1)$ in third case, indicates this Wolf space is exceptional in more than one sense!

Conventions for SU(3)

Inclusion $\mathrm{SU}(3) \subset \mathrm{G}$, for each of the Wolf spaces, depends on choice.

Example ($\mathrm{Gr}_2(\mathbb{C}^4)$)

Here $\mathrm{G} = \mathrm{SU}(4)$ and we have used the obvious choice

$$\mathrm{SU}(3) \cong \left\{ \begin{pmatrix} A & \\ & 1 \end{pmatrix} : A \in \mathrm{SU}(3) \right\} \subset \mathrm{SU}(4).$$

In any case, we shall always fix a basis of $\mathfrak{su}(3)$ s.t. its dual basis $e^1, \dots, e^8$ satisfies the following structure equations:

$$de^1 = -e^{23} - e^{45} + 2e^{67}, de^2 = e^{13} + e^{46} - e^{57} - \sqrt{3}e^{58},$$
$$de^3 = -e^{12} - e^{47} + \sqrt{3}e^{48} - e^{56}, de^4 = e^{15} - e^{26} + e^{37} - \sqrt{3}e^{38},$$
$$de^5 = -e^{14} + e^{27} + \sqrt{3}e^{28} + e^{36}, de^6 = -2e^{17} + e^{24} - e^{35},$$
$$de^7 = 2e^{16} - e^{25} - e^{34}, de^8 = -\sqrt{3}(e^{25} - e^{34}).$$

Orbit map

For each Wolf space, we have symmetric decomposition with

$$\mathfrak{su}(3) \subset \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

Choosing $Z \in \mathfrak{p} \cap \mathfrak{su}(3)^{\perp}$, $\mathrm{SU}(3)$-orbits of $\gamma(t) = \exp(tZ)$ are

$$\iota_t \colon \mathrm{SU}(3) \to \mathrm{G}/\mathrm{K}, \quad g \mapsto g\gamma(t)\mathrm{K}.$$

Using left translation, we can identify ι_{t*} with the map

$$\mathfrak{su}(3) \to \mathfrak{p} \colon \; X \mapsto [\mathrm{Ad}_{\gamma(t)^{-1}}(X)]_{\mathfrak{p}}.$$

Altogether, cohom. 1 action gives map

$$\mathfrak{su}(3) \oplus \mathbb{R} \to \mathfrak{p} \colon \quad X \mapsto [\mathrm{Ad}_{\gamma(t)^{-1}}(X)]_{\mathfrak{p}}, \frac{\partial}{\partial t} \mapsto Z$$

which can be used to pull back Wolf's frame on $\mathfrak{p}$ to $\mathfrak{su}(3) \oplus \mathbb{R}$ so as to better understand cohomogeneous-one nature of the Wolf spaces.

$\mathbb{HP}(2)$: pulled back Wolf frame on $\mathfrak{su}(3) \oplus \mathbb{R}$

$$\tilde{e}^1 = 4\sqrt{2}\cos(2t)e^6,\ \tilde{e}^2 = -4\sqrt{2}\cos(2t)e^7,$$
$$\tilde{e}^3 = 4\sqrt{2}dt,\ \tilde{e}^4 = \frac{4\sqrt{6}}{3}\sin(2t)e^8,$$
$$\tilde{e}^5 = 4\cos(t)(e^2 + e^4),\ \tilde{e}^6 = 4\cos(t)(e^3 + e^5)$$
$$\tilde{e}^7 = 4\sin(t)(e^2 - e^4),\ \tilde{e}^8 = 4\sin(t)(e^3 - e^5).$$

Upshot:

- Generic stabiliser $\mathrm{U}(1)$ with Lie algebra spanned by e_1
- $t = 0$ singular stabiliser $\mathrm{U}(2)$ with Lie algebra

$$\langle e_1, e_2 - e_4, e_3 - e_5, e_8 \rangle$$

- $t = \pi/4$ singular stabiliser $\mathrm{SU}(2)$ with Lie algebra

$$\langle e_1, e_6, e_7 \rangle.$$

$\mathrm{Gr}_2(\mathbb{C}^4)$: pulled back Wolf frame $\mathfrak{su}(3) \oplus \mathbb{R}$

$$2\sqrt{2}\cos(t)(e^2+e^4),\ 2\sqrt{2}\cos(t)(e^3+e^5),$$
$$4dt,\ -\frac{4\sqrt{3}}{3}\sin(2t)e^8,$$
$$4e^6,\ -4e^7,$$
$$-2\sqrt{2}\sin(t)(e^2-e^4),\ 2\sqrt{2}\sin(t)(e^3-e^5).$$

Upshot:

- Generic stabiliser $\mathrm{U}(1)$ with Lie algebra spanned by e_1
- $t=0$ singular stabiliser $\mathrm{U}(2)$ with Lie algebra

$$\langle e_1, e_2-e_4, e_3-e_5, e_8\rangle$$

- $t=\pi/2$ singular stabiliser $\mathrm{U}(2)$ with Lie algebra

$$\langle e_1, e_2+e_4, e_3+e_5, e_8\rangle.$$

$G_2/SO(4)$: pulled back Wolf frame $\mathfrak{su}(3) \oplus \mathbb{R}$

$$\frac{\sqrt{2}}{2}(\cos(t)^3 - \sin(t)^3)e^2 + \frac{\sqrt{2}}{2}(\cos(t)^3 + \sin(t)^3)e^4,$$
$$-\frac{\sqrt{2}}{2}(\cos(t)^3 - \sin(t)^3)e^3 - \frac{\sqrt{2}}{2}(\cos(t)^3 + \sin(t)^3)e^5,$$
$$-e^6,\ e^7,\ \sqrt{3}dt,\ -\sin(2t)e^8,$$
$$-\sqrt{\tfrac{3}{8}}\sin(2t)(\sin(t) - \cos(t))e^2 - \sqrt{\tfrac{3}{8}}\sin(2t)(\sin(t) + \cos(t))e^4,$$
$$-\sqrt{\tfrac{3}{8}}\sin(2t)(\sin(t) - \cos(t))e^3 - \sqrt{\tfrac{3}{8}}\sin(2t)(\sin(t) + \cos(t))e^5.$$

Upshot:

- Generic stabiliser $U(1)$ with Lie algebra spanned by e_1
- $t = 0$ singular stabiliser $U(2)$ with Lie algebra

$$\langle e_1, e_2 - e_4, e_3 - e_5, e_8 \rangle$$

- $t = \pi/4$ singular stabiliser $SO(3)$ with Lie algebra

$$\langle e_1, e_2, e_3 \rangle.$$

The three basic models

Removing one singular orbit, we are left with vector bundle

$$\mathbb{V} = \mathrm{SU}(3) \times_{\mathrm{H}} V$$

over $\mathrm{SU}(3)/\mathrm{H}$ corresponding to one of 3 models:

H	SU(3)/H	$\mathfrak{su}(3)/\mathfrak{h}$	V
$\mathrm{SU}(2)$	S^5	$\mathbb{R} \oplus \mathbb{H}$	Σ^2
$\mathrm{SO}(3)$	L	$S^2_0(\mathbb{R}^3)$	$\mathbb{R}^3$
$\mathrm{U}(2)$	$\mathbb{CP}(2)$	$[\![\Lambda^{1,0} K]\!]$	$[\![\Lambda^{1,0}]\!]$

(1-dim. det. rep.)

Proposition

The vector bundle $\mathbb{V} = \mathrm{SU}(3) \times_{\mathrm{U}(2)} V$ over $\mathbb{CP}(2)$ admits 3 distinct invariant quaternionic Kähler structures.

By the above

$$\mathbb{HP}(2) \setminus S^5 \cong_{\mathrm{SU}(3)} \mathrm{Gr}_2(\mathbb{C}^4) \setminus \mathbb{CP}(2) \cong_{\mathrm{SU}(3)} \mathrm{G}_2/\mathrm{SO}(4) \setminus \mathrm{L}.$$

The vector bundle therefore admits 3 quaternionic Kähler metrics with different holonomy.

Principal orbits and the Killing field X

On open set, away from singular orbits, each case gives us manifold $\mathrm{SU}(3)/\mathrm{U}(1) \times I$ with tangent space decomposing as

$$\mathrm{U}(1) \circlearrowleft \mathbb{R}^8 \cong 2\mathbb{R} \oplus 2V_1 \oplus V_2.$$

Follows from our $\mathfrak{su}(3)$ structure equations, using $\mathrm{U}(1)$ is generated by e_1.

Right translation on $\mathrm{SU}(3)$ induces action of $\mathrm{U}(1)$ generated by e_8, and associated fundamental vector field corresponds to

$$X = e_8.$$

- This is our Kvf on $\mathbb{HP}(2)$ and $\mathrm{Gr}_2(\mathbb{C}^4)$: $\mathcal{L}_X \Omega_{qK} = 0$.
- In case of $\mathrm{G}_2/\mathrm{SO}(4)$, X satisfies generalisation of Killing condition: $d(\|X\|^2) \wedge \mathcal{L}_X \Omega_{qK} = 0$.

(different nature of X will play key role in the following)

Perturbing p-forms

Let $\alpha \in \Lambda^p(\mathbb{R}^n)^*$ and consider the (affine) "perturbation" by a fixed p-form δ:

$$\beta(\lambda) = \alpha + \lambda\delta, \quad \lambda \in \mathbb{R}.$$

Question

When do α and $\beta(\lambda)$ lie in the same $\mathrm{GL}(n, \mathbb{R})$-orbit for all λ?

Nilpotent perturbations

Proposition

Let $A \in \mathfrak{gl}(n, \mathbb{R})$. If the associated derivation satisfies $A \cdot A \cdot \alpha = 0$ then

$$\beta(\lambda) = \alpha + \lambda A \cdot \alpha$$

lies in the same orbit as α for all $\lambda \in \mathbb{R}$.

This follows from direct computations.

In above case we shall say that β is **nilpotent perturbation** of α.

(One can show that w.l.o.g. A can be taken to be nilpotent)

Perturbations invariant by U(1)

In the quaternionic setting, imposing invariance by group action, things can be described very simply. Concretely, we have already seen tangent space decomp.

$$\mathrm{U}(1) \circlearrowleft \mathbb{R}^8 \cong 2\mathbb{R} \oplus 2V_1 \oplus V_2.$$

Proposition

Let Ω be an invariant quaternionic 4-form on $\mathrm{U}(1) \circlearrowleft \mathbb{R}^8$. Then there is o.n.b. $E_1, \ldots, E_8$ s.t.

$$E_1, E_8 \in 2\mathbb{R},\ 2V_1 = \langle E_2, E_3 \rangle \oplus \langle E_4, E_5 \rangle,\ V_2 = \langle E_6, E_7 \rangle$$

and the space of invariant nilpotent perturbations is generated by the 4-form

$$A \cdot \Omega = E^8 \wedge (E_1 \lrcorner \Omega).$$

Perturbations of the Wolf spaces

Lemma

On each of the three Wolf spaces invariant closed nilpotent perturbations of the quaternionic Kähler structure have the form

$$\widetilde{\Omega} = \Omega_{qK} + dh \wedge (e_8 \lrcorner \Omega),$$

where h is a smooth $\mathrm{SU}(3)$-invariant function.

Away from singular orbits, using (generalised) Killing condition and our characterisation of invariant nilpotent perturbations, we find that closed nilpotent perturbations have the form $f(t)dt \otimes e_8$.

Analysing when $f(t)dt \otimes e_8$ extends to each singular orbit, we find that this precisely amounts to condition that a primitive h of $f(t)dt$ is a smooth $\mathrm{SU}(3)$-invariant function.

Main result

Theorem

The exceptional Wolf space $\mathrm{G}_2/\mathrm{SO}(4)$ *admits* $\mathrm{SU}(3)$*-invariant non-Einstein positive closed* $\mathrm{Sp}(2)\mathrm{Sp}(1)$*-structures.*

By previous Lemma any smooth $\mathrm{SU}(3)$-invariant function h defines a closed perturbation via $dh \otimes e_8 = f(t)dt \otimes e_8$. To verify that we get non-Einstein examples, we compute the Ricci tensor associated with $\widetilde{\Omega}$ which equals

$$\begin{pmatrix}
8-\frac{1}{3}t(2t)^2f(t)^2 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{6}\frac{\sqrt{3}f(t)^2(3+c(4t))}{c(2t)^2} & -\frac{1}{3}t(2t)f'(t)-4f(t) \\
0 & 8-\frac{1}{3}t(2t)^2f(t)^2 & 0 & 0 & 0 & 0 & -\frac{1}{3}t(2t)f'(t)-4f(t) & \frac{1}{6}\frac{\sqrt{3}f(t)^2(3+c(4t))}{c(2t)^2} \\
0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 8-\frac{4}{3}t(2t)^2f(t)^2 & \frac{4}{3}t(2t)^2\sqrt{3}f(t) & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{4}{3}t(2t)^2\sqrt{3}f(t) & 8 & 0 & 0 \\
-\frac{1}{6}\frac{\sqrt{3}f(t)^2(3+c(4t))}{c(2t)^2} & -\frac{1}{3}t(2t)f'(t)-4f(t) & 0 & 0 & 0 & 0 & 8+\frac{1}{3}t(2t)^2f(t)^2 & 0 \\
-\frac{1}{3}t(2t)f'(t)-4f(t) & \frac{1}{6}\frac{\sqrt{3}f(t)^2(3+c(4t))}{c(2t)^2} & 0 & 0 & 0 & 0 & 0 & 8+\frac{1}{3}t(2t)^2f(t)^2
\end{pmatrix}$$

where $t(\cdot) = \tan(\cdot)$ and $c(\cdot) = \cos(\cdot)$. Finally, note that scalar curvature is

$$s = -\frac{4}{3}\tan(2t)^2 f(t)^2 + 64$$

s.t. can get $s > 0$ (non-constant) by choosing h suitably.

Remarks on main result

- Our $\mathrm{SU}(3)$-invariant functions on $\mathrm{G}_2/\mathrm{SO}(4)$ correspond to smooth even $\pi/2$-periodic functions: possible to have h real-analytic.
- Restriction of closed 4-form to (singular orbit) $\mathbb{CP}^2$, a quaternionic submanifold, determines its cohomology class. As both Ω_{qK} and $\widetilde{\Omega}$ restrict to standard volume form, we have

$$[\Omega_{qK}] = [\widetilde{\Omega}] \in H^4(\mathrm{G}_2/\mathrm{SO}(4)).$$

Perturbing $\mathbb{HP}(2)$ and $\mathrm{Gr}_2(\mathbb{C}^4)$: rigidity

In these cases, the fact that $X = e_8$ is Kvf implies that closed perturbed $\mathrm{Sp}(2)\mathrm{Sp}(1)$-structures are related to the Wolf structure by $\mathrm{SU}(3)$-equivariant isometry that corresponds to replacing e^8 by $e^8 + h'(t)dt$.

Upshot: perturbations just lead to other ways of expressing Wolf space structures.

PSU(3)-structures

In my discussion of irreducible symmetric spaces with cohomogeneous-one $\mathrm{SU}(3)$-action, I left out one.

$\mathrm{SU}(3) = \mathrm{SU}(3)^2/\Delta\mathrm{SU}(3)$ is cohom. 1 with respect to consimilarity action:

$$\mathrm{SU}(3) \times \mathrm{SU}(3) \to \mathrm{SU}(3)\colon\ (g, h) \mapsto gh\bar{g}^{-1} = ghg^T$$

and admits a compatible special geometry defined by the harmonic stable 3-form

$$\gamma = \sum_{j=1}^{8} e^j \wedge de^j.$$

As for Wolf spaces, cohom. 1 action gives map

$$\mathfrak{su}(3) \oplus \mathbb{R} \to \mathfrak{su}(3)\colon\ X \mapsto \mathrm{Ad}(\gamma(t)^{-1})(X) + X^T,\ \frac{\partial}{\partial t} \mapsto e_1$$

that can be used to pull back above frame e_j.

SU(3): pulled back adapted frame and perturbations

$$\tilde{e}^1 = dt,\ \tilde{e}^8 = 2e^8$$
$$\tilde{e}^2 = (\cos(t) - 1)e^2 + \sin(t)e^3,\ \tilde{e}^3 = -\sin(t)e^2 + (\cos(t) - 1)e^3$$
$$\tilde{e}^4 = (\cos(t) + 1)e^4 + \sin(t)e^5,\ \tilde{e}^5 = -\sin(t)e^4 + (\cos(t) + 1)e^5$$
$$\tilde{e}^6 = (\cos(2t) + 1)e^6 - \sin(2t)e^7,\ \tilde{e}^7 = \sin(2t)e^6 + (\cos(2t) + 1)e^7.$$

- Generic stabiliser $\mathrm{U}(1)$ with Lie algebra spanned by e_1
- $t = 0$ singular stabiliser $\mathrm{SO}(3)$ with Lie algebra

$$\langle e_1, e_2, e_3 \rangle$$

- $t = \pi/2$ singular stabiliser $\mathrm{SU}(2)$ with Lie algebra

$$\langle e_1, e_6, e_7 \rangle.$$

$\mathrm{U}(1)$-invariant nilpotent perturbations do not produce new harmonic $\mathrm{PSU}(3)$-forms.

Cohomogeneous-one Spin(7)-manifolds

Recall 4-form Ω_{-1} with stabiliser $\mathrm{Spin}(7)$, briefly mentioned earlier on. Complete $\mathrm{Spin}(7)$-manifolds (obtainable via *Hitchin flow*) are known to exist on the two models

$$\mathrm{SU}(3) \times_{\mathrm{U}(2)} \mathbb{C}^2 \quad \text{and} \quad \mathrm{SU}(3) \times_{\mathrm{SU}(2)} \Sigma^2,$$

and from that point of view fit into our analysis.

Question

What about the third model $\mathrm{SU}(3) \times_{\mathrm{SO}(3)} \mathbb{R}^3$?

Proposition

There exists no globally defined invariant $\mathrm{Spin}(7)$-form (parallel or not) on the vector bundle $\mathbb{V} = \mathrm{SU}(3) \times_{\mathrm{SO}(3)} \mathbb{R}^3$.

This follows by writing down ("dictionary" of) invariant 4-forms on the above bundle and understanding what are the possible stabilisers at zero section.

Quotient constructions and G_2-holonomy metrics

There are connections between our work and that of:

- Atiyah-Witten on *M-theory dynamics on a G_2-manifold*.
- Gambioli-Nagatomo-Salamon on $\mathrm{U}(1)$-quotients of $\mathbb{HP}(2)$ and $\mathrm{Gr}_2(\mathbb{C}^4)$.

Taking $\mathrm{U}(1)$ generated by Kvf $X = e_8$ latter fits into our framework and can be verified via our methods.

Example

Have good description of $\mathbb{HP}(2)/\mathrm{U}(1) \cong_{\mathrm{SU}(3)} S^7$ and therefore of $S^7 \setminus \mathbb{CP}(2) \cong_{\mathrm{SU}(3)} \Lambda^2_-(\mathbb{CP}(2))$.

In particular, we can directly relate Wolf structure on $\mathbb{HP}(2)$ to Bryant-Salamon G_2-structure on $\Lambda^2_-(\mathbb{CP}(2))$.

Question

Can these observations of relations between specific special holonomy manifolds be generalised?

Thank you and

HAPPY BIRTHDAY, NIGEL!

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