(Joint work with Nan-Kuo Ho, Khoa Dang Nguyen and Eugene Xia)

The genus 2 moduli space

Lisa Jeffrey

Mathematics Department, University of Toronto

September 6, 2016

1. Character varieties

Let Σ_k be an oriented 2-manifold of genus k. Let G be a compact Lie group. For us, $G = SU(2)$

Definition

The representation variety in genus k is

$$R_k = \{g_1, \ldots, g_k, h_1, \ldots, h_k | \prod_{j=1}^{k}[g_j, h_j] = 1\}$$

The corresponding character variety is

$$M_k = \{g_1, \ldots, g_k, h_1, \ldots, h_k | \prod_{j=1}^{k}[g_j, h_j] = 1\}/G$$

where G acts on R_k by conjugation.

Here

$$[g, h] := ghg^{-1}h^{-1}$$

Example: In genus 2

$$M_2 = \{(g_1, h_1, g_2, h_2) \in G^4 | [g_1, h_1] = [g_2, h_2]^{-1}\}/G$$

Theorem

(Narasimhan-Ramanan 1969)

The character variety M_2 is smooth and is homeomorphic to $\mathbb{C}P^3$.

The proof of this theorem uses algebraic geometry methods rather than symplectic methods.

Suhyoung Choi (2011) gave a new proof of Narasimhan-Ramanan's result using topological techniques rather than methods from symplectic geometry.

Theorem

(Narasimhan-Ramanan 1969)

The character variety M_2 is smooth and is homeomorphic to $\mathbb{C}P^3$.

The proof of this theorem uses algebraic geometry methods rather than symplectic methods.

Suhyoung Choi (2011) gave a new proof of Narasimhan-Ramanan's result using topological techniques rather than methods from symplectic geometry.

Complex projective space $\mathbb{C}P^3$ is a toric manifold (it admits an effective Hamiltonian action of $U(1)^3$). The moment polytope of this action is a tetrahedron.

2. Goldman flows

Goldman (1986) constructed Hamiltonian flows on the character varieties M_k as follows, once a pants decomposition of the surface Σ_k is chosen. Let $C_1, \ldots, C_{3k-3}$ be a collection of disjoint simple closed curves in Σ_k.

2. Goldman flows

Goldman (1986) constructed Hamiltonian flows on the character varieties M_k as follows, once a pants decomposition of the surface Σ_k is chosen. Let $C_1, \ldots, C_{3k-3}$ be a collection of disjoint simple closed curves in Σ_k. Any simple closed curve C gives rise to a collection of functions f_C on R_k defined as follows.

$$f_C(\rho) = \mathrm{Trace}\ \mathrm{Hol}_C(\rho)$$

if $\rho \in R_k$ where $\mathrm{Trace}\ \mathrm{Hol}_C$ is the trace of the holonomy of ρ around the curve C (in other words if $\rho([C])$ is conjugate to the diagonal matrix with eigenvalues $e^{i\theta(C)}, e^{-i\theta(C)}$ then $\mathrm{Trace}\ \mathrm{Hol}_C([C]) = 2\cos(\theta(C))$.)

Goldman constructed the Hamiltonian flows of these functions, as follows. If C is a separating curve in Σ_k, then

$$(\Xi\rho)(t) = \exp(tc) \cdot \rho \cdot \exp(-tc)$$

where $\exp(c)$ is chosen to be in the maximal torus containing $\rho([C]))$. Here $c \in \mathrm{Lie}(T)$ and $t \in \mathbb{R}$.

If C is a nonseparating curve in Σ_k, then

$$(\Xi\rho)(t) = \rho \cdot \exp(tc)$$

($\exp(tc)$ as above).

Another way of saying this is [J-Weitsman 1992]:

If A is a flat connection on Σ with holonomy $\rho(A)$ around C, then $(\Xi A)(t)$ is the image of A under a gauge transformation on $\Sigma \setminus \{C\}$ which is equal to the identity on one boundary component C_+ of $\Sigma \setminus \{C\}$ but equal to an element $\exp(tc)$ on the other boundary component C_- (where c is in the Lie algebra of the maximal torus and exp is the exponential map). Ξ is the Hamiltonian flow of the function $\rho \mapsto \mathrm{Trace}\rho([C])$ (the trace of the holonomy around C).

Another way of saying this is [J-Weitsman 1992]:

If A is a flat connection on Σ with holonomy $\rho(A)$ around C, then $(\Xi A)(t)$ is the image of A under a gauge transformation on $\Sigma \setminus \{C\}$ which is equal to the identity on one boundary component C_+ of $\Sigma \setminus \{C\}$ but equal to an element $\exp(tc)$ on the other boundary component C_- (where c is in the Lie algebra of the maximal torus and exp is the exponential map). Ξ is the Hamiltonian flow of the function $\rho \mapsto \mathrm{Trace}\rho([C])$ (the trace of the holonomy around C). These glue back together to form a flat connection on Σ because $\exp(tc)$ is in the stabilizer of the restriction of A to C.

Another way of saying this is [J-Weitsman 1992]:

If A is a flat connection on Σ with holonomy $\rho(A)$ around C, then $(\Xi A)(t)$ is the image of A under a gauge transformation on $\Sigma \setminus \{C\}$ which is equal to the identity on one boundary component C_+ of $\Sigma \setminus \{C\}$ but equal to an element $\exp(tc)$ on the other boundary component C_- (where c is in the Lie algebra of the maximal torus and exp is the exponential map). Ξ is the Hamiltonian flow of the function $\rho \mapsto \mathrm{Trace}\rho([C])$ (the trace of the holonomy around C). These glue back together to form a flat connection on Σ because $\exp(tc)$ is in the stabilizer of the restriction of A to C. However the gauge transformation does not glue back together to form a gauge transformation on Σ, so the image is not gauge equivalent to A. This determines the Hamiltonian flow on the moduli space.

Goldman proved that if C_1 and C_2 are disjoint, then the corresponding Hamiltonian flows commute. J-Weitsman (1993) showed that the function Hol_C is the moment map for a Hamiltonian S^1 action where defined. It is not well defined when $\rho([C])$ is in the center of $SU(2)$.

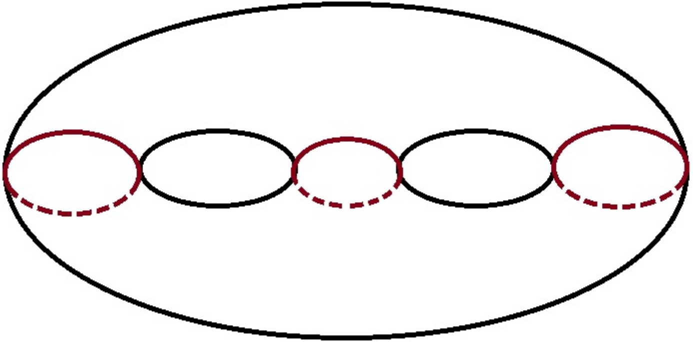
Goldman proved that if C_1 and C_2 are disjoint, then the corresponding Hamiltonian flows commute. J-Weitsman (1993) showed that the function Hol_C is the moment map for a Hamiltonian S^1 action where defined. It is not well defined when $\rho([C])$ is in the center of $SU(2)$. More precisely the map $\rho \mapsto \mathrm{Hol}_C(\rho)$ is a well-defined continuous function on M_2, but it is not differentiable at ρ for which $\rho([C])$ is in the center of $SU(2)$.

When (C_1, C_2, C_3) are the boundary components of a pair of pants (corresponding to the pants decomposition of Σ_2 whose corresponding trivalent graph is the letter Θ) the image of the moment maps $\mu = (\mathrm{Hol}_{C_1}, \mathrm{Hol}_{C_2}, \mathrm{Hol}_{C_3})$ is also a tetrahedron (for $0 \le t_j \le \pi$) determined by the variables $t_1, t_2, t_3 \in [0\pi]$, subject to

$$|t_1 - t_2| \le t_3 \le t_1 + t_2,$$

$$t_1 + t_2 + t_3 \le 2\pi$$

(J-Weitsman 1993).



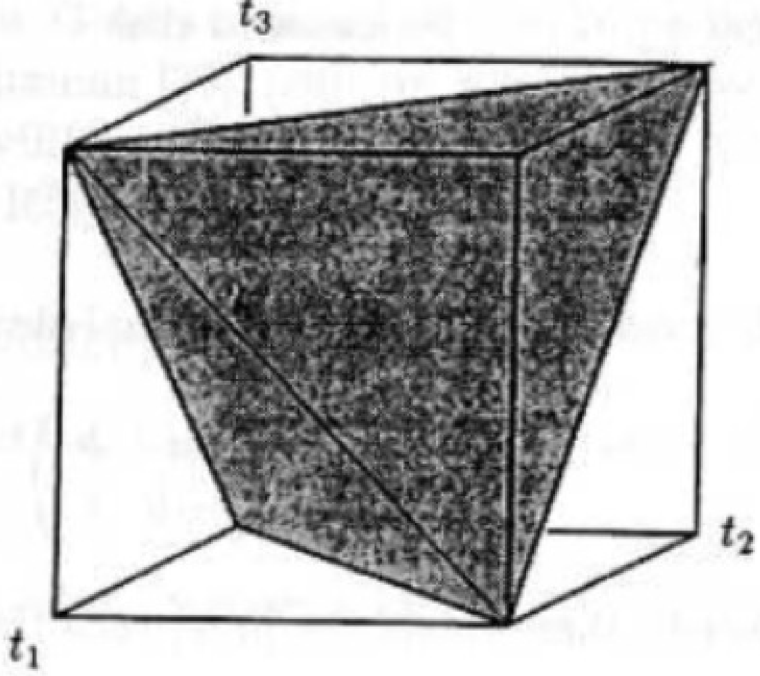
Note that this tetrahedron is the convex hull of the four vertices

$$\pi(0,0,0), \pi(1,1,0), \pi(1,0,1), \pi(0,1,1).$$

Note that this tetrahedron is the convex hull of the four vertices

$$\pi(0,0,0), \pi(1,1,0), \pi(1,0,1), \pi(0,1,1).$$

This is not the same as the tetrahedron Δ' which is the Newton polytope of $\mathbb{C}P^3$, which is the convex hull of the vertices $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$. These polytopes are not identified by an element of $SL(2,\mathbb{Z})$.


t_3
t_2
t_1

3. Duistermaat real locus

Theorem [Duistermaat 1983] Let M^{2n} be a symplectic manifold equipped with an antisymplectic involution τ and a Hamiltonian action of a torus $U(1)^n$ compatible with the involution.

Then the fixed point set of the involution is a Lagrangian submanifold of M.

Definition [O'Shea-Sjamaar]: The torus action is compatible with the involution τ if

$$\tau(ux) = \sigma(u)\tau(x)$$

for all $u \in T$ and $x \in M$ and an involution $\sigma : T \to T$. (For example, σ could be complex conjugation).

The torus action for the first and second copies of T are

$$u_1 \cdot (g_1, h_1, g_2, h_2) = (g_1 u_1, h_1, g_2, h_2)$$

resp.

$$u_2 \cdot (g_1, h_1, g_2, h_2) = (g_1, h_1, g_2 u_2, h_2)$$

The third torus action is

$$(g_1, h_1, g_2, h_2) \mapsto (e^{tX} g_1, h_1, e^{tY} g_2, h_2)$$

where X is the vector field

$$X(g_1, h_1, g_2, h_2) = h_2 h_1 - (h_2 h_1)^{-1}$$

and Y is the vector field

$$Y(g_1, h_1, g_2, h_2) = h_1 h_2 - (h_1 h_2)^{-1}$$

This means almost no involutions are compatible with the torus action. The only family of involutions compatible with the torus action is the following. Let

$$(g_1, h_1, g_2, h_2) = (e^{\lambda_3 X} g_1^s e^{\lambda_1 X_1}, h_1, e^{\lambda_3 Y} g_2^s e^{\lambda_2 X_2}, h_2).$$

Here $h_1 = e^{X_1}$, $h_2 = e^{X_2}$, $X = h_2 h_1 - (h_2 h_1)^{-1}$,
$Y = h_1 h_2 - (h_1 h_2)^{-1}$. Here, $\lambda_1, \lambda_2, \lambda_3$ are arbitrary real numbers. Let s be a section of $M^0 \to \Delta^0$. Then

$$s(\mu([g_1, h_1, g_2, h_2]) = [g_1^s, h_1, g_2^s, h_2];$$

this defines g_1^s and g_2^s (which depend on the section s).
Then

$$\tau([g_1, h_1, g_2, h_2]) = (e^{-\lambda_3 X} g_1^s e^{-\lambda_1 X_1}, h_1, e^{-\lambda_3 Y} g_2^s e^{-\lambda_2 X_2}, h_2).$$

(In other words the involution τ flips the sign of some of the variables.)

This means almost no involutions are compatible with the torus action. The only family of involutions compatible with the torus action is the following. Let

$$(g_1, h_1, g_2, h_2) = (e^{\lambda_3 X} g_1^s e^{\lambda_1 X_1}, h_1, e^{\lambda_3 Y} g_2^s e^{\lambda_2 X_2}, h_2).$$

Here $h_1 = e^{X_1}$, $h_2 = e^{X_2}$, $X = h_2 h_1 - (h_2 h_1)^{-1}$,
$Y = h_1 h_2 - (h_1 h_2)^{-1}$. Here, $\lambda_1, \lambda_2, \lambda_3$ are arbitrary real numbers.
Let s be a section of $M^0 \to \Delta^0$. Then

$$s(\mu([g_1, h_1, g_2, h_2]) = [g_1^s, h_1, g_2^s, h_2];$$

this defines g_1^s and g_2^s (which depend on the section s).
Then

$$\tau([g_1, h_1, g_2, h_2]) = (e^{-\lambda_3 X} g_1^s e^{-\lambda_1 X_1}, h_1, e^{-\lambda_3 Y} g_2^s e^{-\lambda_2 X_2}, h_2).$$

(In other words the involution τ flips the sign of some of the variables.)

Moreover, the image of the fixed point set under the moment map for the torus action is the image of M under the moment map.

Example 1:

$\mathbb{C}P^3$ is a symplectic manifold equipped with an antisymplectic involution (complex conjugation). The standard action of $U(1)^3$ on $\mathbb{C}P^3$ is compatible with the involution. The fixed point set of the involution is $\mathbb{R}P^3$.

Example 2:

M_2 is a symplectic variety equipped with the antisymplectic involution τ described above. The fixed point set of the involution is

$$\{(g_1, I, g_2, I)\}/G$$

(the set where $h_1 = h_2 = I$).

Here I is the identity matrix.

This fixed point set is the set $(G \times G)/G$.

Example 2:

M_2 is a symplectic variety equipped with the antisymplectic involution τ described above. The fixed point set of the involution is

$$\{(g_1, I, g_2, I)\}/G$$

(the set where $h_1 = h_2 = I$).

Here I is the identity matrix.

This fixed point set is the set $(G \times G)/G$. It is connected and compact.

Moment map image

The torus action is well defined on the preimage of the interior of the tetrahedron. So we can construct a bijective map from the preimage of the interior of the tetrahedron to an open dense subset of $\mathbb{C}P^3$.

We may define the map from M_2 to $\mathbb{C}P^3$ on the preimage of the interior of the tetrahedron.

We may define the map from M_2 to $\mathbb{C}P^3$ on the preimage of the interior of the tetrahedron.

The map sends the first Lagrangian submanifold into $\mathbb{R}P^3$. We use the Hamiltonian $U(1)^3$ actions to identify orbits of points in these Lagrangian submanifolds.

The region where the map is not defined is as follows.

1. preimage of vertices $g_1, g_2 \in Z(G)$: the preimage is

$$\{h_1, h_2) \in G \times G\}/G$$

(in other words the values of h_1 and h_2 are unconstrained)

The dimension is 3.

The region where the map is not defined is as follows.

1. preimage of vertices $g_1, g_2 \in Z(G)$: the preimage is

$$\{h_1, h_2) \in G \times G\}/G$$

(in other words the values of h_1 and h_2 are unconstrained)

The dimension is 3.

2. Preimage of edges (one of g_1 or g_2 in $Z(G)$): If $g_1 \in Z(G)$, then the preimage is $\{h_1 \in G\}$, $[h_2, g_2] = 1$ (so h_2 is in the same maximal torus T as g_2).

This is $(G/T) \times (T \times T)/W$.

The dimension is 4.

Moment maps

Let Δ^o be the interior of the tetrahedron. The map μ from $\mu^{-1}(\Delta^o)$ to Δ^o is a moment map for a torus action.

Also the map ν from $\mathbb{C}P^3$ to Δ' is a moment map for the action of $U(1)^3$ on $\mathbb{C}P^3$. Hence so is its restriction to $\nu^{-1}((\Delta')^o)$.

Moment maps

Let Δ^o be the interior of the tetrahedron. The map μ from $\mu^{-1}(\Delta^o)$ to Δ^o is a moment map for a torus action.

Also the map ν from $\mathbb{C}P^3$ to Δ' is a moment map for the action of $U(1)^3$ on $\mathbb{C}P^3$. Hence so is its restriction to $\nu^{-1}((\Delta')^o)$.

This means these two moment maps must be equal, up to an additive constant (by uniqueness of moment maps). Equivalently both moment maps provide a system of action-angle variables.

Moment maps

Let Δ^o be the interior of the tetrahedron. The map μ from $\mu^{-1}(\Delta^o)$ to Δ^o is a moment map for a torus action.

Also the map ν from $\mathbb{C}P^3$ to Δ' is a moment map for the action of $U(1)^3$ on $\mathbb{C}P^3$. Hence so is its restriction to $\nu^{-1}((\Delta')^o)$.

This means these two moment maps must be equal, up to an additive constant (by uniqueness of moment maps). Equivalently both moment maps provide a system of action-angle variables.

It follows that there is a bijective map from $\mu^{-1}(\Delta^o)$ to $\nu^{-1}((\Delta')^o)$, in other words $\mu^{-1}(\Delta^o)$ is an open dense subset of $\mathbb{C}P^3$.

Properties of the torus action

Theorem 1: [Goldman 1986] The torus action on an open dense subset of M_2 is well-defined.

Theorem 2:

For all points (g_1, h_1, g_2, h_2) whose images under the moment map are on the boundary of the tetrahedron, $[h_1, h_2] = 1$. Moreover, if the images under the moment map are in the interior of the tetrahedron, $[h_1, h_2] \neq 1$.

Theorem 3. The points in the interior $\mu^{-1}(\Delta^o)$ are all nonabelian representations. (This follows from the second part of Theorem 2.)

Theorem 4. The points on the boundary (the complement of $\mu^{-1}(\Delta^o)$) are all abelian representations

Proof: We use the fact that h_1 and h_2 commute, hence they are in the same maximal torus (the diagonal $U(1)$). Now impose $[g_1, h_1] = [g_2, h_2]^{-1}$. If $[g_1; h_1; g_2; h_2]$ is an irreducible representation, when h_1 and h_2 commute, then we get a contradiction; we subdivide according to the value of θ_1 and θ_2 where $Trace(h_1) = 2\cos\theta_1$, $Trace(h_2) = 2\cos\theta_2$, In all cases we find that h_1, h_2 or $h_1 h_2$ is in the center of $SU(2)$, which contradicts our assumption of irreducibility.

It follows that the J-Weitsman Hamiltonian torus action can be extended to the preimage of the interior of each face of the tetrahedron (everywhere where none of the g_j or h_j is in the center of $SU(2)$). This is clear because the torus actions can be defined provided we can identify a unique maximal torus containing the holonomy of the flat connection around the specified curve C.

Theorem 5. The interior $\mu^{-1}(\Delta^0)$ is isomorphic to $T^3 \times \Delta^0$.

(Proof: (J-Weitsman 1992) showed that the preimage of Δ^0 under the moment map is a 6-dimensional symplectic manifold with a free Hamiltonian action of T^3. In other words $\mu^{-1}(\Delta^o)$ is a bundle over Δ^0. Because Δ^o is contractible, it follows that this is a trivial bundle. $\square$

Main Theorem:

The interior $\mu^{-1}(\Delta^0)$ is an open dense set in the moduli space.

Proof: Because the set of irreducible representations is smooth, the previous theorems imply that $\mu^{-1}(\Delta^0)$ is dense in the set of irreducible representations. But the set of irreducible representations is dense in M_2, so $\mu^{-1}(\Delta^o)$ is dense in M_2.

The set where the torus action cannot be defined has dimension ≤ 4.

It follows that the union of $\mu^{-1}(\Delta^o)$ and the preimages of the interiors of the faces of the tetrahedron is dense in M_2.

This is the subset where the map to $\mathbb{C}P^3$ can be defined.

References:

S. Choi, Spherical triangles and the two components of the SO(3)-character space of the fundamental group of a closed surface of genus 2 (*Int. J. Math.*, 2011)

J.J. Duistermaat, Convexity and tightness for restrictions of Hamiltonian functions to fixed point sets of an antisymplectic involution. *Trans. Amer. Math. Soc.*, 1983.

W. Goldman, Invariant functions in Lie groups and Hamiltonian flows of surface group representations. (*Inventiones Math.*, 1986)

L. Jeffrey, J. Weitsman, Bohr-Sommerfeld orbits in the moduli space of flat connections and the Verlinde dimension formula. (*Commun. Math. Phys.*, 1992)

M.S. Narasimhan, S. Ramanan, Moduli of vector bundles on a compact Riemann surface. *Ann. Math.*, 1969.