Ergodic complex structures

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Ergodic complex structures

DEFINITION: Let M be a smooth manifold. **A complex structure** on M is an endomorphism $I \in \operatorname{End} TM$, $I^2 = -\operatorname{Id}_{TM}$, such that the eigenspace bundles of I are **involutive**, that is, satisfy $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$.

Let Comp be the space of such tensors equipped with a topology of convergence of all derivatives.

DEFINITION: The diffeomorphism group Diff is a Fréchet Lie group acting on Comp in a natural way. A complex structure is called **ergodic** if its Diff-orbit is dense in Comp.

REMARK: The "moduli space" of complex structures (if it exists) is identified with Comp / Diff; **existence of ergodic complex structures guarantees that the quotient Comp / Diff has no Hausdorff open subsets,** because all open sets of the quotient intersect.

THEOREM: Let M be a compact torus, $\dim_{\mathbb{C}} M \geqslant 2$, or a maximal holonomy hyperkähler manifold (to be explained later). **A complex structure on M is ergodic if and only if $\operatorname{Pic}(M)$ is not of maximal rank.**

Teichmüller spaces

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by Comp the space of complex structures on M, and let Teich := Comp / $\operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

REMARK: In all known cases Teich is **a finite-dimensional complex space** (Kodaira-Spencer-Kuranishi-Douady), but often **non-Hausdorff**.

DEFINITION: A **Calabi-Yau manifold** is a compact, Kähler manifold M with $c_1(M) = 0$.

THEOREM: (Bogomolov-Tian-Todorov) Teich **is a complex manifold when M is Calabi-Yau**.

Definition: Let $\operatorname{Diff}(M)$ be the group of diffeomorphisms of M. We call $\Gamma := \operatorname{Diff}(M)/\operatorname{Diff}_0(M)$ **the mapping class group**. The quotient Teich $/\Gamma$ **is identified with the set of equivalence classes of complex structures.**

REMARK: This terminology is **standard for curves.**

Holomorphically symplectic manifolds

DEFINITION: A **holomorphic symplectic form** is a non-degenerate, closed, holomorphic 2-form.

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: This produces a triple of symplectic forms on M: $\omega_I(\cdot,\cdot) = g(\cdot, I\cdot), \omega_J(\cdot,\cdot) = g(\cdot, J\cdot), \omega_K(\cdot,\cdot) = g(\cdot, K\cdot)$.

CLAIM: A hyperkähler manifold is holomorphically symplectic: $\omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

Proof: It's closed and has Hodge type (2,0), hence holomorphic. It is non-degenerate because ω_J and ω_K are non-degenerate. ■

REMARK: Converse is also true: any holomorphic symplectic compact Kähler manifold is hyperkähler.

Calabi-Yau theorem

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A compact hyperkähler manifold M is called **simple**, or **IHS**, or **maximal holonomy**, if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be compact and of maximal holonomy.

Hilbert schemes

THEOREM: (a special case of Enriques-Kodaira classification)
Let M be a compact complex surface which is hyperkähler. **Then M is either a torus or a K3 surface.**

DEFINITION: A **Hilbert scheme** $M^{[n]}$ of a complex surface M is a classifying space of all ideal sheaves $I \subset \mathcal{O}_M$ for which the quotient $\mathcal{O}_M/I$ has dimension n over $\mathbb{C}$.

REMARK: A Hilbert scheme **is obtained as a resolution of singularities** of the symmetric power $\operatorname{Sym}^n M$.

THEOREM: (Beauville) **A Hilbert scheme of a hyperkähler surface is hyperkähler.**

EXAMPLES.

EXAMPLE: **A Hilbert scheme of K3** is of maximal holonomy and hyperkähler.

EXAMPLE: Let T be a torus. Then it acts on its Hilbert scheme freely and properly by translations. For $n = 2$, the quotient $T^{[n]}/T$ is a Kummer K3-surface. For $n > 2$, a universal covering of $T^{[n]}/T$ is called **a generalized Kummer variety**.

REMARK: There are 2 more "sporadic" examples of compact hyperkähler manifolds, constructed by K. O'Grady. **All known maximal holonomy hyperkaehler manifolds are these 2 and two series:** Hilbert schemes of K3, and generalized Kummer.

REMARK: For hyperkähler manifolds, it is convenient to take for Teich **the space of all complex structures of hyperkähler type**, that is, **holomorphically symplectic and Kähler**. It is open in the usual Teichmüller space. We shall use this notation further on.

Computation of the mapping class group

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. **Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$,** and $c > 0$ a rational number.

DEFINITION: The form q is called **Bogomolov-Beauville-Fujiki form**. It has signature $(3, b_2 - 3)$, positive on $\langle \omega_I, \omega_J, \omega_K \rangle$, and negative on the primitive (1,1)-classes.

THEOREM: (Sullivan) Let M be a compact, simply connected Kähler manifold, $\dim_{\mathbb{C}} M \geqslant 3$. Denote by Γ_0 the group of automorphisms of an algebra $H^*(M,\mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Then **the natural map $\Gamma := \operatorname{Diff}(M)/\operatorname{Diff}_0 \longrightarrow \Gamma_0$ has finite kernel, and its image has finite index in Γ_0.**

THEOREM: Let M be a simple hyperkähler manifold, and Γ its mapping class group. Then
(i) $\Gamma\big|_{H^2(M,\mathbb{Z})}$ **is a finite index subgroup of $O(H^2(M,\mathbb{Z}),q)$.**
(ii) The map $\Gamma \longrightarrow O(H^2(M,\mathbb{Z}),q)$ **has finite kernel.**

REMARK: Sullivan's theorem implies that the mapping class group for $\dim_{\mathbb{C}} M \geqslant 3$, $\pi_1(M) = 0$, **is an arithmetic lattice**. Very much unlike the Teichmüller group!

The period map

Remark: For any $J \in$ Teich, (M, J) is also a simple hyperkähler manifold, hence $H^{2,0}(M, J)$ is one-dimensional.

Definition: Let Per : Teich $\longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ map J to a line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. The map Per : Teich $\longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ is called **the period map**.

REMARK: Per **maps** Teich **into an open subset of a quadric,** defined by

$$\mathbb{P}\text{er} := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \ \mid \ q(l, l) = 0, q(l, \bar{l}) > 0\}.$$

It is called **the period space** of M.

REMARK: $\mathbb{P}er = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1) = \text{Gr}_{++}(H^2(M, \mathbb{R}))$ (Grassmannian of positive 2-dimensional oriented planes). Indeed, the group $SO(H^2(M, \mathbb{R}), q) = SO(b_2 - 3, 3)$ acts transitively on $\mathbb{P}$er, and $SO(2) \times SO(b_2 - 3, 1)$ is a stabilizer of a point.

THEOREM: (Bogomolov) For any hyperkähler manifold, **period map is locally a diffeomorphism.**

Period space as a Grassmannian of positive 2-planes

PROPOSITION: The period space

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M,\mathbb{C}) \ \ | \ \ q(l,l) = 0, q(l,\bar{l}) > 0\}.$$

is identified with $SO(b_2-3,3)/SO(2)\times SO(b_2-3,1)$, which is a Grassmannian of positive oriented 2-planes in $H^2(M,\mathbb{R})$.

Proof. Step 1: Given $l \in \mathbb{P}H^2(M,\mathbb{C})$, **the space generated by** $\operatorname{Im} l, \operatorname{Re} l$ **is 2-dimensional,** because $q(l,l) = 0, q(l,\bar{l})$ implies that $l \cap H^2(M,\mathbb{R}) = 0$.

Step 2: This 2-dimensional plane is positive, because $q(\operatorname{Re} l, \operatorname{Re} l) = q(l+\bar{l}, l+\bar{l}) = 2q(l,\bar{l}) > 0$.

Step 3: Conversely, for any 2-dimensional positive plane $V \in H^2(M,\mathbb{R})$, **the quadric** $\{l \in V \otimes_{\mathbb{R}} \mathbb{C} \ \ | \ \ q(l,l) = 0\}$ **consists of two lines;** a choice of a line is determined by orientation. ■

REMARK: Let $W \subset H^2(M,\mathbb{R})$ be a 2-plane associated with a manifold (M,I). Then $W^\perp = H^{1,1}_I(M,\mathbb{R})$. Since Per is locally a diffeomorphism, $H^{1,1}_I(M) \cap H^2(M,\mathbb{Z})$ **is generally empty.**

COROLLARY: A general deformation of a given hyperkähler manifold **has no complex curves and no divisors.**

Proof: The corresponding cohomology group is 0. ■

Birational Teichmüller moduli space

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: (Huybrechts) Two points $I, I' \in$ Teich **are non-separable if and only if there exists a bimeromorphism $(M, I) \longrightarrow (M, I')$ which is non-singular in codimension 2 and acts as identity on $H^2(M)$.**

REMARK: This is possible only if (M, I) and (M, I') contain a rational curve. **General hyperkähler manifold has no curves;** ones which have curves belong to a countable union of divisors in Teich.

DEFINITION: The space $\operatorname{Teich}_b := \operatorname{Teich}/\sim$ is called **the birational Teichmüller space** of M.

THEOREM: **(Torelli theorem for hyperkähler manifolds)**
The period map $\operatorname{Teich}_b \xrightarrow{\operatorname{Per}} \mathbb{P}\mathrm{er}$ is a diffeomorphism, for each connected component of Teich_b.

Ergodic complex structures

DEFINITION: Let (M, μ) be a space with measure, and G a group acting on M preserving measure. This action is **ergodic** if all G-invariant measurable subsets $M' \subset M$ satisfy $\mu(M') = 0$ or $\mu(M \backslash M') = 0$.

CLAIM: Let M be a manifold, μ a Lebesgue measure, and G a group acting on M ergodically. **Then the set of non-dense orbits has measure 0.**

Proof. Step 1: Consider a non-empty open subset $U \subset M$. Then $\mu(U) > 0$, hence $M' := G \cdot U$ satisfies $\mu(M \backslash M') = 0$. For any orbit $G \cdot x$ not intersecting U, $x \in M \backslash M'$. Therefore, the set Z_U of such orbits has measure 0.

Step 2: Choose a countable base $\{U_i\}$ of topology on M. Then the set of points in dense orbits is $M \backslash \bigcup_i Z_{U_i}$. ■

DEFINITION: Let M be a complex manifold, Teich its Techmüller space, and Γ the mapping group acting on Teich **An ergodic complex structure** is a complex structure with dense Γ-orbit.

CLAIM: Let (M, I) be a manifold with ergodic complex structure, and I' another complex structure. **Then there exists a sequence of diffeomorphisms ν_i such that $\nu_i^*(I)$ converges to I'.**

Ergodicity of the mapping class group action

DEFINITION: **A lattice** in a Lie group is a discrete subgroup $\Gamma \subset G$ such that G/Γ has finite volume with respect to Haar measure.

THEOREM: (Calvin C. Moore, 1966) Let Γ be a lattice in a non-compact simple Lie group G with finite center, and $H \subset G$ a non-compact subgroup. **Then the left action of Γ on G/H is ergodic.**

THEOREM: Let $\mathbb{P}\text{er}$ be a component of a birational Teichmüller space, and Γ its monodromy group. Let $\mathbb{P}\text{er}_e$ be a set of all points $L \subset \mathbb{P}\text{er}$ such that the orbit $\Gamma \cdot L$ is dense (such points are called **ergodic**). **Then $Z := \mathbb{P}\text{er} \setminus \mathbb{P}\text{er}_e$ has measure 0.**

Proof. Step 1: Let $G = SO(b_2 - 3, 3)$, $H = SO(2) \times SO(b_2 - 3, 1)$. **Then Γ-action on G/H is ergodic,** by Moore's theorem.

Step 2: Ergodic orbits are dense, becuse the union of non-ergodic orbits has measure 0. ■

REMARK: Generic deformation of M has no rational curves, and no non-trivial birational models. Therefore, **outside of a measure zero subset,** $\text{Teich} = \text{Teich}_b$. This implies that **almost all complex structures on M are ergodic.**

Ratner's theorem

DEFINITION: Let G be a connected Lie group equipped with a Haar measure. **A lattice** $\Gamma \subset G$ is a discrete subgroup of finite covolume (that is, G/Γ has finite volume).

DEFINITION: A unipotent element in a Lie group is an exponent of a nilpotent element of its Lie algebra.

THEOREM: (Ratner's theorem)
Let $H \subset G$ be a Lie subroup generated by unipotents, and $\Gamma \subset G$ an arithmetic lattice. Then **the closure $\overline{\Gamma \cdot x}$ of any Γ-orbit in G/H is an orbit of a Lie subgroup $S \subset G$ containing xHx^{-1} such that $xSx^{-1} \cap \Gamma \subset G$ is a lattice.**

EXAMPLE: Let V be a real vector space with a non-degenerate bilinear symmetric form of signature $(3,k)$, $k > 0$, $G := SO^+(V)$ a connected component of the isometry group, $H \subset G$ a subgroup fixing a given positive 2-dimensional plane, $H \cong SO^+(1,k) \times SO(2)$, and $\Gamma \subset G$ an arithmetic lattice. Consider the quotient $\mathbb{P}\mathrm{er} := G/H$. **Then a closure of $\Gamma \cdot J$ in G/H is an orbit of a closed connected Lie group $S \supset H$.**

Characterization of ergodic complex structures

CLAIM: Let $G = SO^+(3,k)$, and $H \cong SO^+(1,k) \times SO(2) \subset G$. Then **any closed connected Lie subgroup $S \subset G$ containing H coincides with G or with H.**

COROLLARY: Let $J \in \mathbb{P}\mathrm{er} = G/H$. Then **either J is ergodic, or its Γ-orbit is closed in $\mathbb{P}\mathrm{er}$.**

REMARK: By Ratner's theorem, in the latter case the H-orbit of J has finite volume in G/Γ. Therefore, **its intersection with Γ is a lattice in H.** This brings

COROLLARY: Let $J \in \mathbb{P}\mathrm{er}$ be such that its Γ-orbit is closed in $\mathbb{P}\mathrm{er}$. Consider its stabilizer $\mathrm{St}(J) \cong H \subset G$. **Then $\mathrm{St}(J) \cap \Gamma$ is a lattice in $\mathrm{St}(J)$.**

COROLLARY: Let J be a non-ergodic complex structure on a hyperkähler manifold, and $W \subset H^2(M,\mathbb{R})$ be a plane generated by $\mathrm{Re}\,\Omega, \mathrm{Im}\,\Omega$. **Then W is rational.**

REMARK: This can be used to show that **any hyperkähler manifold is Kobayashi non-hyperbolic.**

Varieties of maximal Picard rank

REMARK: Since $H^{1,1}(M,I)\cap H^2(M,\mathbb{Z}) = \operatorname{Pic}(M,I)$ and $H^{1,1}(M,I)\cap H^2(M,\mathbb{Z}) = W^\perp$, W **is rational if and only if** $\operatorname{Pic}(M,I)$ **has maximal possible rank.**

REMARK: Same is true for a complex torus (same argument).

THEOREM: Let (M,I) be a complex manifold or a compact torus of dimension >1. **Then** I **ergodic if and only if** $\operatorname{rk}\operatorname{Pic}(M,I)$ **is not maximal.**

Kobayashi pseudometric

REMARK: The results further on are from a joint work by Ljudmila Kamenova, Steven Lu, Misha Verbitsky

DEFINITION: A **pseudometric** on a space M is a function $\operatorname{Sym}^2(M) \longrightarrow \mathbb{R}^{\geqslant 0}$ satisfying the triangle inequality (almost like a metric, but can vanish anywhere).

DEFINITION: The **Kobayashi pseudometric** on a complex manifold M is the supremum of all pseudometric on M such that any holomorphic map from the Poincaré disk to M is distance-nonincreasing.

THEOREM: Let $\pi: \mathcal{M} \longrightarrow X$ be a smooth holomorphic family, which is trivialized as a smooth manifold: $\mathcal{M} = M \times X$, and d_x the Kobayashi metric on $\pi^{-1}(x)$. **Then $d_x(m, m')$ is upper continuous on x.** ■

COROLLARY: Denote the diameter of the Kobayashi pseudometric by $\operatorname{diam}(d_x) := \sup_{m,m'} d_x(m, m')$. **Then $\operatorname{diam}: X \longrightarrow \mathbb{R}^{\geqslant 0}$ is upper continuous.**

Vanishing of Kobayashi pseudometric

THEOREM: Let (M, I) be a complex manifold with vanishing Kobayashi pseudometric. Then **the Kobayashi pseudometric vanishes for all ergodic complex structures in the same deformation class.**

Proof: Let $\operatorname{diam}: \text{Comp} \longrightarrow \mathbb{R}^{\geqslant 0}$ map a complex structure J to the diameter of the Kobayashi pseudometric on (M, J). Let J be an ergodic complex structure. The set of points $J' = \nu(J) \in \text{Comp}$, $\nu \in \text{Diff}$, is dense, because J is ergodic. By upper semi-continuity, $0 = \operatorname{diam}(I) \geqslant \inf_{J'=\nu(J)} \operatorname{diam}(J') = \operatorname{diam}(J)$.
■

EXAMPLE: Let M be a projective K3 surface. Then the Kobayashi metric on M vanishes. **Since all non-projective K3 are ergodic,** the Kobayashi metric vanishes on non-projective K3 surfaces as well.

THEOREM: Let M be a compact simple hyperkähler manifold. Assume that a deformation of M admits a holomorphic Lagrangian fibration and the Picard rank of M is not maximal. **Then the Kobayashi pseudometric on M vanishes.**

THEOREM: Let M be a Hilbert scheme of K3. **Then the Kobayashi pseudometric on M vanishes.**